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Research article

Invariant vector fields on contact metric manifolds under $\mathcal{D}$-homothetic deformation

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Abstract: In this paper, we study some vector fields on a contact metric manifold which are invariant under a $\mathcal{D}$-homothetic deformation.

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1. Introduction

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold of dimension $2n + 1$, $n \geq 1$. The so called $\mathcal{D}$-homothetic deformation introduced by Tanno in [17] is defined by

$$\bar{\phi} = \phi,\ \bar{\xi} = \frac{1}{a}\xi,\ \bar{\eta} = a\eta,\ \bar{g} = ag + a(a-1)\eta \otimes \eta \tag{1.1}$$

for certain positive constant a. It is easily seen that any $\mathcal{D}$-homothetic deformation deforms a contact metric structure into another contact metric structure. The study of invariants under a $\mathcal{D}$-homothetic deformation on a contact metric manifold is rather interesting and have been investigated by many authors. For example, any $\mathcal{D}$-homothetic deformation preserves K-contact, Sasakian (see [1]), contact strongly pseudo-convex CR (see [11]), the extended contact Bochner curvature tensor (see [5]), (k, μ)-contact structure (see [2, 3]), the Jacobi (k, μ)-contact structure (see [8]) and local ϕ-symmetry (see [4]). Very recently, some necessary conditions for a Ricci almost soliton invariant under a $\mathcal{D}$-homothetic deformation on a contact metric manifold is provided in [6]. Some other invariant properties and geometric conditions under a $\mathcal{D}$-homothetic deformation can also be seen in [10, 12].

A vector field V on a Riemanian manifold (M, g) is said to be conformal if $\mathcal{L}_V g = \rho g$, where ρ denotes a smooth function on M and $\mathcal{L}$ is the Lie differentiation. In particular, a conformal vector field

V is said to be homothetic if $\rho \in \mathbb{R}$ and is said to be Killing if $\rho = 0$. The geometry of conformal and Killing vector fields on contact metric manifolds have been studied in [14, 15]. In this paper, we present a sufficient and necessary condition for a conformal vector field invariant under a $\mathcal{D}$-homothetic deformation. As an application, some conditions for a holomorphically planar conformal vector field being invariant are provided. In addition, a complete η-Einstein K-contact metric manifold admitting a non-trivial generalized Ricci vector field is studied. We also show that a generalized Ricci vector field cannot be invariant under any $\mathcal{D}$-homothetic deformation on a contact metric manifold.

2. Contact metric manifolds

All notations adopted throughout this paper follow D. E. Blair [1]. A smooth manifold M of dimension $2n + 1$ is said to be a contact manifold if there exits on it a global 1-form η such that $\eta \wedge (\mathrm{d}\eta)^n \neq 0$ everywhere. If on M there exits a Riemannian metric g compatible with the contact structure, M is said to be a contact metric manifold. This is equivalent to that there exist a $(1, 1)$-type and $(1, 0)$-type tensor fields ϕ and ξ respectively such that

$$\phi^2 = -I + \eta \otimes \xi, \ \eta = g(\xi, \cdot), \ \mathrm{d}\eta = \Phi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X and Y, where Φ denotes the fundamental 2-form defined by $\Phi(X, Y) = g(X, \phi Y)$. A contact metric manifold is said to be K-contact if ξ is Killing and a Sasakian manifold if the contact structure is normal (see [1]).

On a contact metric manifold M^{2n+1}, we denote by $l = R(\cdot, \xi)\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi\phi$ respectively, where R denotes the Riemannian curvature tensor (defined by $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$) and $\mathcal{L}$ is the Lie differentiation. One can check that both l and h are symmetric; and h is trace-free and anti-commutes with ϕ. Using these notations, the following equation holds (see [1]):

$$\nabla\xi = -\phi - \phi h. \tag{2.3}$$

Applying (2.3) we see that a contact metric manifold is K-contact if and only if $h = 0$. Such a condition is sufficient for a contact metric manifold to be Sasakian for dimension three.

3. Invariant vector fields

If a geometric condition or property is preserved under a $\mathcal{D}$-homothetic deformation, it is said to be invariant. In this section, we give some invariant vector fields on contact metric manifolds under $\mathcal{D}$-homothetic deformation. Notice that when $a = 1$, (1.1) is just the identity transformation. Therefore, throughout this paper, for any $\mathcal{D}$-homothetic deformation, a is assumed to be a positive constant and not equal to 1.

A vector field V on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called an infinitesimal contact transformation if

$$\mathcal{L}_V\eta = \sigma\eta$$

for certain smooth function σ (see [16]). In particularly, a vector field V on a contact metric manifold is said to be a strictly infinitesimal contact transformation if $\mathcal{L}_V\eta = 0$. A vector field V on a contact metric manifold is said to be an infinitesimal automorphism if it leaves ϕ, ξ, η and g invariant.

Theorem 3.1. *A conformal vector field on contact metric manifolds is invariant under a non-identity $\mathcal{D}$-homothetic deformation if and only if it is an infinitesimal automorphism.*

Proof. Suppose a vector field V on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is conformal, we write $\mathcal{L}_V g = \rho g$ with ρ a smooth function. In view of (1.1), we have

$$\mathcal{L}_V \bar{g} = a\rho g + a(a-1)((\mathcal{L}_V \eta) \otimes \eta + \eta \otimes (\mathcal{L}_V \eta)) \tag{3.1}$$

If V is conformal for the metric $\bar{g}$, i.e., $\mathcal{L}_V \bar{g} = \bar{\rho}\bar{g}$, it follows from the above relation and (1.1) that

$$\rho g + (a-1)((\mathcal{L}_V \eta) \otimes \eta + \eta \otimes (\mathcal{L}_V \eta)) = \bar{\rho} g + (a-1)\bar{\rho}\eta \otimes \eta. \tag{3.2}$$

Since V is conformal for the metric g, we get $\rho = (\mathcal{L}_V g)(\xi, \xi) = 2\eta(\nabla_\xi V)$ because of (2.3). Using (2.3) we also have $(\mathcal{L}_V \eta)\xi = \eta(\nabla_\xi V)$. Therefore, the action of (3.2) on (ξ, ξ) gives $\rho = \bar{\rho}$ due to $a \neq 0$. In view of $a \neq 1$, now (3.2) becomes

$$(\mathcal{L}_V \eta) \otimes \eta + \eta \otimes (\mathcal{L}_V \eta) = \rho \eta \otimes \eta. \tag{3.3}$$

The action of (3.3) on $(\xi, \phi X)$ implies $(\mathcal{L}_V \eta)\phi X = 0$ for any vector field X and this shows $\mathcal{L}_V \eta = \frac{1}{2}\rho\eta$ because of $(\mathcal{L}_V \eta)\xi = \eta(\nabla_\xi V) = \frac{1}{2}\rho$. This means that V is an infinitesimal contact transformation. It has been proved in [15, Theorem 1] that if a conformal vector field on a contact metric manifold is an infinitesimal contact transformation, then it is an infinitesimal automorphism. The application of this result gives $\rho = 0$ and also the "only if" part proof of the theorem.

Conversely, if a conformal vector field on contact metric manifolds is an infinitesimal automorphism, using $\mathcal{L}_V \eta = 0$ in (3.1) we have $\mathcal{L}_V \bar{g} = a\rho g$. Recalling again the result shown in [15, Theorem 1] or [16], we also have $\rho = 0$ and this implies $\mathcal{L}_V \bar{g} = 0$. □

From proof of the above theorem, we have

Corollary 3.1. *If a conformal vector field on contact metric manifolds is invariant under a non-identity $\mathcal{D}$-homothetic deformation, then it is Killing.*

On a contact metric manifold, a holomorphically planar conformal vector (for short, HPCV) field (introduced by Sharma in [13]) is defined as a vector field V satisfying

$$\nabla_X V = \alpha X + \beta \phi X \tag{3.4}$$

for any vector field X and certain two smooth functions α and β. It has been proved in [13] that if a complete and connected K-contact metric manifold M admits a non-zero HPCV field V, then either V is a constant multiple of ξ, or M is isometric to a unit sphere. By skew-symmetry of ϕ with respect to g, one can check that an HPCV field is necessarily a conformal vector field.

Theorem 3.2. *An HPCV field V on contact metric manifolds of dimension > 3 is invariant under a non-identity $\mathcal{D}$-homothetic deformation if and only if V is a constant multiple of ξ, $a = 0$ and the manifold is K-contact.*

Proof. Suppose V is a holomorphically planar conformal vector field, we write $\nabla_X V = \alpha X + \beta\phi X$ for any vector field X. For any $\mathcal{D}$-homothetic deformation on a contact metric manifold, from (1.1) and the Koszul formula we have

$$\bar{\nabla}_X Y = \nabla_X Y - (a-1)(\eta(X)\phi Y + \eta(Y)\phi X) - \frac{a-1}{a} g(\phi hX, Y)\xi$$

for any vector fields X, Y, where $\bar{\nabla}$ is the Levi-Civita connection of the metric $\bar{g}$. Replacing Y by V in the above equation gives

$$\bar{\nabla}_X V = aX + b\phi X - (a-1)(\eta(X)\phi V + \eta(V)\phi X) - \frac{a-1}{a} g(\phi hX, V)\xi$$

for any vector field X.

If V is also a holomorphically planar conformal vector field for the new contact metric structure (1.1), i.e., $\bar{\nabla}_X V = \bar{\alpha} X + \bar{\beta}\bar{\phi} X$, combining this with the previous relation and using (1.1) we have

$$\alpha X + \beta\phi X - (a-1)(\eta(X)\phi V + \eta(V)\phi X) - \frac{a-1}{a} g(\phi hX, V)\xi = \bar{\alpha} X + \bar{\beta}\phi X \tag{3.5}$$

for any vector field X. Replacing X by ξ in (3.5) gives $\alpha\xi - (a-1)\phi V = \bar{\alpha}\xi$ and this implies $\alpha = \bar{\alpha}$ and $\phi V = 0$, where we have used the assumption $a \neq 1$.

It has been proved by A. Ghosh in [7, Lemma 3] that if a contact metric manifold of dimension > 3 admits a non-zero HPCV field V such that $\phi V = 0$, then M is K-contact. Using $h = 0$, $\alpha = \bar{\alpha}$ and $\phi V = 0$, (3.5) becomes $\beta - (a-1)\eta(V) = \bar{\beta}$ because X is an arbitrary vector field. A. Ghosh in [7, Lemma 1] proved that for any HPCV field on a contact metric manifold of dimension > 3, the associated function b is constant. Thus, $\phi V = 0$ shows $V = \frac{\beta - \bar{\beta}}{a-1}\xi$ with $\frac{\beta - \bar{\beta}}{a-1}$ a constant. Moreover, for any HPCV field V, according to (3.4) we have $\mathcal{L}_V g = 2ag$, i.e., V is conformal. Following Theorem 3.1, if V is invariant under a $\mathcal{D}$-homothetic deformation, then it is Killing and hence we have $a = 0$. The proof for "if" part is easy to check. □

Let (M, g) be a Riemannian manifold and Ric its Ricci tensor which is defined by $\mathrm{Ric}(X, Y) = \mathrm{trace}\{Z \to R(Z, X)Y\}$. We denote by Q the associated Ricci operator defined by $\mathrm{Ric}(X, Y) = g(QX, Y)$. A vector field V on M is said to be a generalized Ricci vector field (see [9]) if

$$\nabla_X V = \mu QX \tag{3.6}$$

for any vector field X and certain smooth function μ, or equivalently, $g(\nabla_\cdot V, \cdot) = \mu\mathrm{Ric}$. In particularly, V is said to be a Ricci vector field if μ in (3.6) is assumed to be a constant. If $V = 0$, (3.6) is meaningless and then a generalized Ricci vector filed is always assumed to be non-zero. Notice that on an Einstein manifold, a generalized Ricci vector field reduces to a concircular one and also a conformal one.

A contact metric manifold is said to be η-Einstein if $\mathrm{Ric} = \alpha g + \beta\eta \otimes \eta$ for some smooth functions α and β. In particular, on a K-contact manifold of dimension > 3, both α and β are constant (see [18]). Moreover, on a K-contact manifold, using $h = 0$ in (2.3) we get $\nabla\xi = -\phi$ and hence $l = Id - \eta \otimes \xi$, and we also have $Q\xi = 2n\xi$.

Theorem 3.3. *If a complete η-Einstein K-contact manifold M of dimension > 3 admits a generalized Ricci vector field, then M is compact and Sasakian.*

Proof. On an η-Einstein K-contact manifold M of dimension greater than three, in view of $Q\xi = 2n\xi$, we write

$$Q = \left(\frac{r}{2n} - 1\right) Id + \left(2n + 1 - \frac{r}{2n}\right)\eta \otimes \xi, \tag{3.7}$$

where r is the constant scalar curvature. Let V on M be a generalized Ricci vector field. Taking the covariant derivative of (3.6) implies that $\nabla_X\nabla_Y V = X(\mu)QY + \mu\nabla_X QY$ for any vector fields X, Y. It follows directly that

$$R(X, Y)V = X(\mu)QY - Y(\mu)QX + \mu((\nabla_X Q)Y - (\nabla_Y Q)X)$$

for any vector fields X, Y. In view of constancy of r, contracting X in the above equation and using the formula div$Q = \frac{1}{2}Dr$ we obtain

$$QV = QD\mu - rD\mu, \tag{3.8}$$

where by Df we mean the gradient of a function f. Comparing (3.8) with (3.7) gives

$$\begin{aligned}&\left(\frac{r}{2n} - 1\right)V + \left(2n + 1 - \frac{r}{2n}\right)\eta(V)\xi \\ =&\left(\frac{r}{2n} - r - 1\right)D\mu + \left(2n + 1 - \frac{r}{2n}\right)\xi(\mu)\xi.\end{aligned} \tag{3.9}$$

Taking the inner product of (3.9) with ξ gives $\eta(V) = (1 - \frac{r}{2n})\xi(\mu)$, which is inserted in (3.9) implying

$$\left(\frac{r}{2n} - 1\right)V = \left(\frac{r}{2n} - r - 1\right)D\mu + \frac{r}{2n}\left(2n + 1 - \frac{r}{2n}\right)\xi(\mu)\xi. \tag{3.10}$$

In view of constancy of r, taking the derivative of (3.10) and using $h = 0$, (2.3), we obtain

$$\begin{aligned}&\left(\frac{r}{2n} - 1\right)\nabla_X V \\ =&\left(\frac{r}{2n} - r - 1\right)\nabla_X D\mu + \frac{r}{2n}\left(2n + 1 - \frac{r}{2n}\right)[X(\xi(\mu))\xi - \xi(\mu)\phi X]\end{aligned}$$

for any vector field X. Recalling that V is a generalized Ricci vector field, from (3.6) and (3.7) we get

$$\nabla_X V = \mu\left(\frac{r}{2n} - 1\right)X + \mu\left(2n + 1 - \frac{r}{2n}\right)\eta(X)\xi.$$

Submitting the above relation into the previous one gives

$$\begin{aligned}&\left(\frac{r}{2n} - r - 1\right)\nabla_X D\mu + \frac{r}{2n}\left(2n + 1 - \frac{r}{2n}\right)[X(\xi(\mu))\xi - \xi(\mu)\phi X] \\ =&\mu\left(\frac{r}{2n} - 1\right)^2 X + \mu\left(\frac{r}{2n} - 1\right)\left(2n + 1 - \frac{r}{2n}\right)\eta(X)\xi\end{aligned} \tag{3.11}$$

for any vector field X.

By using the Poincare lemma (i.e., $d^2 = 0$) we see that $g(\nabla_X D\mu, Y)$ is symmetric with respect to X and Y, and hence it follows from (3.11) that

$$\frac{r}{2n}\left(2n + 1 - \frac{r}{2n}\right)[X(\xi(\mu))\eta(Y) + 2\xi(\mu)g(X, \phi Y) - Y(\xi(\mu))\eta(X)] = 0. \tag{3.12}$$

for any vector fields X, Y.

In view of (3.12), we discuss the following several cases. First, if the constant scalar curvature $r = 0$, (3.7) becomes $Q = -Id + (2n+1)\eta \otimes \xi$. It was proved by Sharma in [13, Proposition 1] that on a complete K-contact η-Einstein manifold M satisfying $\mathrm{Ric} = \alpha g + \beta \eta \otimes \eta$, if $\alpha > -2$, then M is compact and Sasakian. Next, in view of (3.12) we consider $r = 2n(2n+1)$ and in this case the manifold is Einstein, i.e., $Q = 2nId$. Following again [13, Proposition 1], the manifold M is compact and Sasakian. Third, if $r \neq 0$ and $r \neq 2n(2n+1)$, from (3.12) we have

$$X(\xi(\mu))\eta(Y) + 2\xi(\mu)g(X, \phi Y) - Y(\xi(\mu))\eta(X) = 0$$

for any vector fields X, Y. Let $X = \phi Y$ in the above relation be two unit vector fields orthogonal to ξ, it follows that $\xi(\mu) = 0$. Using this in (3.11) we get

$$\begin{aligned}&\left(\frac{r}{2n} - r - 1\right)\nabla_X D\mu\\ =&\mu\left(\frac{r}{2n} - 1\right)^2 X + \mu\left(\frac{r}{2n} - 1\right)\left(2n + 1 - \frac{r}{2n}\right)\eta(X)\xi\end{aligned} \tag{3.13}$$

for any vector field X. Taking the inner product of (3.13) with ξ, and using $\xi(\mu) = 0$, $h = 0$, and (2.3) we obtain

$$\begin{aligned}&\left(\frac{r}{2n} - r - 1\right)\phi X(\mu)\\ =&\mu\left(\frac{r}{2n} - 1\right)^2 \eta(X) + \mu\left(\frac{r}{2n} - 1\right)\left(2n + 1 - \frac{r}{2n}\right)\eta(X)\end{aligned} \tag{3.14}$$

for any vector field X. Replacing ϕX by X in (3.14) we obtain

$$\left(\frac{r}{2n} - r - 1\right)\phi^2 D\mu = 0. \tag{3.15}$$

The above equation gives either $r = \frac{2n}{1-2n}$ or $\phi^2 D\mu = 0$. For the former case, as similar with the above situation, applying again [13, Proposition 1], the manifold M is compact and Sasakian. For the later case, in view of $\xi(\mu) = 0$, we see that μ is a constant. Using this in (3.10) we have

$$\left(\frac{r}{2n} - 1\right)V = 0$$

for any vector field X. Since we have assumed that V is non-zero, it follows that $r = 2n$. As similar with the above situation, applying again [13, Proposition 1], the manifold M is compact and Sasakian. □

Let M^{2n+1} be a K-contact metric manifold and V its generalized Ricci vector field, i.e., $\nabla_X V = \mu QX$. Note that $h = 0$ on a K-contact metric manifold, thus, using (1.1) for any $\mathcal{D}$-homothetic deformation (1.1) we have

$$\bar{\nabla}_X Y = \nabla_X Y - (a-1)(\eta(X)\phi Y + \eta(Y)\phi X) \tag{3.16}$$

for any vector fields X, Y. Using (3.16), we have $\bar{\nabla}_X V = \mu QX - (a-1)(\eta(X)\phi V + \eta(V)\phi X)$ and hence

$$\begin{aligned}\bar{g}(\bar{\nabla}_X V, Y) =&a\mu \mathrm{Ric}(X, Y) + 2na(a-1)\mu\eta(X)\eta(Y)\\ &- a(a-1)(\eta(X)g(\phi V, Y) + \eta(V)g(\phi X, Y))\end{aligned}$$

for any vector fields X, Y, where we have used $Q\xi = 2n\xi$. Suppose V is also a generalized Ricci vector field for the metric $\bar{g}$ in (1.1), i.e., $\bar{g}(\bar{\nabla}_X V, Y) = \bar{\mu}\overline{\mathrm{Ric}}(X, Y)$. Recalling that the Ricci tensor for the metric $\bar{g}$ in (1.1) is given by (see also [6]):

$$\overline{\mathrm{Ric}}(X, Y) = \mathrm{Ric}(X, Y) - 2(a-1)g(X, Y) + 2(a-1)(na+n+1)\eta(X)\eta(Y)$$

for any vector fields X, Y. Combining this relation with the previous one gives that

$$\begin{aligned}&(\bar{\mu} - a\mu)\mathrm{Ric}(X, Y)\\ =&2\bar{\mu}(a-1)g(X, Y) - a(a-1)(\eta(X)g(\phi V, Y) + \eta(V)g(\phi X, Y))\\ &+ 2na(a-1)\mu\eta(X)\eta(Y) - 2\bar{\mu}(a-1)(na+n+1)\eta(X)\eta(Y)\end{aligned}$$

for any vector fields X, Y. Notice that the Ricci tensor Ric is symmetric. When the $\mathcal{D}$-homothetic deformation (1.1) is not identity, it follows from the above relation that

$$\eta(X)g(\phi V, Y) + \eta(V)g(\phi X, Y) = \eta(Y)g(\phi V, X) + \eta(V)g(\phi Y, X)$$

for any vector fields X, Y. Let Y in the above relation be ξ. This shows $\phi V = 0$. Using this back in the above relation we obtain $\eta(V)g(X, \phi Y) = 0$ for any vector fields X, Y. Letting $X = \phi Y$ be two unit vector fields orthogonal to ξ we obtain $\eta(V) = 0$. Finally, the operation of ϕ on $\phi V = 0$ implies $V = \eta(V)\xi = 0$. Based on these calculations, we have

Theorem 3.4. *On a K-contact metric manifold, a generalized Ricci vector field can not be invariant under any non-identity $\mathcal{D}$-homothetic deformation.*

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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