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Research article

On an integral and consequent fractional integral operators via generalized convexity

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Abstract: Fractional calculus operators are very useful in basic sciences and engineering. In this paper we study an integral operator which is directly related with many known fractional integral operators. A new generalized convexity namely exponentially $(\alpha, h-m)$-convexity is defined which has been applied to obtain the bounds of unified integral operators. A generalized Hadamard inequality is established for the generalized convex functions. The established theorems reproduce several known results.

Keywords: integral operators; fractional integral operators; convex functions; bounds
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1. Introduction

Convex functions have wide applications in mathematical analysis, optimization theory, mathematical statistics, graph theory and many other subjects. Convex function is expressed and visualized in different ways, its analytic representation (1.1) provides motivation to define new concepts and notions. It is generalized in different forms, the $(h-m)$-convex function is defined as follows:

Definition 1. *[1] Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : [0,b] \to \mathbb{R}$ is called $(h-m)$-convex function, if f is non-negative and for all $x, y \in [0,b]$, $m \in [0,1]$ and $t \in (0,1)$, one has*

$$f(tx + m(1-t)y) \leq h(t)f(x) + mh(1-t)f(y). \tag{1.1}$$

By selecting suitable function h and particular value of parameter m, the above definition produces the functions comprise in the following remark:

Remark 1. *(i) If $m = 1$, then h-convex function can be obtained.*
(ii) If $h(t) = t$, then m-convex function can be obtained.
(iii) If $h(t) = t$ and $m = 1$, then convex function can be obtained.
(iv) If $h(t) = 1$ and $m = 1$, p-function can be obtained.
(v) If $h(t) = t^s$ and $m = 1$, then s-convex function can be obtained.
(vi) If $h(t) = \frac{1}{t}$ and $m = 1$, then Godunova-Levin function can be obtained.
(vii) If $h(t) = \frac{1}{t^s}$ and $m = 1$, then s-Godunova-Levin function of second kind can be obtained.

Another generalization of convex function is called (α, m)-convex function defined as follows:

Definition 2. *[2] A function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$ if*

$$f(tx + m(1 - t)y) \leq t^{\alpha} f(x) + m(1 - t^{\alpha})f(y) \tag{1.2}$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Remark 2. *(i) If (α, m)=$(1, m)$, then (1.2) gives the definition of m-convex function.*
(ii) If (α, m)=$(1, 1)$, then (1.2) gives the definition of convex function.
(iii) If (α, m)=$(1, 0)$, then (1.2) gives the definition of star-shaped function.

Next, we give definition of (s, m)-convex function.

Definition 3. *[3] A function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be (s, m)-convex function, where $(s, m) \in [0, 1]^2$ if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have*

$$f(ta + m(1 - t)b) \leq t^s f(a) + m(1 - t)^s f(b).$$

The following definition of generalized convexity unifies the aforementioned definitions:

Definition 4. *[1] Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : [0, b] \to \mathbb{R}$ is called $(\alpha, h - m)$-convex function, if f is non-negative and for all $x, y \in [0, b], t \in (0, 1)$ and $(\alpha, m) \in [0, 1]^2$, one has*

$$f(tx + m(1 - t)y) \leq h(t^{\alpha})f(x) + mh(1 - t^{\alpha})f(y). \tag{1.3}$$

Next, we give definition of exponentially (s, m)-convex function.

Definition 5. *[4] Let $s \in [0, 1]$ and $I \subseteq [0, \infty)$ be an interval. A function $f : I \to \mathbb{R}$ is said to be exponentially (s, m)-convex in second sense, if*

$$f(tx + m(1 - t)y) \leq t^s \frac{f(x)}{e^{\alpha x}} + m(1 - t)^s \frac{f(y)}{e^{\alpha y}} \tag{1.4}$$

holds for all $m \in [0, b]$ and $\alpha \in \mathbb{R}$.

The above definition provides several kinds of convexities as follows:

Remark 3. *i) If we take $m = 1$, then exponentially s-convex function defined by Mehreen et al. in [5], can be achieved.*
ii) If we take $s = m = 1$, then exponentially convex function defined by Awan et al. in [6], can be achieved.
iii) If we take $\alpha = 0$, then (s, m)-convex function defined by Efthekhari in [7], can be achieved.
iv) If we take $\alpha = 0$ and $m = 1$, then s-convex function defined by Hudzik in [8], can be achieved.
v) If we take $\alpha = 0$ and $s = 1$, then m-convex function defined by Toader in [9], can be achieved.
vi) If we take $\alpha = 0$ and $s = m = 1$, then convex function (9), can be achieved.

We will unify all above generalizations of convex functions in a single notion which will be called exponentially $(\alpha, h - m)$-convex function (see Definition 12). Further we will use this generalized convexity for getting bounds of a unified integral operator. In the following, we give the definition of this unified integral operator and definitions of some of associated fractional integral operators.

Definition 6. *[10] Let $f, g : [a, b] \longrightarrow \mathbb{R}$, $0 < a < b$, be the functions such that f be positive and $f \in L_1[a, b]$, and g be differentiable and strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a, \infty)$ and $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then for $x \in [a, b]$ the left and right integral operators are defined by*

$$({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f)(x, \omega; p) = \int_a^x K_x^y(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) f(y) d(g(y)), \tag{1.5}$$

$$({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f)(x, \omega; p) = \int_x^b K_y^x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) f(y) d(g(y)), \tag{1.6}$$

where the involved kernel is defined by

$$K_x^y(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) = \frac{\phi(g(x) - g(y))}{g(x) - g(y)} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(y))^{\mu}; p) \tag{1.7}$$

and $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}$ is the Mittag-Leffller function given in (1.18).

For suitable settings of functions ϕ, g and certain values of parameters included in Mittag-Leffler function, several recently defined known fractional integrals can be reproduced, see [11, Remarks 6 & 7].

Definition 7. *[12] Let $f \in L_1[a, b]$. Then Riemann-Liouville fractional integrals of order μ where $\Re(\mu) > 0$ are defined by:*

$$I_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x - t)^{\mu - 1} f(t)\, dt, \quad x > a, \tag{1.8}$$

$$I_{b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t - x)^{\mu - 1} f(t)\, dt, \quad x < b, \tag{1.9}$$

where $\Gamma(.)$ is the gamma function.

An analogue k-fractional Riemann-Liouville integral operators are given in next definition.

Definition 8. *[13] Let $f \in L_1[a,b]$. Then the k-fractional Riemann-Liouville integrals of order μ where $\Re(\mu) > 0,\ k > 0$ are defined by:*

$$I^k_{a^+} f(x) = \frac{1}{k\Gamma_\kappa(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} f(t)\,dt,\ \ x > a, \tag{1.10}$$

$$I^k_{b^-} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1} f(t)\,dt,\ \ x < b, \tag{1.11}$$

where $\Gamma_k(.)$ is defined in [14].

Definition 9. *[12] Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order μ where $\Re(\mu) > 0$ are defined by:*

$$^{\mu}_{g}I_{a^+} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (g(x)-g(t))^{\mu-1} g'(t) f(t) dt,\ \ x > a, \tag{1.12}$$

$$^{\mu}_{g}I_{b^-} f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (g(t)-g(x))^{\mu-1} g'(t) f(t) dt,\ \ x < b. \tag{1.13}$$

Definition 10. *[15] Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order μ where $\Re(\mu), k > 0$ are defined by:*

$$^{\mu}_{g}I^k_{a^+} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1} g'(t) f(t) dt,\ \ x > a, \tag{1.14}$$

$$^{\mu}_{g}I^k_{b^-} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (g(t)-g(x))^{\frac{\mu}{k}-1} g'(t) f(t) dt,\ \ x < b. \tag{1.15}$$

A fractional integral operator containing an extended generalized Mittag-Leffler function in its kernel is defined as follows:

Definition 11. *[16] Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operators $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f$ are defined by:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p) f(t) dt, \tag{1.16}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f\right)(x;p) = \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(t-x)^{\mu};p) f(t) dt, \tag{1.17}$$

where

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{t^n}{(l)_{n\delta}} \tag{1.18}$$

is the extended generalized Mittag-Leffler function.

Fractional integrals have a great importance in the field of mathematical inequalities. In recent decades many researchers introduced new fractional integral operators which have been used to obtain several types of fractional integral inequalities, see [12, 13, 17–25]. The objective of this paper is to obtain the bounds of a unified integral operator utilizing exponentially $(\alpha, h-m)$-convexity which are associated with many fractional integral inequalities.

In Section 2 we will list some properties of the kernel involved in the unified integral operators, which will be helpful in proving the main results of the paper. In Section 3 by using exponentially $(\alpha, h-m)$-convex functions, upper bounds of unified integral operators (1.5) and (1.6) are obtained. Furthermore, by using condition of symmetry, two sided (upper an lower) bounds in the form of Hadamard inequality are obtained. Also we establish a related inequality by using exponentially $(\alpha, h-m)$-convexity of function $|f'|$ and by defining integral operators for convolution of two functions.

2. Properties of the kernel $K_m^n(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)$

Here we give some properties of kernel given in (1.7), which will be used for further results.

P1: Let g and $\frac{\phi}{m}$ be increasing functions. Then for $m < t < n$, $m, n \in [a, b]$, the kernel $K_m^n(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)$ satisfies the following inequality:

$$K_t^m(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(t) \leq K_n^m(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(t). \tag{2.1}$$

It can be proved from the following two straightforward inequalities:

$$\frac{\phi(g(t) - g(m))}{g(t) - g(m)} g'(t) \leq \frac{\phi(g(n) - g(m))}{g(n) - g(m)} g'(t), \tag{2.2}$$

$$E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(t) - g(m))^{\mu}; p) \leq E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(n) - g(m))^{\mu}; p). \tag{2.3}$$

The reverse of inequality (2.1) holds when g and $\frac{\phi}{m}$ are of opposite monotonicity.

P2: Let g and $\frac{\phi}{m}$ be increasing functions. If $\phi(0) = \phi'(0) = 0$, then for $m, n \in [a, b]$, $m < n$, $K_n^m(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \geq 0$.

P3: For $p, q \in \mathbb{R}$ and for real valued functions ϕ_1 and ϕ_2 we have
$K_n^m(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; p\phi_1 + q\phi_2) = pK_n^m(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi_1) + qK_n^{mm}(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi_2)$.

3. Main results

First we define a generalized convexity namely exponentially $(\alpha, h-m)$-convexity as follows:

Definition 12. *Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. A function $f : [0, b] \to \mathbb{R}$ will be called exponentially $(\alpha, h-m)$-convex function, if for all $x, y \in [0, b], t \in (0, 1)$, $(\alpha, m) \in [0, 1]^2$ and $\eta \in \mathbb{R}$ one has*

$$f(tx + m(1-t)y) \leq h(t^{\alpha})\frac{f(x)}{e^{\eta x}} + mh(1 - t^{\alpha})\frac{f(y)}{e^{\eta y}}. \tag{3.1}$$

Remark 4. *All kinds of convex functions which are defined in the introduction section are deducible from above definition.*

The following result provides upper bound for unified integral operators of $(\alpha, h-m)$-convex functions.

Theorem 1. *Let $f : [a,b] \longrightarrow \mathbb{R}$, $0 \le a < mb$ be a positive integrable exponentially $(\alpha, h-m)$-convex function, $m \neq 0$. Let $g : [a,b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function. Then for unified integral operators the following inequality holds:*

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\right)(x,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x,\omega;p) \le K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi)(x-a) \tag{3.2}$$
$$\left(\frac{f(a)}{e^{\eta a}} H^a_x(z^\alpha, h; g') + \frac{mf\left(\dfrac{x}{m}\right)}{e^{\eta(\frac{x}{m})}} H^a_x(1-z^\alpha, h; g')\right) + K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi)(b-x)$$
$$\times\left(\frac{f(b)}{e^{\eta b}} H^x_b(z^\alpha, h; g') + \frac{mf\left(\dfrac{x}{m}\right)}{e^{\eta(\frac{x}{m})}} H^x_b(1-z^\alpha, h; g')\right),$$

where $H^a_x(z^\alpha, h; g') = \int_0^1 h(z^\alpha)g'(x-z(x-a))dz$ and $H^a_x(1-z^\alpha, h; g') = \int_0^1 h(1-z^\alpha)g'(x-z(x-a))dz$.

Remark 5. *(i) If $\alpha = 1, m = 1$, $\eta = 0$ and $h(z) = z$ in (3.2), then [11, Theorem 8] can be obtained.*
(ii) If $p = \omega = \eta = 0$, $\alpha = 1$ and $h(t) = t^s$ in (3.2), then [26, Theorem 1] can be obtained.
(iii) If $h(z) = z$, $p = \omega = \eta = 0$ and $g(z) = z$ in (3.2), then [27, Theorem 1] can be obtained.
(iv) If $h(t) = t^s$, $\alpha = 1$, $p = \omega = \eta = 0$ and $g(z) = z$ in (3.2), then [3, Theorem 1] can be obtained.
(v) If $\phi(t) = \Gamma(\mu)t^\mu$, $g(x) = x$ and $p = \omega = \eta = 0$, $\alpha = 1, m = 1$ and $h(z) = z$ in (3.2), then [28, Corollary 1] can be obtained.
(vi) If $\phi(t) = \Gamma(\mu)t^\mu$, $p = \omega = \eta = 0$, $\alpha = 1, m = 1$ and $h(z) = z$ in (3.2), then [29, Corollary 1] can be obtained.

Theorem 2. *With assumptions of Theorem 1, if $h \in L_\infty[0,1]$ and $f \in L_\infty[a,b]$, then unified integral operators are bounded and continuous.*

Definition 13. *A function $f : [a, mb] \longrightarrow \mathbb{R}$ will be called exponentially m-symmetric with $m \in (0,1]$ if the following equation holds:*

$$\frac{f(x)}{e^{\eta x}} = \frac{f(\frac{a+b-x}{m})}{e^{\eta(\frac{a+b-x}{m})}}, \ \eta \in \mathbb{R}.$$

The following result provides generalized Hadamard inequality for exponentially $(\alpha, h-m)$-convex functions.

Theorem 3. *The conditions on f, g and ϕ are same as in Theorem 1 and in addition if f is exponentially m-symmetric, then we have*

$$\frac{h(\eta)f\left(\dfrac{a+b}{2}\right)}{h\left(\dfrac{1}{2^{\alpha}}\right)+mh\left(\dfrac{2^{\alpha}-1}{2^{\alpha}}\right)}\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b,\omega;p)\right) \tag{3.3}$$
$$\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p)\leq 2K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(b-a)$$
$$\times\left(\frac{f(a)}{e^{\eta a}}H^a_b(z^{\alpha},h;g')+\frac{mf\left(\dfrac{b}{m}\right)}{e^{\eta(\frac{b}{m})}}H^a_b(1-z^{\alpha},h;g')\right).$$

Remark 6. *(i) If $\alpha=1,m=1$, $h(z)=z$ and $\eta=0$ in (3.3), then [11, Theorem 22] can be obtained.*
(ii) If $h(z)=z$, $p=\omega=\eta=0$ and $g(z)=z$ in (3.3), then [27, Theorem 3] can be obtained.
(iii) If $p=\omega=\eta=0$, $\alpha=1$ and $h(t)=t^s$ in (3.3), then [26, Theorem 3] can be obtained.

Theorem 4. *Let $f:[a,b]\longrightarrow\mathbb{R}$, $0\leq a<mb$ be a differentiable function and $|f'|$ be $(\alpha,h-m)$-convex. Let $g:[a,b]\longrightarrow\mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function on $[a,b]$. Then for unified integral operators the following inequality holds:*

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}(f*g)\right)(x,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}(f*g)\right)(x,\omega;p)\right|\leq K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi) \tag{3.4}$$
$$\times(x-a)\left(\frac{|f'(a)|}{e^{\eta a}}H^a_x(z^{\alpha},h;g')+\frac{m\left|f'\left(\dfrac{x}{m}\right)\right|}{e^{\eta(\frac{x}{m})}}H^a_x(1-z^{\alpha},h;g')\right)$$
$$+K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(b-x)\left(\frac{|f'(b)|}{e^{\eta b}}H^x_b(z^{\alpha},h;g')+\frac{m\left|f'\left(\dfrac{x}{m}\right)\right|}{e^{\eta(\frac{x}{m})}}H^x_b(1-z^{\alpha},h;g')\right).$$

where

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f*g\right)(x,\omega;p):=\int_a^x K^t_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)f'(t)d(g(t)),$$
$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f*g\right)(x,\omega;p):=\int_x^b K^x_t(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)f'(t)d(g(t)).$$

Remark 7. *(i) If $\alpha=1,m=1$, $\eta=0$ and $h(z)=z$ in (3.4), then [11, Theorem 25] can be obtained.*
(ii) If $p=\omega=0$, $\alpha=1$, $\eta=0$ and $h(t)=t^s$ in (3.4), then [26, Theorem 2] can be obtained.
(iii) If $h(z)=z$, $p=\omega=0=\eta=0$ and $g(z)=z$ in (3.4), then [27, Theorem 2] can be obtained.
(iv) If $h(t)=t^s$,$\alpha=1$, $p=\omega=0=\eta=0$ and $g(z)=z$ in (3.4), then [3, Theorem 3] can be obtained.

4. Proofs of main results

In this section we give the proofs of the results stated in aforementioned section.

Proof of Theorem 1. By ($\mathbf{P_1}$), the following inequalities hold:

$$K_x^t(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi)g'(t) \leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi)g'(t), \;\; a < t < x, \tag{4.1}$$

$$K_t^x(E_{\nu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi)g'(t) \leq K_b^x(E_{\nu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi)g'(t), \;\; x < t < b. \tag{4.2}$$

Using exponentially $(\alpha, h-m)$-convexity of f, we have

$$f(t) \leq h\left(\frac{x-t}{x-a}\right)^{\alpha} \frac{f(a)}{e^{\eta a}} + mh\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right) \frac{f\left(\frac{x}{m}\right)}{e^{\eta(\frac{x}{m})}}, \tag{4.3}$$

$$f(t) \leq h\left(\frac{t-x}{b-x}\right)^{\alpha} \frac{f(b)}{e^{\eta b}} + mh\left(1-\left(\frac{t-x}{b-x}\right)^{\alpha}\right) \frac{f\left(\frac{x}{m}\right)}{e^{\eta(\frac{x}{m})}}. \tag{4.4}$$

From (4.1) and (4.3), the following integral inequality holds true:

$$\begin{aligned} \int_a^x K_x^t(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi)d(g(t)) &\leq \frac{f(a)}{e^{\eta a}} K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi) \\ &\times \int_a^x h\left(\frac{x-t}{x-a}\right)^{\alpha} d(g(t)) + \frac{mf\left(\frac{x}{m}\right)}{e^{\eta(\frac{x}{m})}} K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi) \\ &\times \int_a^x h\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right) d(g(t)). \end{aligned} \tag{4.5}$$

By using (1.5) of Definition 6 on left hand side, and by setting $z = \dfrac{x-t}{x-a}$ on right hand side, the following inequality is obtained:

$$\begin{aligned} \left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(x,\omega;p) &\leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi)(x-a) \\ &\times \left(\frac{f(a)}{e^{\eta a}} \int_0^1 h(z^{\alpha}) g'(x-z(x-a))dz + \frac{mf\left(\frac{x}{m}\right)}{e^{\eta(\frac{x}{m})}} \int_0^1 h(1-z^{\alpha}) g'(x-z(x-a))dz\right). \end{aligned} \tag{4.6}$$

Above inequality can be written as

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(x,\omega;p) \leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi)(x-a)\left(\frac{f(a)}{e^{\eta a}} H_x^a(z^{\alpha}, h; g') + \frac{mf\left(\frac{x}{m}\right)}{e^{\eta(\frac{x}{m})}} H_x^a(1-z^{\alpha}, h; g')\right). \tag{4.7}$$

On the other hand, multiplying (4.2) and (4.4), by using (1.6) of Definition 6 on left hand side and integrating over $(x, b]$ on right hand side, we obtain:

$$\begin{aligned} \left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\right)(x,\omega;p) &\leq K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g;\phi)(b-x) \\ &\times \left(\frac{f(b)}{e^{\eta b}} \int_0^1 h(z^{\alpha}) g'(x+z(b-x))dz + \frac{mf\left(\frac{x}{m}\right)}{e^{\eta(\frac{x}{m})}} \int_0^1 h(1-z^{\alpha}) g'(x+z(b-x))dz\right). \end{aligned} \tag{4.8}$$

Above inequality can be written as

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x,\omega;p) \leq K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(b-x)\left(\frac{f(b)}{e^{\eta b}}H^x_b(z^\alpha,h;g') + \frac{mf\left(\frac{x}{m}\right)}{e^{\eta(\frac{x}{m})}}H^x_b(1-z^\alpha,h;g')\right). \quad (4.9)$$

By adding (4.7) and (4.9), (3.2) can be obtained. □

Proof of Theorem 2. From (4.7) we have

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p)\right| \leq M^{\alpha,m}_{h,K^a_b}\|f\|_\infty,$$

where $M^{\alpha,m}_{h,K^a_b} = \frac{1}{e^{\eta a}}K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(g(b)-g(a))(m+1)\|h\|_\infty$.

Similarly, from (4.9) the following inequality holds:

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x,\omega;p)\right| \leq M^{\alpha,m}_{h,K^a_b}\|f\|_\infty.$$

The boundedness with linearity provides the continuity. □

Remark 8. *(i) If $\eta = 0, \alpha = 1$ and $h(z) = z$, then the results hold for boundedness and continuity for m-convexity.*
(ii) If we put $\eta = 0, \alpha = 1$, $h(z) = z$, and $m = 1$ then the results hold for boundedness and continuity for convexity.

Lemma 1. *Let $f : [a, mb] \longrightarrow \mathbb{R}$, $0 \leq a \leq mb$, be an exponentially $(\alpha, h-m)$-convex function. If f is exponentially m-symmetric, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \left(h\left(\frac{1}{2^\alpha}\right) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\right)\frac{f(x)}{e^{\eta x}} \quad x \in [a,b]. \quad (4.10)$$

Proof. Since f is an exponentially $(\alpha, h-m)$-convex, the following inequality is valid:

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right)\frac{f(x)}{e^{\eta x}} + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\frac{f(\frac{a+b-x}{m})}{e^{\eta(\frac{a+b-x}{m})}}.$$

By using exponentially m-symmetry of f in above inequality, we get (4.10). □

Remark 9. *(i) If $h(x) = x$, $\alpha = m = 1$ and $\eta = 0$ in (4.10), then [28, Lemma 1] can be obtained.*
(ii) If $\alpha = 1$, $h(t) = t^s$ and $\eta = 0$ in (4.10), then [26, Lemma 1] can be obtained.
(iii) If $h(x) = x$ and $\eta = 0$ in (4.10), then [2, Lemma 1]
(iv) If $\alpha = 1$ and $\eta = 0$ in (4.10), then [30, Lemma 1]

□

Proof of Theorem 3. By $(\mathbf{P_1})$, the following inequalities hold:

$$K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x) \leq K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x), \quad a < x < b, \quad (4.11)$$

$$K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x) \leq K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x) \quad a < x < b. \quad (4.12)$$

Using exponentially $(\alpha, h-m)$-convexity of f, we have

$$f(t) \leq h\left(\frac{x-a}{b-a}\right)^{\alpha} \frac{f(a)}{e^{\eta a}} + mh\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right) \frac{f\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}}. \tag{4.13}$$

Multiplying (4.11) and (4.13) and integrating the resulting inequality over $[a, b]$, we obtain:

$$\begin{aligned}\int_a^b K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) d(g(x)) &\leq \frac{f(a)}{e^{\eta a}} K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \\ &\times \int_a^b h\left(\frac{x-a}{b-a}\right)^{\alpha} d(g(x)) + \frac{mf\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}} K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \\ &\times \int_a^b h\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right) d(g(x)).\end{aligned}$$

By using (1.5) of Definition 6 on left hand side, and by setting $z = \dfrac{x-t}{x-a}$ on right hand side, the following inequality is obtained:

$$\begin{aligned}\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(b, \omega; p) &\leq K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(b-a) \\ &\times \left(\frac{f(a)}{e^{\eta a}} \int_0^1 h(z^{\alpha}) g'(x - z(b-a)) dz + \frac{mf\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}} \int_0^1 h(1-z^{\alpha}) g'(x - z(b-a)) dz\right).\end{aligned} \tag{4.14}$$

Above inequality can be written as

$$\begin{aligned}\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(b, \omega; p) &\leq K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(b-a) \\ &\times \left(\frac{f(a)}{e^{\eta a}} H_b^a(z^{\alpha}, h; g') + \frac{mf\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}} H_b^a(1-z^{\alpha}, h; g')\right).\end{aligned} \tag{4.15}$$

Adopting the same pattern of simplification as we did for (4.11) and (4.13), the following inequality can be observed for (4.13) and (4.12):

$$\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\right)(a, \omega; p) \leq K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(b-a) \left(\frac{f(a)}{e^{\eta a}} H_b^a(z^{\alpha}, h; g') + \frac{mf\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}} H_b^a(1-z^{\alpha}, h; g')\right). \tag{4.16}$$

By adding (4.15) and (4.16), following inequality can be achieved:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p)\leq 2K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(b-a) \tag{4.17}$$
$$\times\left(\frac{f(a)}{e^{\eta a}}H^a_b(z^\alpha,h;g')+\frac{mf\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}}H^a_b(1-z^\alpha,h;g')\right).$$

Multiplying both sides of (4.10) by $K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)d(g(x))$, and integrating over $[a,b]$ we have

$$f\left(\frac{a+b}{2}\right)\int_a^b K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)d(g(x))$$
$$\leq\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)\int_a^b K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)\frac{f(x)}{e^{\eta x}}d(g(x)).$$

From Definition 6, the following inequality is obtained:

$$\frac{h(\eta)f\left(\frac{a+b}{2}\right)}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p)\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p). \tag{4.18}$$

Similarly, multiplying both sides of (4.10) by $K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)d(g(x))$, and integrating over $[a,b]$ we have

$$\frac{h(\eta)f\left(\frac{a+b}{2}\right)}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b,\omega;p)\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p). \tag{4.19}$$

By adding (4.18) and (4.19) following inequality is obtained:

$$\frac{h(\eta)f\left(\frac{a+b}{2}\right)}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b,\omega;p)\right) \tag{4.20}$$
$$\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p).$$

Using (4.17) and (4.20), the inequality (3.3) can be achieved. □

Proof of Theorem 4.

Using exponentially $(\alpha,h-m)$-convexity of $|f'|$ we have

$$|f'(t)|\leq h\left(\frac{x-t}{x-a}\right)^\alpha\frac{|f'(a)|}{e^{\eta a}}+mh\left(1-\left(\frac{x-t}{x-a}\right)^\alpha\right)\frac{\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta(\frac{x}{m})}}. \tag{4.21}$$

The inequality (4.21) can be written as follows:

$$-\left(h\left(\frac{x-t}{x-a}\right)^{\alpha}\frac{|f'(a)|}{e^{\eta a}}+mh\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\frac{\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta\left(\frac{x}{m}\right)}}\right)\leq f'(t) \tag{4.22}$$
$$\leq\left(h\left(\frac{x-t}{x-a}\right)^{\alpha}\frac{|f'(a)|}{e^{\eta a}}+mh\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\frac{\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta\left(\frac{x}{m}\right)}}\right).$$

First we consider the second inequality of (4.22)

$$f'(t)\leq\left(h\left(\frac{x-t}{x-a}\right)^{\alpha}\frac{|f'(a)|}{e^{\eta a}}+mh\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\frac{\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta\left(\frac{x}{m}\right)}}\right). \tag{4.23}$$

Multiplying (4.1) and (4.23) and integrating over $[a,x]$, we obtain

$$\int_a^x K_x^t(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi)d(g(t))\leq\frac{|f'(a)|}{e^{\eta a}}K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi)$$
$$\times\int_a^x h\left(\frac{x-t}{x-a}\right)^{\alpha}d(g(t))+\frac{m\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta\left(\frac{x}{m}\right)}}K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi)$$
$$\times\int_a^x h\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)d(g(t)).$$

By using (1.5) of Definition 6 on left hand side, and by setting $z=\dfrac{x-t}{x-a}$ on right hand side, the following inequality is obtained:

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi)(x-a)\left(\frac{|f'(a)|}{e^{\eta a}}\right.$$
$$\left.\times\int_0^1 h(z^{\alpha})g'(x-z(x-a))dz+\frac{m\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta\left(\frac{x}{m}\right)}}\int_0^1 h(1-z^{\alpha})g'(x-z(x-a))dz\right).$$

Above inequality can be written as

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi)(x-a) \tag{4.24}$$
$$\times\left(\frac{|f'(a)|}{e^{\eta a}}H_x^a(z^{\alpha},h;g')+\frac{m\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta\left(\frac{x}{m}\right)}}H_x^a(1-z^{\alpha},h;g')\right).$$

If we consider the left hand side from the inequality (4.22), and adopt the same pattern as we did for the right hand side inequality, we have

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p) \geq -K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(x-a) \tag{4.25}$$
$$\times\left(\frac{|f'(a)|}{e^{\eta a}}H^a_x(z^\alpha,h;g') + m\frac{\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta(\frac{x}{m})}}H^a_x(1-z^\alpha,h;g')\right).$$

From (4.24) and (4.25), following inequality is observed:

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p)\right| \leq K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(x-a) \tag{4.26}$$
$$\times\left(\frac{|f'(a)|}{e^{\eta a}}H^a_x(z^\alpha,h;g') + \frac{m\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta(\frac{x}{m})}}H^a_x(1-z^\alpha,h;g')\right).$$

Now using exponentially $(\alpha, h-m)$-convexity of we have

$$|f'(t)| \leq h\left(\frac{t-x}{b-x}\right)^\alpha\frac{|f'(b)|}{e^{\eta b}} + mh\left(1-\left(\frac{t-x}{b-x}\right)^\alpha\right)\frac{\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta(\frac{x}{m})}}. \tag{4.27}$$

On the same pattern as we did for (4.1) and (4.21), one can get following inequality from (4.2) and (4.27):

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x,\omega;p)\right| \leq K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(b-x)\left(\frac{|f'(b)|}{e^{\eta b}}\right.$$
$$\left.\times\int_0^1 h(z^\alpha)g'(b-z(b-x))dz + \frac{m\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta(\frac{x}{m})}}\int_0^1 h(1-z^\alpha)g'(b-z(b-x))dz\right).$$

Above inequality can be written as

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x,\omega;p) \leq K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)(b-x) \tag{4.28}$$
$$\times\left(\frac{|f'(b)|}{e^{\eta b}}H^x_b(z^\alpha,h;g') + \frac{m\left|f'\left(\frac{x}{m}\right)\right|}{e^{\eta(\frac{x}{m})}}H^x_b(1-z^\alpha,h;g')\right).$$

By adding (4.26) and (4.28), inequality (3.4) can be achieved. □

5. Applications of main results

In this section by applying Theorem 1 we give some interesting consequences. The reader can obtain the applications of Theorems 2, 3 and 4.

Some Hadamard Inequalities for exponentially $(\alpha, h-m)$-convex functions:

By applying Theorem 3 we give fractional Hadamard inequalities for exponentially $(\alpha, h-m)$-convex functions.

Corollary 1. *If $\phi(t) = \frac{\Gamma(\mu)t^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$ and $p = \omega = 0$ in (3.3), then the Hadamard inequality for fractional integral operators (defined in [15]) of exponentially $(\alpha, h-m)$-convex functions holds as follows:*

$$\frac{2h(\eta)f\left(\frac{a+b}{2}\right)(g(b)-g(a))^{\mu/k}}{k\Gamma_k(\mu)\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)} \leq {}_g^{\mu}I_{b_-}^{k} f(a) + {}_g^{\mu}I_{a^+}^{k} f(b)$$
$$\leq \frac{2(b-a)(g(b)-g(a))^{\frac{\mu}{k}-1}}{k\Gamma_k(\mu)}\left(\frac{f(a)}{e^{\eta a}}H_b^a(z^\alpha, h; g') + \frac{mf\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}}H_b^a(1-z^\alpha, h; g')\right),$$

$\mu \geq k$.

Corollary 2. *If $k = 1$ in Corollary 1, then the Hadamard inequality for fractional integral operators (defined in [12]) of exponentially $(\alpha, h-m)$-convex functions holds as follows:*

$$\frac{2h(\eta)f\left(\frac{a+b}{2}\right)(g(b)-g(a))^{\mu}}{\Gamma(\mu)\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)} \leq {}_g^{\mu}I_{b_-} f(a) + {}_g^{\mu}I_{a^+} f(b)$$
$$\leq \frac{2(b-a)(g(b)-g(a))^{\mu-1}}{\Gamma(\mu)}\left(\frac{f(a)}{e^{\eta a}}H_b^a(z^\alpha, h; g') + \frac{mf\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}}H_b^a(1-z^\alpha, h; g')\right).$$

Corollary 3. *If $g(x) = x$ in Corollary 1, then the Hadamard inequality for fractional integral operators (defined in [13]) of exponentially $(\alpha, h-m)$-convex functions holds as follows:*

$$\frac{2h(\eta)f\left(\frac{a+b}{2}\right)(b-a)^{\mu/k}}{\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)k\Gamma_k(\mu)} \leq {}^{\mu}I_{b_-}^{k} \frac{f(a)}{e^{\eta a}} + {}^{\mu}I_{a^+}^{k} f(b)$$
$$\leq \frac{2(b-a)^{\frac{\mu}{k}+1}}{k\Gamma_k(\mu)}\left(f(a)\int_0^1 h(z^\alpha)dz + \frac{mf\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}}\int_0^1 h(1-z^\alpha)dz\right), \mu \geq k.$$

Corollary 4. *If $g(x) = x$ in Corollary 2, then the Hadamard inequality for fractional integral operators (defined in [12]) of exponentially $(\alpha, h-m)$-convex functions holds as follows:*

$$\frac{2h(\eta)f\left(\frac{a+b}{2}\right)(b-a)^{\mu}}{\left(h\left(\frac{1}{2^{\alpha}}\right)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\right)\Gamma(\mu)} \leq {}^{\mu}I_{b_-}f(a) + {}^{\mu}I_{a^+}f(b)$$
$$\leq \frac{2(b-a)^{\mu+1}}{\Gamma(\mu)}\left(\frac{f(a)}{e^{\eta a}}\int_0^1 h(z^{\alpha})dz + \frac{mf\left(\frac{b}{m}\right)}{e^{\eta(\frac{b}{m})}}\int_0^1 h(1-z^{\alpha})dz\right).$$

6. Conclusions

This paper provides estimates of an integral operator via generalized convexity namely exponentially $(\alpha, h-m)$-convexity. The given results consist of bounds of this generalized integral operator fo exponentially $(\alpha, h-m)$-convex functions. All the results hold for various associated fractional integral operators and notions of convexities; namely $(\alpha, h-m)$-convexity, $(h-m)$-convexity, (α, m)-convexity, (s, m)-convexity and related convex functions. The reader can get the results for several kinds of fractional integral operators of convex and related functions given in Remarks 1–3.

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Conflict of interest

It is declared that all authors have equal contribution and that, they have no competing interests.

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