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Research article

New results on complex conformable integral

Francisco Martínez[1,]*, Inmaculada Martínez[1], Mohammed K. A. Kaabar[2] and Silvestre Paredes[1]

[1] Department of Applied Mathematics and Statistics, Technological University of Cartagena, Spain

[2] Department of Mathematics and Statistics, Washington State University, Pullman, WA 99163, USA

* **Correspondence:** Email: f.martinez@upct.es**;** Tel: +34968325586**.**

Abstract: A new theory of analytic functions has been recently introduced in the sense of conformable fractional derivative. In addition, the concept of fractional contour integral has also been developed. In this paper, we propose and prove some new results on complex fractional integration. First, we establish necessary and sufficient conditions for a continuous function to have antiderivative in the conformable sense. Finally, some of the well-known Cauchy's integral theorems will also be the subject of the extension that we do in this paper.

Keywords: conformable fractional derivative; conformable fractional integral; fractional contour integrals; fractional analytic functions; Cauchy's integral theorem
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1. Introduction

Integrals are very important in studying the functions of a complex variable. The integration theory is notable for its mathematical elegance. The theorems are generally concise and powerful, and their proofs are simple. The theory has also multiple applications in applied mathematics, engineering, and natural sciences, [1].

Fractional Calculus has attracted a great interest from mathematicians, scientists, and engineers due to the fact that many systems can be modeled in the context of fractional-order derivatives better than modeling them using the classical integer-order derivatives because many systems and phenomena exhibit memory effect and have internal damping [25]. The main idea of Fractional Calculus was initiated at the end of the 17th century when the French mathematician L'Hôpital proposed an interesting research question about the half-order derivative [25]. This was the point where the discussion started going into discovering this new mathematical idea in more details, and recently this topic of research created a revolution in applied mathematics and sciences. Various fractional derivative definitions have been formulated, and the most common ones in research studies are the Caputo fractional derivative and Riemann-Liouville fractional derivative.

In this current paper, our main objective is to establish, in the fractional context, important results on the existence of antiderivatives of a function of a complex variable in a certain domain, such as Cauchy-Goursat Theorem. In this way, we extend the research done in [2].

In this complex analysis fractional formulation, we will use the definitions of conformable derivatives and integrals, introduced by Khalil et al. [3]. From this new definition of derivative, important elements of the mathematical analysis of functions of a real variable have been established, among which we can highlight: Rolle's Theorem, Mean Value Theorem, chain rule, fractional power series expansion, fractional integration by parts formulas, and fractional single Laplace transform and double Laplace transform definitions, [3–5,25]. This new definition is a limit-based definition that satisfies the properties of the usual derivatives.

The conformable partial derivative of the order $\alpha \in (0,1]$ of the real-valued functions of several variables and conformable gradient vector are defined, as well as the conformable version of Clairaut's Theorem for partial derivatives of conformable fractional orders is proven in [6]. In [7], the conformable Jacobian matrix is defined; the chain rule for multivariable conformable derivative is given; the relation between conformable Jacobian matrix and conformable partial derivatives is revealed. In [8], two new results on homogeneous functions involving their conformable partial derivatives are introduced, specifically, the homogeneity of the conformable partial derivatives of a homogeneous function and the conformable version of Euler's Theorem. Furthermore, in a short time, several research studies have been conducted on the theory and applications of fractional differential equations based on this newly defined fractional derivative, [9–21,25]. In addition, both physical and geometrical meanings of conformable derivatives have been interpreted in [24,26], respectively. While many of the classical and well-known fractional derivatives can offer many advantages in modelling various physical systems, there are some disadvantages associated with those fractional derivatives such as the difficulty of obtaining analytical solutions and the need to implement special numerical techniques to overcome this challenge. However, conformable derivative provides a great help in finding analytical solutions in a very simple and effective way comparing to the classical fractional derivatives such as Grünwald-Letnikov, Caputo, and many others (we refer to [25] for more information about some analytical methods for solving wave equation in the context of conformable derivative).

In relation to complex fractional analysis, a theory of analytic functions in the conformable sense is developed in [22,23]. Furthermore, the definition of conformable integral along a contour and its first properties are introduced in [2].

The paper is organized as follows: In Section 2, the main concepts of conformable fractional calculus are presented. In Section 3, an important property of moduli of fractional contour integrals is presented. In Section 4, we establish the necessary and sufficient conditions for a continuous function to have $\alpha-$antiderivative. In Section 5, we develop the conformable version of some of the well-known Cauchy Theorems, specifically, the Cauchy-Goursat theorem, the Cauchy theorem for star-shaped domains, and the Morera theorem. In Section 6, we present several examples that show how the physical concepts, such as the circulation and the net flow of the velocity vector field of a two-dimensional fluid flow, can be modified appropriately to directly acclimatize the perception of fractional contour integrals.

2. Basic definitions and tools

We state in this section some definitions, remarks, and theorems which are important for our research study:

Remark 2.1. Let $\alpha \in [0,1)$. Then, $1-\alpha \geq 0$. So, $z^{1-\alpha} = r^{1-\alpha} e^{i(1-\alpha)\theta}$ is well defined for all $z \neq 0$. Further, the function $f(z) = z^{1-\alpha}$ is analytic on $C - (-\infty, 0]$.

Remark 2.2. [27]. On one hand, $\forall\, m \in N$, assume that $r = |z|$ and $\Phi = arg\, z$ where $z \in C$, then the De Moivre's formula can be defined as: $z^m = r^m\big(cos(m\Phi) + i\, sin(m\Phi)\big)$. On the other hand, assume now that $\Omega = \rho\big(cos(\Psi) + i\, sin(\Psi)\big)$ where $\neq 0$; $\Omega \in C$, then there exists a number of m such that the m^{th} roots of $\Omega = z^m$ can be expressed as follows:

$$z^{\frac{1}{m}} = \left(re^{i\Phi}\right)^{\frac{1}{m}} = (r)^{\frac{1}{m}}\left(e^{i\Phi}\right)^{\frac{1}{m}} = r^{\frac{1}{m}}\left(cos\left(\frac{2k\pi + \Phi}{m}\right) + i\, sin\left(\frac{2k\pi + \Phi}{m}\right)\right).$$

Let $\alpha \in [0,1)$. Then $1-\alpha \geq 0$. So, $z^{1-\alpha} = r^{1-\alpha} e^{i(1-\alpha)\theta}$ is well defined for all $z \neq 0$. Further, the function $f(z) = z^{1-\alpha}$ is analytic on $C - (-\infty, 0]$.

Definition 2.1. [22]. Let $f: D \to C$ be a given function and $z_0 \in D - \{0\}$ (which is an open set). Then, f is called α −differentiable at z_0 if

$$\lim_{\epsilon \to 0} \frac{f\left(z_0 + \varepsilon\, z_0^{1-\alpha}\right) - f(z_0)}{\varepsilon}. \tag{1}$$

In this case, we write $(T_\alpha f)(z_0)$ for such fractional derivative. Also, we use $f^{(\alpha)}(z_0)$ as an alternative way of writing the fractional derivative. If f is α-differentiable in some neighborhood of $z = 0$, and $\lim_{z\to 0}(T_\alpha f)(z)$ exists, then we define: $(T_\alpha f)(0) = \lim_{z \to 0}(T_\alpha f)(z)$.

As a consequence of the above definition, the following useful theorem is obtained:

Theorem 2.1. [23]. Let $f: D \to C$ be a given function and $z_0 \in D - \{0\}$ (which is an open set). If f is α-differentiable at $z_0 > 0$, then f is continuous at z_0.

One can easily prove the following:

Theorem 2.2. [22, 23]. The fractional derivative in Definition 2.1 satisfies the following:

(i) $T_\alpha(\lambda) = 0$, for all constant functions such that $f(t) = \lambda$.
(ii) $T_\alpha(f+g) = (T_\alpha f) + (T_\alpha g)$.
(iii) $T_\alpha(fg) = f(T_\alpha g) + g(T_\alpha f)$.
(iv) $T_\alpha\left(\frac{f}{g}\right) = \frac{g(T_\alpha f) - f(T_\alpha g)}{g^2}$.
(v) $T_\alpha\left(\frac{1}{\alpha} z^\alpha\right) = 1$.
(vi) $T_\alpha\left(e^{\frac{1}{\alpha} z^\alpha}\right) = e^{\frac{1}{\alpha} z^\alpha}$.
(vii) $T_\alpha\left(\sin\left(\frac{1}{\alpha} z^\alpha\right)\right) = cos\left(\frac{1}{\alpha} z^\alpha\right)$.
(viii) $T_\alpha\left(\cos\left(\frac{1}{\alpha} z^\alpha\right)\right) = -sin\left(\frac{1}{\alpha} z^\alpha\right)$.
(ix) In addition, if f is differentiable, then $(T_\alpha f)(z) = z^{1-\alpha} f'(z)$.

Motivated by the research work done in [27], to have a good visualization for some the selected functions in Theorem 2.2, Figures from 1 to 4 show the plot of the following functions:

$$f(z) = \frac{1}{\alpha} z^\alpha; f(z) = e^{\frac{1}{\alpha} z^\alpha};\ f(z) = sin\left(\frac{1}{\alpha} z^\alpha\right); f(z) = cos\left(\frac{1}{\alpha} z^\alpha\right)$$

at different chosen values of $\alpha = 0.25; 0.50; 0.75; 0.95$, respectively.

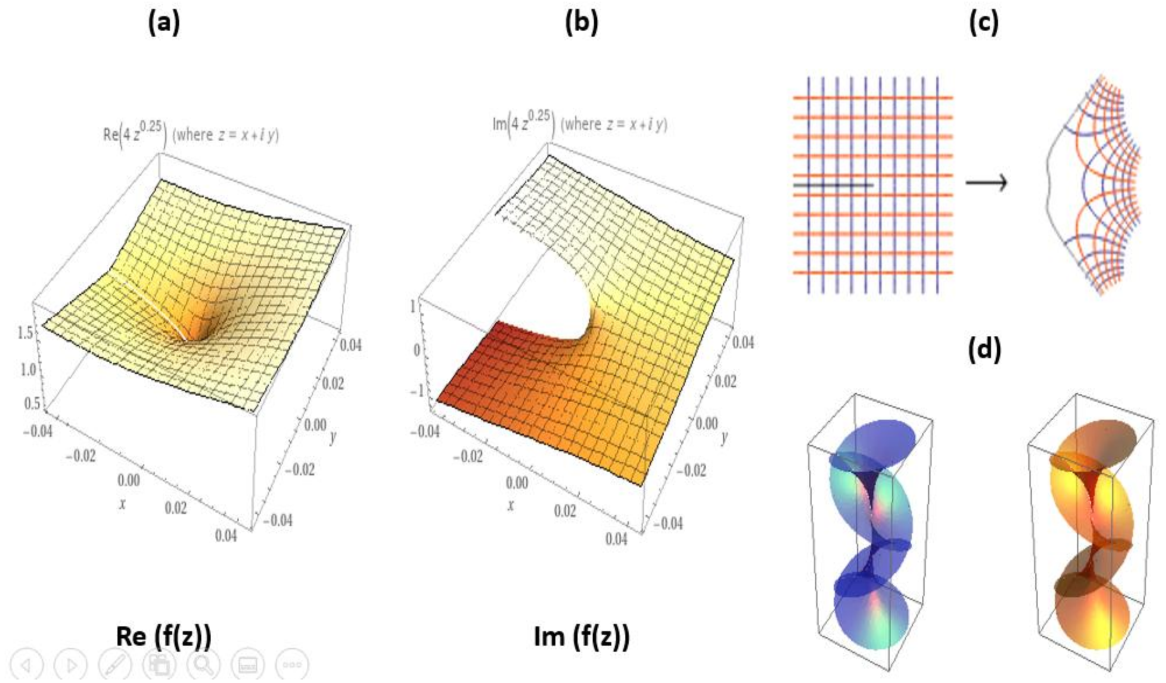

Figure 1. The plot of real part (a), imaginary part (b), complex map (c), and Riemann surface (d) of the function: $f(z) = \frac{1}{\alpha} z^{\alpha}$ at $\alpha = 0.25$.

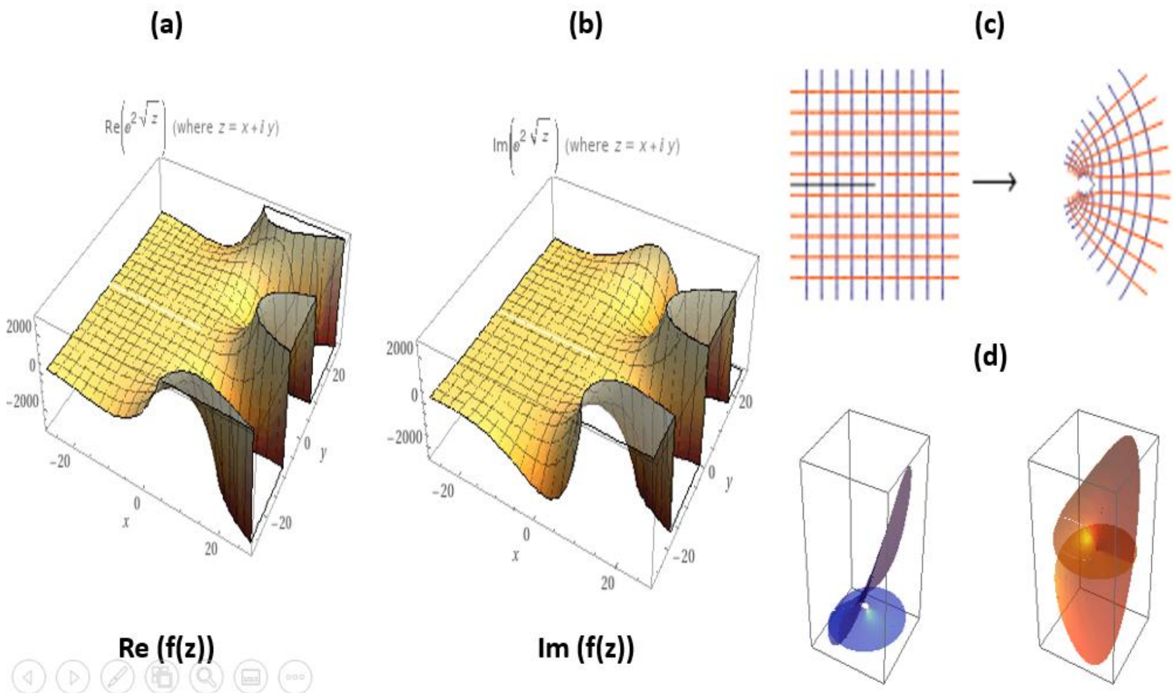

Figure 2. The plot of real part (a), imaginary part (b), complex map (c), and Riemann surface (d) of the function: $f(z) = e^{\frac{1}{\alpha} z^{\alpha}}$ at $\alpha = 0.50$.

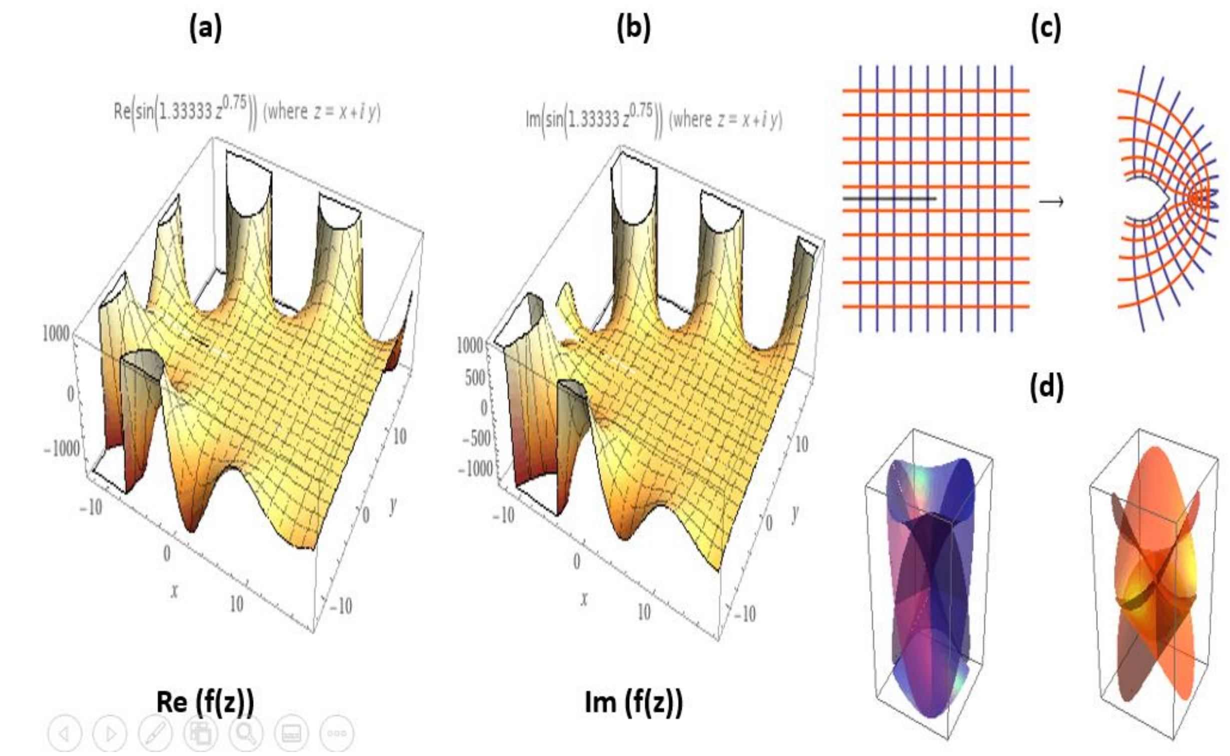

Figure 3. The plot of real part (a), imaginary part (b), complex map (c), and Riemann surface (d) of the function: $f(z) = sin\left(\frac{1}{\alpha} z^{\alpha}\right)$ at $\alpha = 0.75$.

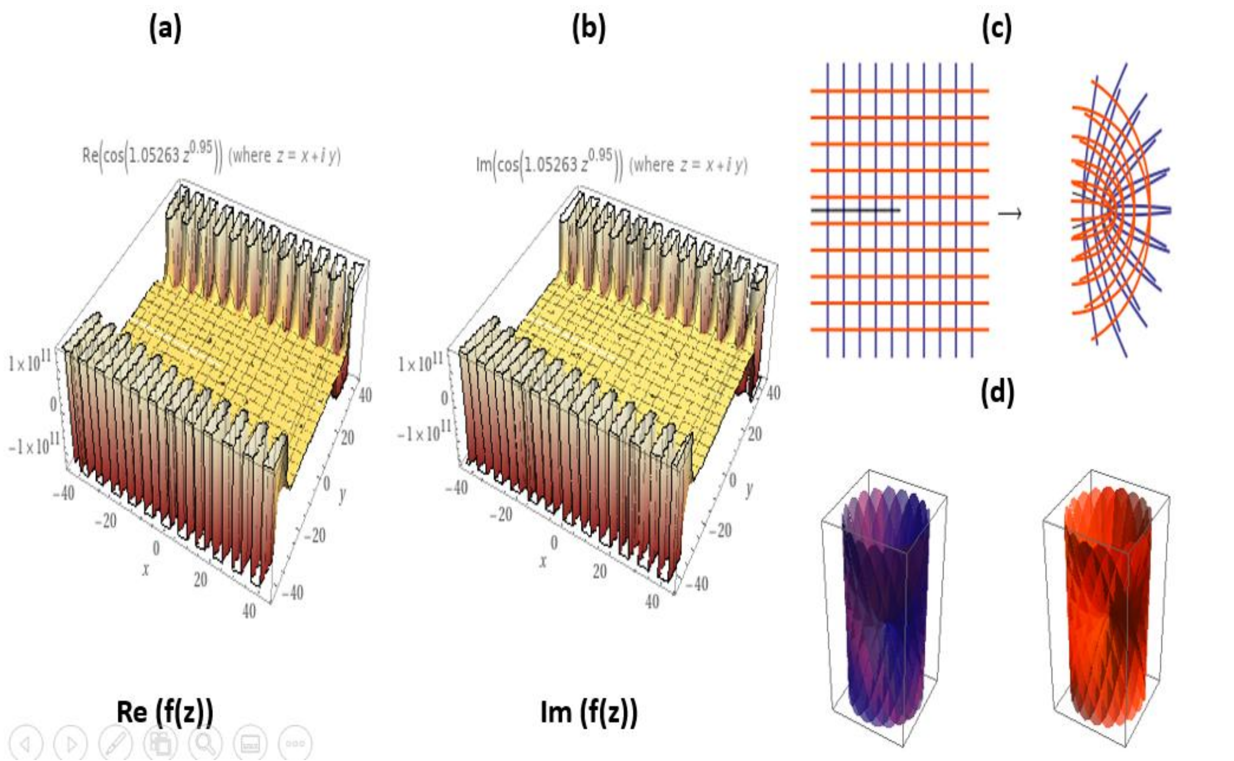

Figure 4. The plot of real part (a), imaginary part (b), complex map (c), and Riemann surface (d) of the function: $f(z) = cos\left(\frac{1}{\alpha} z^{\alpha}\right)$ at $\alpha = 0.95$.

Definition 2.2. [22]. Let $f: D \subset C \rightarrow C$ such that f is α −differentiable at z_0 and in some disk around z_0. Then, f is called α −analytic at z_0. If f is α −analytic at each z in a region D, then we say f is α −analytic on D.

The following definition is the α-fractional integral of a complex-valued function of a complex variable:

Definition 2.3. Let $\gamma: [a, b] \rightarrow C$ be a contour, with $a > 0$ and $\gamma^* \subset C - (-\infty. 0]$. Let $f: \gamma^* \rightarrow C$ be continuous. We define the contour $\alpha-$ integral of f along γ as soon as we understand the complex number [2],

$$\int_{\gamma} f(z) \frac{dz}{z^{1-\alpha}} = \int_{a}^{b} f(\gamma(t)) \frac{(T_{\alpha}\gamma)(t)}{(\gamma(t))^{1-\alpha}} \frac{dt}{t^{1-\alpha}}. \tag{2}$$

Note that, since γ is a contour, $(T_{\alpha}\gamma)(t)$ is also piecewise continuous on the closed interval $[a, b]$; hence, the existence of integral (2) is ensured.

3. Upper bounds for moduli of contour α −integrals

In this section, we consider a contour $\gamma: [a, b] \rightarrow C$, with $a > 0$ and $\gamma^* \subset C - (-\infty. 0]$.

Theorem 3.1. Let $\gamma: [a, b] \rightarrow C$ be a contour. For a continuous function $f: \gamma^* \rightarrow C$, we have:

$$\left|\int_{\gamma} f(z) \frac{dz}{z^{1-\alpha}}\right| \leq max \ \left\{\left|\frac{f(z)}{z^{1-\alpha}} : z \in \gamma^* \right|\right\} L(\gamma), \tag{3}$$

where $L(\gamma)$ is the length of the contour γ.

Proof. By theorem 4.4 in [2], we have:

$$\left|\int_{\gamma} f(z) \frac{dz}{z^{1-\alpha}}\right| = \left|\int_{a}^{b} f(\gamma(t)) \frac{(T_{\alpha}\gamma)(t)}{(\gamma\,(t))^{1-\alpha}} \frac{dt}{t^{1-\alpha}}\right| \leq \int_{a}^{b} |f(\gamma(t))| \left|\frac{(T_{\alpha}\gamma)(t)}{(\gamma(t))^{1-\alpha}}\right| \frac{dt}{t^{1-\alpha}}$$

$$\leq max \left\{\left|\frac{f(z)}{z^{1-\alpha}} : z \in \gamma^*\right|\right\} \int_{a}^{b} |(T_{\alpha}\gamma)(t)| \frac{dt}{t^{1-\alpha}} = max \left\{\left|\frac{f(z)}{z^{1-\alpha}} : z \in \gamma^*\right|\right\} L(\gamma).$$

Remark 3.1. In practice, it is not necessary to determine

$$max \left\{\left|\frac{f(z)}{z^{1-\alpha}} : z \in \gamma^*\right|\right\} = max \left\{\left|\frac{f(\gamma(t))}{(\gamma(t))^{1-\alpha}} : t \in [a, b]\right|\right\},$$

because it is very often to have an easy estimate as follows:

$$\left|\frac{f(z)}{z^{1-\alpha}}\right| \leq M, \forall z \in \gamma^*$$

Hence, we have $\max \ \left\{\left|\frac{f(z)}{z^{1-\alpha}} : z \in \gamma^*\right|\right\} \leq M$. By using (3), we obtain the following:

$$\left|\int_{\gamma} f(z) \frac{dz}{z^{1-\alpha}}\right| \leq M \cdot L(\gamma),$$

which in practice is just as useful as (3).

Example 3.1. Let γ denote the line segment from $z = 2i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin. Show that

$$\left|\int_{\gamma} \frac{z^{3/2}}{z^9}\frac{dz}{z^{1/2}}\right| \leq 32\sqrt{2}$$

Solution. The midpoint of γ is clearly the closest on γ to the origin. The distance of that midpoint from the origin is clearly $\sqrt{2}$, and the length of γ is $2\sqrt{2}$.
Hence, if z is any point of γ, $|z| \geq \sqrt{2}$. This means that, for such a point $\left|\frac{z^{3/2}}{z^9}\frac{1}{z^{1/2}}\right| \leq 16$. Consequently, by taking $M = 16$ and $L(\gamma) = 2\sqrt{2}$, we have:

$$\left|\int_{\gamma} \frac{z^{3/2}}{z^9}\frac{dz}{z^{1/2}}\right| \leq M \cdot L(\gamma) = 32\sqrt{2}$$

4. α −Antiderivatives

Although the value of a contour α −integral of a function $f(z)$ from a fixed point z_1 to a fixed point z_2 depends generally on the path taken, there are certain functions whose α −integrals from z_1 to z_2 have values that are independent of path. The following theorem is useful in determining when fractional integration is independent of path, and moreover, when an α −integral around a closed path has a value of zero.

To prove the theorem, we shall discover a fractional extension of the fundamental theorem of calculus that simplifies the evaluation of many contour α −integrals. That extension involves the concept of an α −antiderivative of a continuous function f in a domain $D \subset C - (-\infty. 0]$, which we define as follows:

Definition 4.1. Let f be a continuous function in a domain $D \subset C - (-\infty. 0]$. If there exists a function $F: D \to C$ such that $(T_\alpha F)(z) = f(z)$ for all z in D, then F is called α −antiderivative (or α −primitive) of f on D.

Remark 4.1. We notice the following:

(i) An α −antiderivative is, of necessary, an α −analytic function.

(ii) An α −antiderivative of a given function f is unique except for an additive complex constant.

Theorem 4.1. Suppose that a function $f(z)$ is continuous on a domain $D \subset C - (-\infty. 0]$. If any one of the following statements is true, then the others hold true.

(iii) $f(z)$ has an α −antiderivative $F(z)$ in D.

(iv) The α −integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value.

(v) The α −integrals of $f(z)$ around closed contours lying entirely in D all have zero value.

Proof. To prove the above theorem, it is sufficient to show that the statement (i) implies statement (ii), which also implies statement (iii), and finally that statement (iii) implies statement (i).
Let assume that statement (i) is true. If a contour γ from z_1 to z_2, lying in D, is just a smooth curve, with parametric representation $\gamma: [a, b] \to C$, with $a > 0$. By the chain rule, [4], we have:

$$\big(T_\alpha(F \circ \gamma)\big)(t) = (T_\alpha F)\big(\gamma(t)\big)\frac{(T_\alpha \gamma)(t)}{\big(\gamma(t)\big)^{1-\alpha}} = f\big(\gamma(t)\big)\frac{(T_\alpha \gamma)(t)}{\big(\gamma(t)\big)^{1-\alpha}}.$$

By the fundamental theorem of calculus which can be extended to be applicable to complex-valued functions of a real variable [2], so it follows that

$$\int_{\gamma} f(z)\frac{dz}{z^{1-\alpha}} = \int_a^b f\big(\gamma(t)\big)\frac{(T_\alpha\gamma)(t)}{\big(\gamma(t)\big)^{1-\alpha}}\frac{dt}{t^{1-\alpha}} = F\big(\gamma(t)\big)\big]_a^b = F\big(\gamma(b)\big) - F\big(\gamma(a)\big)$$

Since $\gamma(a) = z_1$ and $\gamma(b) = z_2$, the value of this contour α −integral can be written as:

$$F(z_2) - F(z_1)$$

As a result, the above value is independent of the contour γ as long as γ extends from z_1 to z_2 and lies in D. That is,

$$\int_{z_1}^{z_2} f(z)\frac{dz}{z^{1-\alpha}} = F(z_2) - F(z_1) = F(z)\big]_{z_1}^{z_2}, \tag{4}$$

where γ is smooth. Expression (4) is also valid when γ is any contour, not necessarily a smooth one, that lies in D. If γ consists of a finite number of smooth curves γ_k $(k = 1,2,\dots,n)$, each γ_k is extending from point z_k to a point z_{k+1}, then

$$\int_{\gamma} f(z)\frac{dz}{z^{1-\alpha}} = \sum_{k=1}^{n}\int_{\gamma_k} f(z)\frac{dz}{z^{1-\alpha}} = \sum_{k=1}^{n}[F(z_{k+1}) - F(z_k)] = F(z_{n+1}) - F(z_1).$$

The fact that statement (ii) follows from statement (i) is now established.
To see that statement (ii) implies statement (iii), we let z_1 and z_2 represent any two points on a closed contour γ lying in D and form two paths, each with initial point z_1 and final point z_2, such that $\gamma = \gamma_1 + (-\gamma_2) = \gamma_1 - \gamma_2$. By assuming that statement (ii) is true, one can write

$$\int_{\gamma_1} f(z)\frac{dz}{z^{1-\alpha}} = \int_{\gamma_2} f(z)\frac{dz}{z^{1-\alpha}}, \tag{5}$$

or

$$\int_{\gamma_1} f(z)\frac{dz}{z^{1-\alpha}} + \int_{-\gamma_2} f(z)\frac{dz}{z^{1-\alpha}} = 0. \tag{6}$$

That is, the α −integral of $f(z)$ around the closed contour $\gamma = \gamma_1 - \gamma_2$ has a zero value.

It remains to show that statement (iii) implies statement (i). We do this by assuming that statement (iii) is true, establishing the validity to statement (ii), and then arriving at statement (i). To see that statement (ii) is true, we let γ_1 and γ_2 represent any two contours, lying in D, from a point z_1 to a point z_2 and let's observe that, in view of statement (iii), Equation (6) holds. Thus, Equation(5) holds. Therefore, integration is independent of path in D, and we can define the function as:

$$F(z) = \int_{z_0}^{z} f(s)\frac{ds}{s^{1-\alpha}},$$

on D. The proof of the theorem is complete once we show that $(T_\alpha F)(z) = f(z)$ everywhere in D. We do this by letting $z + \Delta z \cdot z^{1-\alpha}$ be any point, distinct from z, lying in some neighbourhood of z that is small enough to be contained in D. Then, we have:

$$F(z + \Delta z \cdot z^{1-\alpha}) - F(z) = \int_{z_0}^{z+\Delta z\cdot z^{1-\alpha}} f(s)\frac{ds}{s^{1-\alpha}} - \int_{z_0}^{z} f(s)\frac{ds}{s^{1-\alpha}} = \int_{z}^{z+\Delta z\cdot z^{1-\alpha}} f(s)\frac{ds}{s^{1-\alpha}},$$

where the path of integration from z to $z + \Delta z \cdot z^{1-\alpha}$ may be selected as a line segment. Since

$$f(z) = \int_{z}^{z+\Delta z\cdot z^{1-\alpha}} \frac{f(z)}{\Delta z \cdot z^{1-\alpha}}\,ds.$$

Hence, we obtain the following:

$$\left|\frac{F(z+\Delta z\cdot z^{1-\alpha})-F(z)}{\Delta z}-f(z)\right|=\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z\cdot z^{1-\alpha}}f(s)\frac{ds}{s^{1-\alpha}}-\int_{z}^{z+\Delta z\cdot z^{1-\alpha}}\frac{f(z)}{\Delta z\cdot z^{1-\alpha}}ds\right|$$

$$=\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z\cdot z^{1-\alpha}}\left(\frac{f(s)}{s^{1-\alpha}}-\frac{f(z)}{z^{1-\alpha}}\right)ds\right|\le\frac{1}{|\Delta z|}\left|\frac{f(s)}{s^{1-\alpha}}-\frac{f(z)}{z^{1-\alpha}}\right||\Delta z\cdot z^{1-\alpha}|$$

$$=\left|\frac{f(s)}{s^{1-\alpha}}-\frac{f(z)}{z^{1-\alpha}}\right||\Delta z\cdot z^{1-\alpha}|\,.$$

Consequently, since f is continuous, we get

$$\lim_{\Delta z\to 0}\left|\frac{F(z+\Delta z\cdot z^{1-\alpha})-F(z)}{\Delta z}-f(z)\right|=0\,,$$

or $(T_\alpha F)(z)=f(z)$.

Remark 4.2. It is easy to prove that the α −primitive F of a continuous function f in a domain $D\subset C-(-\infty.0]$ satisfies a Lipschitz condition. In fact, we can write the following:

$$F(z)=\int_{z_0}^{z}f(s)\frac{ds}{s^{1-\alpha}}\,,$$

on D. Since f is a continuous function in D, we have

$$\left|\frac{f(z)}{z^{1-\alpha}}\right|\le K,\forall z\in D$$

So,

$$|F(z_2)-F(z_1)|=\left|\int_{z_1}^{z_2}f(s)\frac{ds}{s^{1-\alpha}}\right|\le\int_{z_1}^{z_2}\left|\frac{f(s)}{s^{1-\alpha}}\right|ds\le K|z_2-z_1|,\forall z_1,z_2\in D$$

Example 4.1. By finding an $\frac{1}{3}$ −antiderivative, we evaluate this integral, where the path is any contour γ between the indicated limits of integration, such that $\gamma^*\subset C-(-\infty.0]$.

$$\int_{\frac{1}{216}}^{-8i}cos\left(3\pi\sqrt[3]{z}\right)\frac{dz}{\sqrt[3]{z^2}}$$

Solution. The function $cos\left(3\pi\sqrt[3]{z}\right)$ has the $\frac{1}{3}$ − antiderivative $\frac{sin\left(3\pi\sqrt[3]{z}\right)}{\pi}$ everywhere in the set $C-(-\infty.0]$. Consequently, from any contour γ from $-8i$ to $\frac{1}{216}$, such that $\gamma^*\subset C-(-\infty.0]$,

$$\int_{\frac{1}{216}}^{-8i}cos\left(3\pi\sqrt[3]{z}\right)\frac{dz}{\sqrt[3]{z^2}}=\frac{sin\left(3\pi\sqrt[3]{z}\right)}{\pi}\Bigg]_{\frac{1}{216}}^{-8i}=\frac{1}{\pi}\left(-1+i\frac{e^{12\pi}-1}{2e^{6\pi}}\right)$$

Example 4.2. By finding an $\frac{1}{2}$ −antiderivative, we evaluate this integral, where the path is any contour γ between the indicated limits of integration, such that $\gamma^*\subset C-(-\infty.0]$.

$$\int_{\frac{1}{4}}^{\frac{1}{64}}e^{i2\pi\sqrt{z}}\frac{dz}{\sqrt{z}}$$

Solution. The function $e^{i2\pi\sqrt{z}}$ has the $\frac{1}{2}$ −antiderivative $\frac{e^{i2\pi\sqrt{z}}}{i\pi}$ everywhere in the set $C-(-\infty.0]$. Consequently, from any contour γ from $\frac{1}{4}$ to $\frac{1}{64}$, such that $\gamma^* \subset C-(-\infty.0]$,

$$\int_{\frac{1}{4}}^{\frac{1}{64}} e^{i2\pi\sqrt{z}}\frac{dz}{\sqrt{z}} = \frac{e^{i2\pi\sqrt{z}}}{i\pi}\Bigg]_{\frac{1}{4}}^{\frac{1}{64}} = \frac{1}{i\pi}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}+1\right) = \frac{1}{i\pi}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}+2}{2}\right)$$

Example 4.3. Using the above theorem, we can show that

$$\int_{\gamma} z^{n-\alpha}\frac{dz}{z^{1-\alpha}} = 0 \quad (n=1,2,\dots)$$

when γ is any closed contour such that $\gamma^* \subset C-(-\infty.0]$, and $\alpha \in (0,1]$.
Solution. Note that the function $z^{n-\alpha}$ $(n=1,2,\dots)$ always has an α −antiderivative in any domain that contains in $C-(-\infty.0]$. So, by the above theorem,

$$\int_{\gamma} z^{n-\alpha}\frac{dz}{z^{1-\alpha}} = 0$$

for any closed contour γ such that $\gamma^* \subset C-(-\infty.0]$, and $\alpha \in (0,1]$.

5. Fractional Cauchy's integral theorem

The usefulness of Theorem 4.1 to prove the existence of α −antiderivatives is doubtful, since to justify that a function has α −antiderivatives in a certain domain Ω, it is necessary to verify that its α −integral along any closed curve in Ω is zero, which does not seem an easy task in practice. This problem in classical complex analysis is solved through theorems that guarantee that under certain conditions, the integral of a function along any curve closed is null. Those theorems are called Cauchy theorems. In those theorems, we consider an open set Ω and a closed curve γ in Ω. Additional hypotheses are assumed in Ω or γ to conclude that $\int_{\gamma} f(z)dz = 0$ for each analytic function f in Ω.

In this section, we will first establish a conformable version of one of the most important classical integral theorems mentioned above, the so-called Cauchy-Goursat theorem.

Theorem 5.1. Let f be α −analytic on a set open $\Omega \subset C-(-\infty.0]$ and let $\Delta(a,b,c)$ be a triangle of vertices a,b,c contained in Ω. Then, we have:

$$\int_{[a,b,c,a]} f(z)\frac{dz}{z^{1-\alpha}} = 0 \tag{7}$$

Proof. We will call $\gamma=[a,b,c,a]$, $\Delta = \Delta(a,b,c)$ and $I=\int_{\gamma} f(z)\frac{dz}{z^{1-\alpha}}$. The objective is to prove that $I=0$. For this, let's consider the midpoints of the sides $a' = \frac{b+c}{2}$, $b' = \frac{a+c}{2}$, $c' = \frac{a+c}{2}$.
We can write I in the following form:

$$I = \int_{\gamma} f(z)\frac{dz}{z^{1-\alpha}} = \int_{[a,b',c',a]} f(z)\frac{dz}{z^{1-\alpha}} + \int_{[c',b,a',c']} f(z)\frac{dz}{z^{1-\alpha}} + \int_{[a',c,b',a']} f(z)\frac{dz}{z^{1-\alpha}} + \int_{[b',c',a',b]} f(z)\frac{dz}{z^{1-\alpha}}$$

This equality is true since there are paths (the ones inside the triangle) which are travelled in opposite directions, so the respective integrals are cancelled (see Figure 5).

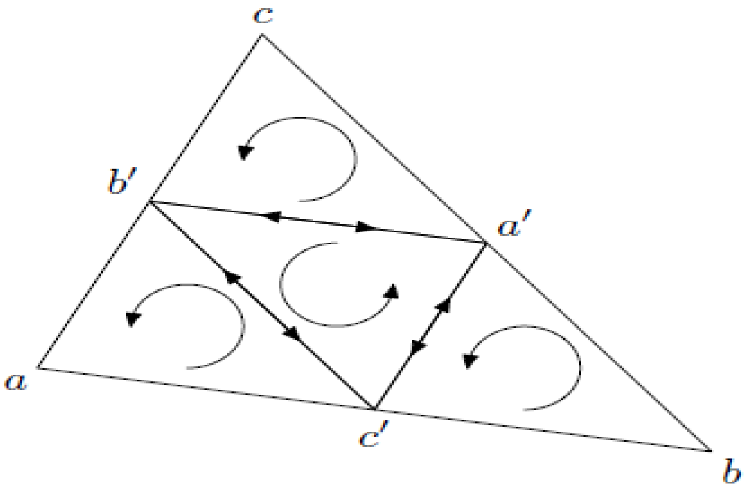

Figure 5. Scheme of the integration path.

If we call these four integrals J_1, J_2, J_3, J_4, and I_1 one of the integrals with the largest modulus from them, and $\gamma_1 = [a_1, b_1, c_1, a_1]$ to the path of I_1, we have:

$$|I| \leq 4|I_1|$$

By repeating the same argument for the triangle $\Delta_1 = \Delta(a_1, b_1, c_1)$, we obtain a sequence of triangles $\Delta_n = \Delta(a_n, b_n, c_n)$ and polygonal $\gamma_n = [a_n, b_n, c_n, a_n]$ with the property that $\Delta_n \subset \Delta_{n-1}$ and

$$diameter(\Delta_n) = \frac{1}{2} diameter(\Delta_{n-1}) = \frac{1}{2^n} diameter(\Delta)$$

$$L(\gamma_n) = \frac{1}{2} L(\gamma_{n-1}) = \frac{1}{2^n} L(\gamma)$$

$$|I_{n+1}| \leq 4|I_n|$$

Let $z_0 \in \bigcap_{n=1}^{\infty} \Delta_n$ (which exists since we are considering the intersection of a sequence decreasing non-empty closed with a succession of diameters converging to zero in complete metric space). Clearly, $z_0 \in \Delta(a, b, c)$. Let's put $p(z) = f(z_0) + (T_\alpha F)(z_0)\frac{z-z_0}{z_0^{1-\alpha}}$ which is a polynomial function. Therefore, it has a α −antiderivative; then $\int_{\gamma_n} p(z)\frac{dz}{z^{1-\alpha}} = 0$.

Hence, we can write the following:

$$I_n = \int_{\gamma_n} f(z)\frac{dz}{z^{1-\alpha}} = \int_{\gamma_n} \left(f(z) - f(z_0) - (T_\alpha F)(z_0)\frac{z-z_0}{z_0^{1-\alpha}}\right)\frac{dz}{z^{1-\alpha}}.$$

If we take

$$z = z_0 + (z - z_0) = z_0 + \frac{(z-z_0)z_0^{1-\alpha}}{z_0^{1-\alpha}},$$

we can also write

$$f(z)-f(z_0)-(T_\alpha F)(z_0)\frac{z-z_0}{z_0^{1-\alpha}}=f\left(z_0+\frac{(z-z_0)z_0^{1-\alpha}}{z_0^{1-\alpha}}\right)-f(z_0)-(T_\alpha F)(z_0)\frac{z-z_0}{z_0^{1-\alpha}}.$$

Given $\varepsilon>0$, due to the α −differentiability of f at z_0, there exists $\delta>0$ such that the open disk $D(z_0,\delta)$ is contained in Ω, and for all $z\in D(z_0,\delta)$, we have:

$$\left|f\left(z_0+\frac{(z-z_0)z_0^{1-\alpha}}{z_0^{1-\alpha}}\right)-f(z_0)-(T_\alpha F)(z_0)\frac{z-z_0}{z_0^{1-\alpha}}\right|<\varepsilon\frac{|z-z_0|}{|z_0^{1-\alpha}|}.$$

If the $diameter(\Delta_n)<\delta$ then $\Delta_n\subset D(z_0,\delta)$. Furthermore, if m_n denotes the minimum value of $|z^{1-\alpha}|$ in γ_n^*, we have:

$$\begin{aligned}|I|\le|I_n|&\le\frac{4^nL(\gamma_n)}{m_n}max\left\{\left|f(z)-f(z_0)-(T_\alpha F)(z_0)\frac{z-z_0}{z_0^{1-\alpha}}\right|:z\in\gamma_n^*\right\}\\&\le\frac{4^n\varepsilon L(\gamma_n)}{m_n|z_0^{1-\alpha}|}max\{[z-z_0]:z\in\gamma_n^*\}\le\frac{4^n\varepsilon L(\gamma_n)}{m_n|z_0^{1-\alpha}|}diameter(\Delta_n)\\&\le\frac{4^n\varepsilon}{m_n|z_0^{1-\alpha}|}\frac{1}{2^n}L(\gamma)\frac{1}{2^n}diameter(\Delta)=\frac{\varepsilon L(\gamma)diameter(\Delta)}{m_n|z_0^{1-\alpha}|}.\end{aligned}$$

Since $\varepsilon>0$ is chosen arbitrary, we get $I=0$.

Remark 5.1. An open set $\Omega\subset C-(-\infty.0]$ is a star-shaped domain with respect to a point $z_0\in\Omega$ if the line segment that unites z_0 with any other point of Ω stays inside Ω, that is, $[z_0.z]^*\subset\Omega$ for all $z\in\Omega$. For example, a disk is a star-shaped domain with respect to any of its points. Of course, any convex set is a star-shaped domain over any of its points, but there are star-shaped domains that are not convex (such as the polygon shown in the Figure 6).

Theorem 5.2 (Fractional Cauchy's theorem for a star-shaped domain). All α −analytical functions in one star-shaped domain have α −antiderivatives in that domain.

Proof. Let $\Omega\subset C-(-\infty.0]$ be a star-shaped domain with respect to z_0 and let f be α −analytic on Ω. We are looking for an α −antiderivative of f, and the most intuitive way to define it is as follows:

$$F(z)=\int_{[z_0,z]}f(s)\frac{ds}{s^{1-\alpha}}.$$

We are going to show that the defined F is certainly α −antiderivative of f in Ω. Since Ω is a star-shaped domain at z_0, the function F is well defined. Let $a\in\Omega$ and $\rho>0$, such that $D(a,\rho)\subset\Omega$. Let us take a point $z\in D(a,\rho)$. Since all the points of the segment $[a,z]$ are contained in Ω, then the segment that joins z_0 with any of these points will be contained in Ω as this is a star-shaped domain. Therefore, the triangle $\Delta(z_0,a,z)$ is totally contained in Ω (see Figure 6).

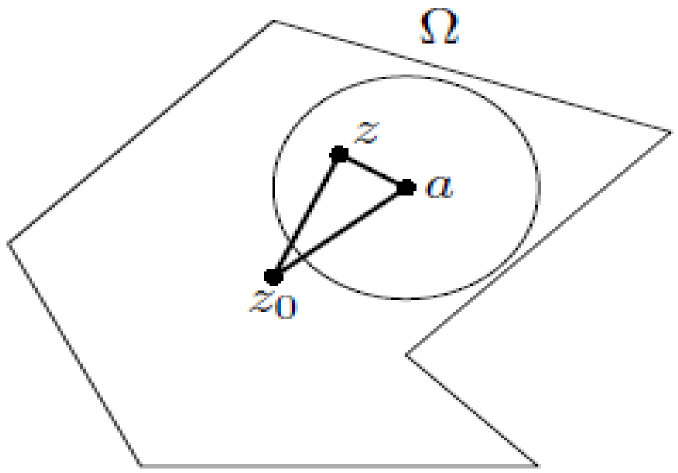

Figure 6. Star-shaped domain.

From the fractional Cauchy-Goursat theorem, we know that

$$\int_{[z_0,a,z,z_0]} f(s)\frac{ds}{s^{1-\alpha}} = 0.$$

This integral can be written as

$$\int_{[z_0,a]} f(s)\frac{ds}{s^{1-\alpha}} + \int_{[a,z]} f(s)\frac{ds}{s^{1-\alpha}} + \int_{[z,z_0]} f(s)\frac{ds}{s^{1-\alpha}} = 0.$$

i.e.,

$$F(z) - F(a) = \int_{[z_0.z]} f(s)\frac{ds}{s^{1-\alpha}} - \int_{[z_0,a]} f(s)\frac{ds}{s^{1-\alpha}} = \int_{[a,z]} f(s)\frac{ds}{s^{1-\alpha}} .$$

From the previous equality, and by following the same reasoning that we use in the proof of the characterization of the existence of α −antiderivatives (see Theorem 4.1), it is easily proven that F is α −differentiable at a and $(T_\alpha F)(a) = f(a)$, which concludes the proof.

Finally, we will establish a result that is a reciprocal of the fractional Cauchy-Goursat theorem for triangles.

Theorem 5.3. ***(Fractional Morera's Theorem).*** For a continuous function $f: \Omega \to C$ on a set open $\Omega \subset C - (-\infty. 0]$, the following conditions are equivalent:

(i) f is α −analytic on Ω.

(ii) $\int_{[a,b,c,a]} f(z)\frac{dz}{z^{1-\alpha}} = 0$ for every triangle $\Delta(a,b,c)$ in Ω.

Proof. (i) ⇒ (ii) is Theorem 5.1.

Let us see then that (i) ⇒ (i). Let z_0 and $r > 0$ such that $D(z_0, r) \subset \Omega$. The disk is a star-shaped domain, and the integral over any triangle contained in it is zero. This allows us to build an α −antiderivative $F(z) = \int_{[z_0.z]} f(s)\frac{ds}{s^{1-\alpha}}$ for all $z \in D(z_0, r)$. F is an α −antiderivative of f on

$D(z_0, r)$, that is, $(T_\alpha F)(z) = f(z)$ for all $z \in D(z_0, r)$ (proof of this is identical to the proof of the existence of primitives in star-shaped domains, see theorem 5.2). Then f is the α −derivative of α −analytic on function on disk $D(z_0, r)$. Therefore, it is an α −analytic function on the mentioned disk; f is particulary an α −differentiable at z_0. Since z_0 is chosen arbitrary, it follows that f is α −analytic on Ω.

6. Application: Circulation and net flow

It is a well-known result that if $f(z) = P(x, y) + iQ(x, y)$ is the complex representation of the velocity vector field $F(x, y) = P(x, y)i + Q(x, y)j$ of a two-dimensional fluid flow, [28], the circulation of F around a closed contour γ is given by

$$Re\left(\int_\gamma \overline{f(z)}dz\right).$$

Likewise, the net flux of F across a closed contour γ can be defined as

$$Im\left(\int_\gamma \overline{f(z)}dz\right).$$

In the following examples we will formulate these physical concepts in terms of fractional contour integrals:

Example 6.1. Suppose the velocity field of a fluid flow is $f(z) = \overline{\sin\left(3\sqrt[3]{z}\right)}$. Compute the circulation and net flux across a closed contour γ, where γ is the square with vertices $z = 1. z = 2 + i, z = 1 + 2i, z = i$.

Solution: We must compute $\int_\gamma \overline{\overline{\sin\left(3\sqrt[3]{z}\right)}}\frac{dz}{\sqrt[3]{z^2}} = \int_\gamma sin\left(3\sqrt[3]{z}\right)\frac{dz}{\sqrt[3]{z^2}}$, and then take the real and imaginary parts of the integral to find the circulation and net flux, respectively. However, since the function $\sin\left(3\sqrt[3]{z}\right)$ is $\frac{1}{3}$ −analytic in $C - (-\infty. 0]$, we immediately have $\int_\gamma sin\left(3\sqrt[3]{z}\right)\frac{dz}{\sqrt[3]{z^2}} = 0$ by the Fractional Cauchy's integral theorem. Therefore, both of the circulation and net flux are zero.

Example 6.2. Suppose the velocity field of a fluid flow is $f(z) = \overline{z}^{\frac{3}{2}}(z - 2i)^2$. Compute the circulation and net flux across a closed contour γ, where γ is the circle $|z - 2i| = 1$.

Solution. Since $\overline{f(z)} = z^{\frac{3}{2}}\overline{(z - 2i)}^2$ and $\gamma(t) = 2i + e^{it}$. $0 \leq t \leq 2\pi$, we have:

$$\int_\gamma z^{\frac{3}{2}}\overline{(z - 2i)}^2 \frac{dz}{z^{\frac{1}{2}}} = \int_0^{2\pi} \left(2i + e^{it}\right)e^{-i2t}ie^{it}dt = -2\int_0^{2\pi} e^{-it}dt + i\int_0^{2\pi} dt = 2\pi i$$

Thus, the circulation around γ is 0, and the net flux across γ is 2π.

7. Conclusions

The main objective of this work is to establish some generalizations in the field of fractional calculus, and to provide some important results on complex integration. The objective has been successfully achieved, so the definition of fractional contour integral has been used to construct some results, such as the necessary and sufficient conditions for a continuous function to have antiderivative in the conformable sense and the extension of some of the well-known Cauchy integral theorems. It seems that the results obtained in this work correspond to the results obtained in the classical case. Finally, we would like to indicate that this research work opens the door for further research studies on developing the complex fractional integration with applications to natural sciences and engineering.

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Conflict of interest

All authors declare no conflicts of interest in this research paper.

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