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## Research article

# Periodic boundary value problem involving sequential fractional derivatives in Banach space 

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#### Abstract

In this paper, by the method of upper and lower solutions coupled with the monotone iterative technique, we investigate the existence and uniqueness results of solutions for a periodic boundary value problem of nonlinear fractional differential equation involving conformable sequential fractional derivatives in Banach space. An example is given to illustrate our main result.


Keywords: fractional differential equation; periodic boundary value problem; monotone iterative; Banach space; existence and uniqueness
Mathematics Subject Classification: 26A33, 34B15

## 1. Introduction

In this work, we consider the following periodic boundary value problem (PBVP for short) in a Banach space $E$

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} x(t)=f\left(t, x(t), \mathcal{D}^{\alpha} x(t)\right), t \in(0,1], 0<\alpha \leq 1,  \tag{1.1}\\
x(0)=x(1), \mathcal{D}^{\alpha} x(0)=\mathcal{D}^{\alpha} x(1),
\end{array}\right.
$$

where $f(t, x, y)$ is a continuous $E$-value function on $[0,1] \times E \times E, \mathcal{D}^{\alpha}$ is the conformable fractional derivative of order $\alpha, \mathcal{D}^{2 \alpha}=\mathcal{D}^{\alpha} \mathcal{D}^{\alpha}$ is the conformable sequential fractional derivative.

Sequential fractional derivative for a sufficiently smooth function $g(t)$ due to Miller and Ross [1] is defined as $\mathcal{D}^{\delta} g(t)=\mathcal{D}^{\delta_{1}} \mathcal{D}^{\delta_{2}} \ldots \mathcal{D}^{\delta_{k}} g(t)$, here $\delta=\left(\delta_{1}, \delta_{2}, \ldots, \delta_{k}\right)$ is a multi-index. In general, the operator $\mathcal{D}^{\delta}$ can be Riemann-Liouville or Caputo or any other kind of differential operators. There is a close connection between the sequential fractional derivatives and the non-sequential derivatives [2]. Many research papers have appeared recently concerning the existence of solutions for fractional
differential equations involving Riemann-Liouville or Caputo sequential fractional derivatives by techniques of nonlinear analysis such as fixed point theorems, coincidence degree continuation theorems and nonlinear alternatives, see, for example, the papers [3-11] and the references therein.

In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives due to their wide range of applications in varied fields of science and engineering. Generally, for most of the fractional differential equations, it is difficult to find exact solutions in closed forms. In most cases, only approximate solutions or numerical solutions can be expected. Therefore, many iterative methods have been designed to be one of the suitable and successful classes of numerical techniques for obtaining the solutions of numerous types of fractional differential equations, see, for instance, [12-16] and the references therein. The monotone iterative technique, combined with the method of upper and lower solutions, is an effective technique for proving the existence of solutions for initial and boundary value problems of nonlinear differential equations. The basic idea of this method is that by choosing upper and lower solutions as two initial iterations, one can construct the monotone sequences for a corresponding linear equation and that converge monotonically to the extremal solutions of the nonlinear equation. Not only does this method give constructive proof for existence theorems but also the monotone behavior of iterative sequences is useful in the treatment of numerical solutions of various initial and boundary value problems. So many authors developed the upper and lower solutions method to investigate fractional differential equations, see, for example, [17-23] and the references therein.

In [2], using the method of upper and lower solutions and its associated monotone iterative technique, the authors considered the existence of minimal and maximal solutions and uniqueness of solution of the following initial value problem for fractional differential equation involving Riemann-Liouville sequential fractional derivative

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} y(t)=f\left(t, y(t), \mathcal{D}^{\alpha} y(t)\right), t \in(0, T], 0<\alpha \leq 1, \\
\left.t^{1-\alpha} y(t)\right|_{t=0}=y_{0},\left.t^{1-\alpha} \mathcal{D}^{\alpha} x(t)\right|_{t=0}=y_{1},
\end{array}\right.
$$

where $f(t, x, y)$ is continuous on $[0,1] \times \mathbb{R} \times \mathbb{R}$.
The nonlinear impulsive fractional differential equation with periodic boundary conditions

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} u(t)=f\left(t, u, \mathcal{D}^{\alpha} u\right), t \in(0, T] \backslash\left\{t_{1}, \ldots, t_{m}\right\}, 0<\alpha \leq 1, \\
\lim _{t \rightarrow 0^{+}} t^{1-\alpha} u(t)=u(1), \lim _{t \rightarrow 0^{+}} t^{1-\alpha} \mathcal{D}^{\alpha} u(t)=\mathcal{D}^{\alpha} u(1), \\
\lim _{t \rightarrow t_{j}^{+}}\left(t-t_{j}\right)^{1-\alpha}\left(u(t)-u\left(t_{j}\right)\right)=I_{j}\left(u\left(t_{j}\right)\right), \\
\lim _{t \rightarrow t_{j}^{+}}\left(t-t_{j}\right)^{1-\alpha}\left(\mathcal{D}^{\alpha} u(t)-\mathcal{D}^{\alpha} u\left(t_{j}\right)\right)=\bar{I}_{j}\left(u\left(t_{j}\right)\right),
\end{array}\right.
$$

is studied in [24], where $\mathcal{D}^{\alpha}$ is the standard Riemann-Liouville fractional derivative and $\mathcal{D}^{2 \alpha}=\mathcal{D}^{\alpha} \mathcal{D}^{\alpha}$ is the Riemann-Liouville sequential fractional derivative. $I_{j}, \bar{I}_{j} \in C(\mathbb{R}, \mathbb{R}), j=1,2, \ldots, m . f$ is continuous at every point $(t, u, v) \in[0,1] \times \mathbb{R} \times \mathbb{R}$. The existence and uniqueness results of solutions are obtained by the method of upper and lower solutions and its associated monotone iterative technique.

A new simple well-behaved definition of the fractional derivative called conformable fractional derivative has been presented very recently in [25]. This new definition is a natural extension of the usual derivative, and it satisfies some similar properties to the integer order calculus such as derivative
of the product of two functions, derivative of the quotient of two functions and the chain rule. In [26] the author developed further the definitions and properties of conformable fractional derivative and integral.
conformable fractional derivative: Given a function $f:[0, \infty) \rightarrow \mathbb{R}$, the conformable fractional derivative of $f$ of order $\alpha \in(0,1]$ is defined by

$$
\mathcal{D}^{\alpha} f(t)=\lim _{\varepsilon \rightarrow 0} \frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}, t>0 .
$$

If there exists conformable fractional derivative of $f$ of order $\alpha \in(0,1]$ in some $(0, a), a>0$, and $\lim _{t \rightarrow 0^{+}} \mathfrak{D}^{\alpha} f(t)$ exists, then define $\mathcal{D}^{\alpha} f(0)=\lim _{t \rightarrow 0^{+}} \mathcal{D}^{\alpha} f(t)$. Evidently, $\mathcal{D}^{\alpha} f(t)=f^{\prime}(t)$ for $\alpha=1$.
conformable fractional integral: Given a function $f:[0, \infty) \rightarrow \mathbb{R}$, the conformable fractional integral of $f$ of order $\alpha \in(0,1]$ is defined by

$$
I^{\alpha} f(t)=\int_{0}^{t} \tau^{\alpha-1} f(\tau) d \tau
$$

The conformable sequential fractional derivative is proposed in [26]. Given a function $f:[0, \infty) \rightarrow$ $\mathbb{R}$ and $\alpha \in(0,1]$, the conformable sequential fractional derivative of $f$ of order $n$ is expressed by

$$
\mathcal{D}^{n \alpha} f(t)=\underbrace{\mathcal{D}^{\alpha} \mathcal{D}^{\alpha} \ldots \mathcal{D}^{\alpha}}_{n \text {-times }} f(t), t>0 .
$$

The physical and geometrical meaning of the conformable fractional derivative is interpreted in [27]. Several applications of the definition have shown the significance of conformable fractional derivative. For example, [28] discussed the potential conformable quantum mechanics, [29] discussed the conformable Maxwell equations, and [30,31] showed that the conformable fractional derivative models present good agreements with experimental data. For recent results on the existence, stability and oscillation of solutions for conformable fractional differential equations, we refer the reader to [32-44].

The existence of solutions for periodic boundary value problem of impulsive conformable fractional integro-differential equation

$$
\left\{\begin{array}{l}
t_{k} \mathcal{D}^{\alpha} x(t)=f(t, x,(F x)(t),(S x)(t)), t \in J=[0, T] \backslash\left\{t_{1}, \ldots, t_{m}\right\}, 0<\alpha \leq 1, \\
x(0)=x(T), x\left(t_{k}^{+}\right)-x\left(t_{k}^{-}\right)=I_{k}\left(x\left(t_{k}\right)\right), k=2, \ldots, m,
\end{array}\right.
$$

is studied in [33], where ${ }_{t_{k}} \mathcal{D}^{\alpha}$ denotes the conformable fractional derivative of order $\alpha$ starting from $t_{k}, f \in C\left(J \times \mathbb{R}^{3}, \mathbb{R}\right),(F x)(t)=\int_{0}^{t} l(t, s) x(s) d s,(S x)(t)=\int_{0}^{T} h(t, s) x(s) d s, l \in C\left(D, \mathbb{R}^{+}\right), D=\{(t, s) \in$ $\left.J^{2}: t \geq s\right\}, h \in C\left(J^{2}, \mathbb{R}^{+}\right), I_{k} \in C(\mathbb{R}, \mathbb{R})$. By the method of upper and lower solutions in reversed order coupled with the monotone iterative technique, some sufficient conditions for the existence of solutions are established.

In [42], applying the upper and lower solutions method and the monotone iterative technique, the authors investigated the existence of solutions to antiperiodic boundary value problem for impulsive conformable fractional functional differential equation

$$
\left\{\begin{array}{l}
t_{k} \mathcal{D}^{\alpha} x(t)=f(t, x, x(\omega(t))), t \in J=[0, T] \backslash\left\{t_{1}, \ldots, t_{m}\right\}, 0<\alpha \leq 1, \\
x(0)=-x(T), x\left(t_{k}^{+}\right)-x\left(t_{k}^{-}\right)=I_{k}\left(x\left(t_{k}\right)\right), k=2, \ldots, m, \\
x(t)=x(0), t \in[-r, 0],
\end{array}\right.
$$

where $f \in C\left(J \times \mathbb{R}^{2}, \mathbb{R}\right), t_{k} \mathcal{D}^{\alpha}$ denotes the conformable fractional derivative of order $\alpha$ starting from $t_{k}$, $\omega \in C\left(J, J^{+}\right), J^{+}=[-r, T], r>0, t-r \leq \omega(t), t \in J$ and $t_{k}<\omega(t) \leq t, t \in\left(t_{k}, t_{k+1}\right], I_{k} \in C(\mathbb{R}, \mathbb{R})$.

However, to the best of our knowledge, the existence of minimal and maximal solutions and uniqueness of solution for fractional PBVP (1.1) involving conformable sequential fractional derivatives in ordered Banach spaces have not been considered up to now. Inspired by above works, we apply the theory of noncompactness measure and the method of upper and lower solutions coupled with the monotone iterative technique to construct two monotone iterative sequences, and then prove that the sequences converge to the extremal periodic solutions of PBVP (1.1), respectively, under some monotonicity conditions and noncompactness measure conditions of $f$. Also, we prove minimal and maximal solutions are equal and thus we obtain the uniqueness of solution. Moreover, we give the existence and uniqueness results of periodic solutions of the non-sequential fractional differential equation.

## 2. Preliminaries

In this section, we introduce some notations, definitions and preliminary facts which are used throughout this paper. Let $J=[0,1]$ and $E$ be an ordered Banach space with the norm $\|\cdot\|$ and the partial order " $\leq$ ", whose positive cone $P=\{x \in E, x \geq \theta\}$ is normal with normal constant $L$, where $\theta$ denotes the zero element of $E$. Generally, $C(J, E)$ denotes the ordered Banach space of all continuous $E$-value functions on the interval $J$ with the norm $\|x\|_{c}=\max _{t \in J}\|x(t)\|$ and the partial order " $\leq$ " deduced by the positive cone $P_{c}=\{x: x \in C(J, E), x(t) \geq \theta\} . P_{c}$ is also normal with the same normal constant $L$. Let $C_{\alpha}(J, E)=\left\{x: x \in C(J, E), \mathcal{D}^{\alpha} x \in C(J, E)\right\}$, evidently, $C_{\alpha}(J, E)$ also is a Banach space with the norm $\|x\|_{\alpha}=\max \left\{\|x\|_{c},\left\|D^{\alpha} x\right\|_{c}\right\}$. A function $x \in C_{\alpha}(J, E)$ is called a solution of $\operatorname{PBVP}$ (1.1) if it satisfies the equation and the boundary conditions in (1.1).

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.
Definition 2.1. ([45]) Let $E$ be a Banach space and let $\Omega_{E}$ be the family of bounded subsets of $E$. The Kuratowski measure of noncompactness is the map $\mu: \Omega_{E} \rightarrow[0, \infty]$ defined by

$$
\mu(B)=\inf \left\{\varepsilon>0: B \subseteq \cup_{i=1}^{n} B_{i}, \operatorname{diam}\left(B_{i}\right) \leq \varepsilon\right\}, \text { here } B \in \Omega_{E} .
$$

Property 2.2. The Kuratowski measure of noncompactness satisfies some properties (for more details see [45]).
(1) $\mu(B)=0 \Leftrightarrow \bar{B}$ is compact ( $B$ is relatively compact), where $\bar{B}$ denotes the closure of $B$.
(2) $\mu(A \cup B)=\max \{\mu(A), \mu(B)\}$.
(3) $\mu(c B)=|c| \mu(B), c \in \mathbb{R}$.

Denote the Kuratovski noncompactness measures of bounded sets in $C(J, E)$ and $C_{\alpha}(J, E)$ by $\mu_{c}$ and $\mu_{c_{\alpha}}$, respectively. For any $H \subset C(J, E)$ and $t \in J$, set $H(t)=\{x(t): x \in H\} \subset E$. If $H$ is bounded in $C(J, E)$, then $H(t)$ is bounded in $E$, and $\mu(H(t)) \leq \mu_{c}(H)$. Furthermore, we have the following well-known result.

Lemma 2.3. ([45]) Let $H \subset C(J, E)$ be bounded and equicontinuous. Then $\mu(H(t))$ is continuous on $J$ and $\mu_{c}(H)=\max _{t \in J} \mu(H(t))$.

We can deduce the following useful result by Lemma 2.3.
Lemma 2.4. Let $H \subset C_{\alpha}(J, E)$ be bounded and equicontinuous. Then

$$
\mu_{c_{\alpha}}(H)=\max \left\{\max _{t \in J} \mu(H(t)), \max _{t \in J} \mu\left(\mathcal{D}^{\alpha} H(t)\right)\right\},
$$

where $\mathcal{D}^{\alpha} H(t)=\left\{\mathcal{D}^{\alpha} x(t) \mid x \in H\right\} \subset E, t \in J$.
Proof. Firstly, we prove that $\mu_{c_{\alpha}}(H) \leq d=$ : $\max \left\{\max _{t \in J} \mu(H(t)), \max _{t \in J} \mu\left(\mathcal{D}^{\alpha} H(t)\right)\right\}$. Noting that $H \subset$ $C(J, E)$ and $\mathcal{D}^{\alpha} H \subset C(J, E)$ are bounded and equicontinuous, by Lemma 2.3, we know

$$
\mu_{c}(H)=\max _{t \in J} \mu(H(t)) \leq d, \mu_{c}\left(\mathcal{D}^{\alpha} H\right)=\max _{t \in J} \mu\left(\mathcal{D}^{\alpha} H(t)\right) \leq d .
$$

Therefore, there exist $V_{1}, V_{2}, \ldots, V_{n} \subset H$ and $W_{1}, W_{2}, \ldots, W_{m} \subset H$ such that $H=\bigcup_{i=1}^{n} V_{i}=\bigcup_{j=1}^{m} W_{j}$ and

$$
\begin{equation*}
\operatorname{diam}_{c} V_{i}<d+\varepsilon, \operatorname{diam}_{c} \mathcal{D}^{\alpha} W_{j}<d+\varepsilon, \quad i=1,2, \ldots, n, j=1,2, \ldots, m \tag{2.1}
\end{equation*}
$$

where $\operatorname{diam}_{c}(\cdot)$ denotes the diameter of the bounded subset of $C(J, E)$. At the same time, for any $x_{1}, x_{2} \in V_{i}$, by (2.1) we obtain

$$
\begin{equation*}
\left\|x_{1}(t)-x_{2}(t)\right\| \leq d+\varepsilon \tag{2.2}
\end{equation*}
$$

Similarly, for $y_{1}, y_{2} \in W_{j}$, we have

$$
\begin{equation*}
\left\|D^{\alpha} y_{1}(t)-\mathcal{D}^{\alpha} y_{2}(t)\right\| \leq d+\varepsilon \tag{2.3}
\end{equation*}
$$

Let $H_{i j}=\left\{x: x \in V_{i}, \mathcal{D}^{\alpha} x \in \mathcal{D}^{\alpha} W_{j}\right\}, i=1,2, \ldots, n, j=1,2, \ldots, m$. According to (2.1), (2.2) and (2.3), we can get

$$
\operatorname{diam}_{c} H_{i j} \leq d+\varepsilon, \quad \operatorname{diam}_{c} \mathcal{D}^{\alpha} H_{i j} \leq d+\varepsilon,
$$

this means $\operatorname{diam}_{c_{\alpha}} H_{i j} \leq d+\varepsilon, i=1,2, \ldots, n, j=1,2, \ldots, m$. Then it follows from $H=\bigcup_{\substack{i=1,2, \cdots, n \\ j=1,2, \cdots, m}} H_{i j}$ that $\mu_{c_{\alpha}}(H) \leq d$.

On the other hand, for any $\varepsilon>0$, there exist $H_{i} \subset H, i=1,2, \ldots, k$ such that $H=\bigcup_{i=1}^{k} H_{i}$ and $\operatorname{diam}_{c_{\alpha}}\left(H_{i}\right) \leq \mu_{c_{\alpha}}(H)+\varepsilon$. Hence, for any $t \in J$ and any $x_{1}, x_{2} \in H_{i}, i=1,2, \ldots, k$, we have

$$
\begin{equation*}
\left\|x_{1}(t)-x_{2}(t)\right\| \leq\left\|x_{1}-x_{2}\right\|_{c} \leq\left\|x_{1}-x_{2}\right\|_{c_{\alpha}} \leq \mu_{c_{\alpha}}(H)+\varepsilon . \tag{2.4}
\end{equation*}
$$

From the fact $H(t)=\bigcup_{i=1}^{k} H_{i}(t)$ and (2.4), we have $\mu(H(t)) \leq \mu_{c_{\alpha}}(H)+\varepsilon$ for any $t \in J$, so $\max _{t \in J} \mu(H(t)) \leq \mu_{c_{\alpha}}(H)+\varepsilon$. Thus, $\max _{t \in J} \mu(H(t)) \leq \mu_{c_{\alpha}}(H)$ since $\varepsilon>0$ is arbitrary. Similarly, it follows that $\max _{t \in J} \mu\left(\mathcal{D}^{\alpha} H(t)\right) \leq \mu_{c_{\alpha}}(H)$. Consequently, the proof of Lemma 2.4 is completed.

The following lemma also will be used in the proof of our main results.
Lemma 2.5. ([46-48]) Let $E$ be a Banach space, $H=\left\{x_{n}\right\} \subset L(J, E)$ be a countable set with $\left\|x_{n}(t)\right\| \leq$ $\rho(t)$ for a.a. $t \in J$ and every $x_{n} \in H$, where $\rho(t) \in L(J)$. Then $\mu(H(t))$ is Lebesgue integrable on $J$, and $\mu\left(\left\{\int_{J} x_{n}(t) d t\right\}\right) \leq 2 \int_{J} \mu(H(t)) d t$.

Conformable calculus satisfies the following properties.
Property 2.6. ([25]) Let $\alpha \in(0,1]$ and $f, g$ be conformable differentiable of order $\alpha$. Then
(1) $\mathcal{D}^{\alpha} C=0$ for all constant functions $f(t)=C$.
(2) $\mathcal{D}^{\alpha}(a f+b g)=a \mathcal{D}^{\alpha}(f)+b \mathcal{D}^{\alpha}(g)$ for all $a, b \in \mathbb{R}$.
(3) $\mathcal{D}^{\alpha}(f g)=f D^{\alpha}(g)+g \mathcal{D}^{\alpha}(f)$.
(4) $D^{\alpha} I^{\alpha} f(t)=f(t)$ for $t>0$, where $f$ is any continuous function in the domain of $\mathcal{I}^{\alpha}$.

Property 2.7. By the definition of conformable fractional integral, it is easy to know $I^{\alpha} f: C(J) \rightarrow$ $C(J)$ for $\alpha \in(0,1]$.

Remark 2.8. ([26, 39, 41]) The function $\exp \left(\lambda \frac{t^{\alpha}}{\alpha}\right)=e^{\lambda^{\frac{\alpha^{\alpha}}{\alpha}}}, t \geq 0$, is called fractional conformable exponential function, where $\alpha \in(0,1]$ and $\lambda \in \mathbb{R}$. The conformable derivative of fractional conformable exponential function is $\mathcal{D}^{\alpha} e^{\lambda^{\frac{\alpha}{\alpha}}}=\lambda e^{\lambda^{\frac{l^{\alpha}}{\alpha}}}$.

Lemma 2.9. ([35,39]) For $0<\alpha \leq 1$, the general solution of the fractional nonhomogeneous equation

$$
\mathcal{D}^{\alpha} x(t)-\lambda x(t)=\sigma(t), t>0,
$$

is expressed by

$$
x(t)=e^{\lambda \frac{l^{\alpha}}{\alpha}}\left[C+I^{\alpha}\left(e^{-\lambda \frac{\alpha^{\alpha}}{\alpha}} \sigma(t)\right)\right],
$$

where $C$ is a constant. Further, the unique solution of the linear initial value problem

$$
\left\{\begin{array}{l}
\mathcal{D}^{\alpha} x(t)-\lambda x(t)=\sigma(t), t \in(0,1] \\
x(0)=x_{0}
\end{array}\right.
$$

has the following form

$$
x(t)=e^{\lambda \frac{\lambda^{\alpha}}{\alpha}}\left[x_{0}+I^{\alpha}\left(e^{-\lambda \frac{\lambda^{\alpha}}{\alpha}} \sigma(t)\right)\right] .
$$

From Lemma 2.9 we can obtain the solution of linear boundary value problem.
Lemma 2.10. For $\sigma(t) \in C(J)$ and $0<\alpha \leq 1$, the unique solution $x \in C(J)$ of the linear boundary value problem

$$
\left\{\begin{array}{l}
\mathcal{D}^{\alpha} x(t)-\lambda x(t)=\sigma(t), t \in(0,1], \lambda \neq 0 \\
x(0)-x(1)=d
\end{array}\right.
$$

has the following form

$$
\begin{equation*}
x(t)=\frac{d e^{\lambda^{\frac{\alpha}{\alpha}}}}{1-e^{\frac{\lambda}{\alpha}}}+\int_{0}^{1} G_{\lambda}(t, s) \sigma(s) d s \tag{2.5}
\end{equation*}
$$

where

The continuity of the solution $x(t)$ in Lemma 2.10 is guaranteed by Property 2.7.

Remark 2.11. It is easy to know from the expression of $G_{\lambda}(t, s)$ that for $\lambda<0$,

$$
0 \leq G_{\lambda}(t, s) \leq \begin{cases}\left(\frac{e^{\frac{\lambda}{\alpha}}}{1-e^{\frac{\alpha}{\alpha}}}+1\right) s^{\alpha-1}=\frac{1}{1-e^{\frac{\lambda}{\alpha}}} s^{\alpha-1}, & s \leq t \\ \frac{1}{1-e^{\frac{\lambda}{\alpha}}} s^{\alpha-1}, & s \geq t\end{cases}
$$

Thus $\int_{0}^{1} G_{\lambda}(t, s) d s \leq \frac{1}{\alpha\left(1-e^{\left.\frac{1}{\alpha}\right)}\right.}$.
Furthermore, from Lemma 2.10 we can deduce the solution of linear PBVP with sequential derivative.

Lemma 2.12. For $\sigma(t) \in C(J), 0<\alpha \leq 1$ and $M, N>0$ such that $M^{2} \geq 4 N$, the solution of the linear PBVP

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} x(t)+M \mathcal{D}^{\alpha} x(t)+N x(t)=\sigma(t), t \in(0,1]  \tag{2.6}\\
x(0)=x(1), \mathcal{D}^{\alpha} x(0)=\mathcal{D}^{\alpha} x(1)
\end{array}\right.
$$

is

$$
\begin{equation*}
x(t)=\int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int_{0}^{1} G_{\lambda_{1}}(s, \tau) \sigma(\tau) d \tau\right) d s \tag{2.7}
\end{equation*}
$$

where

$$
\lambda_{1}=\frac{-M-\sqrt{M^{2}-4 N}}{2} \leq \lambda_{2}=\frac{-M+\sqrt{M^{2}-4 N}}{2}<0 .
$$

Proof. Let $\left(D^{\alpha}-\lambda_{2}\right) x(t)=y(t), t \in(0,1]$. Then the problem (2.6) is equivalent to

$$
\left\{\begin{array}{l}
\left(\mathcal{D}^{\alpha}-\lambda_{2}\right) x(t)=y(t), t \in(0,1]  \tag{2.8}\\
x(0)=x(1)
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\left(\mathcal{D}^{\alpha}-\lambda_{1}\right) y(t)=\sigma(t), t \in(0,1]  \tag{2.9}\\
y(0)=y(1)
\end{array}\right.
$$

By Lemma 2.10, we obtain that the problems (2.8) and (2.9) have the following representation of solutions

$$
\begin{equation*}
x(t)=\int_{0}^{1} G_{\lambda_{2}}(t, s) y(s) d s \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
y(t)=\int_{0}^{1} G_{\lambda_{1}}(t, s) \sigma(s) d s \tag{2.11}
\end{equation*}
$$

respectively. Substituting (2.11) into (2.10), we get (2.7).
The following comparison result plays an important role in the proofs of our main results.
Lemma 2.13. Let $M^{2} \geq 4 N, M, N>0$ and

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} x(t)+M \mathcal{D}^{\alpha} x(t)+N x(t) \geq 0, t \in(0,1]  \tag{2.12}\\
x(0) \geq x(1), \mathcal{D}^{\alpha} x(0) \geq \mathcal{D}^{\alpha} x(1)
\end{array}\right.
$$

Then $x(t) \geq 0$ on $J$.

Proof. Rewrite (2.12) as follows

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} x(t)+M \mathcal{D}^{\alpha} x(t)+N x(t)=\sigma(t) \geq 0, t \in(0,1]  \tag{2.13}\\
x(0)-x(1)=k \geq 0, \mathcal{D}^{\alpha} x(0)-\mathcal{D}^{\alpha} x(1)=l \geq 0
\end{array}\right.
$$

Then the problem (2.13) is equivalent to

$$
\left\{\begin{array}{l}
\left(\mathcal{D}^{\alpha}-\lambda_{2}\right) x(t)=y(t), t \in(0,1] \\
x(0)-x(1)=k
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\left(\mathcal{D}^{\alpha}-\lambda_{1}\right) y(t)=\sigma(t), t \in(0,1], \\
y(0)-y(1)=l-\lambda_{2} k .
\end{array}\right.
$$

In view of $\lambda_{2}<0, l-\lambda_{2} k \geq 0, \sigma(t) \geq 0, \lambda_{1}<0$ and (2.5), we can obtain $y(t) \geq 0$. Furthermore, $\lambda_{2}<0, k \geq 0, y(t) \geq 0$ and (2.5) ensure $x(t) \geq 0$ on $J$.

## 3. Main results

Definition 3.1. Let $v, w \in C_{\alpha}(J, E)$. $v$ is called a lower solution of PBVP (1.1) if it satisfies

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} v(t) \leq f\left(t, v(t), \mathcal{D}^{\alpha} v(t)\right), t \in(0,1]  \tag{3.1}\\
v(0) \leq v(1), \mathcal{D}^{\alpha} v(0) \leq \mathcal{D}^{\alpha} v(1),
\end{array}\right.
$$

and $w$ is called an upper solution of $\operatorname{PBVP}(1.1)$ if it satisfies

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} w(t) \geq f\left(t, w(t), \mathcal{D}^{\alpha} w(t)\right), t \in(0,1]  \tag{3.2}\\
w(0) \geq w(1), \mathcal{D}^{\alpha} w(0) \geq \mathcal{D}^{\alpha} w(1)
\end{array}\right.
$$

In the following, we assume that $v(t) \leq w(t), t \in J$. Define the ordered interval in space $C(J, E)$

$$
[v, w]=\{x(t) \in C(J, E), v(t) \leq x(t) \leq w(t), t \in J\} .
$$

Denote $D_{1}(t)=\mathcal{D}^{\alpha} v(t)+\lambda_{2}(w(t)-v(t))$ and $D_{2}(t)=\mathcal{D}^{\alpha} w(t)-\lambda_{2}(w(t)-v(t))$. We work with the following conditions on the function $f$ in (1.1).
(H1) There exist constants $M, N>0$ with $M^{2} \geq 4 N$ such that

$$
f\left(t, w, \mathcal{D}^{\alpha} w\right)-f\left(t, v, \mathcal{D}^{\alpha} v\right) \geq-M\left(\mathcal{D}^{\alpha} w-\mathcal{D}^{\alpha} v\right)-N(w-v), \quad t \in J
$$

where $v, w \in C_{\alpha}(J, E)$ are lower and upper solutions of (1.1).
(H2) There exist constants $M, N>0$ with $M^{2} \geq 4 N$ such that

$$
f\left(t, x_{2}, y_{2}\right)-f\left(t, x_{1}, y_{1}\right) \geq-M\left(y_{2}-y_{1}\right)-N\left(x_{2}-x_{1}\right), \quad t \in J,
$$

where $v \leq x_{1} \leq x_{2} \leq w$ and $D_{1}(t) \leq y_{i}(t) \leq D_{2}(t), i=1,2$.
(H3) There exists a constant $K \geq 0$ such that

$$
\mu\left(\left\{f\left(t, x_{n}, y_{n}\right)+N x_{n}(t)+M y_{n}(t)\right\}\right) \leq K\left(\mu\left(\left\{x_{n}(t)\right\}\right)+\mu\left(\left\{y_{n}(t)\right\}\right)\right), \quad t \in J,
$$

for any monotonic sequence $\left\{x_{n}\right\} \subset[v, w]$ and any sequence $\left\{y_{n}\right\}$ such that $D_{1}(t) \leq y_{n}(t) \leq D_{2}(t)$. Moreover,

$$
\frac{8 K\left(1-\lambda_{2}\right)}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)}<1
$$

(H4) There exist constants $\bar{M}, \bar{N}>0$ such that

$$
f\left(t, x_{2}, y_{2}\right)-f\left(t, x_{1}, y_{1}\right) \leq \bar{M}\left(y_{2}-y_{1}\right)+\bar{N}\left(x_{2}-x_{1}\right), t \in J,
$$

where $v \leq x_{1} \leq x_{2} \leq w$ and $D_{1}(t) \leq y_{i}(t) \leq D_{2}(t), i=1,2$. Moreover,

$$
\frac{8 \bar{K}\left(1-\lambda_{2}\right)}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)}<1
$$

here $\bar{K}=\max \{L(\bar{M}+M), L(\bar{N}+N)\}$ and $L$ is the normal constant of cone $P$.
Remark 3.2. The condition (H1) and Lemma 2.10 guarantee that $\mathcal{D}^{\alpha}(w(t)-v(t))-\lambda_{2}(w(t)-v(t)) \geq$ $\theta, t \in J$. In fact, let $z(t)=\mathcal{D}^{\alpha}(w(t)-v(t))-\lambda_{2}(w(t)-v(t))$, then by (3.1), (3.2) and (H1), for $t \in(0,1]$,

$$
\begin{aligned}
\mathcal{D}^{\alpha} z(t)-\lambda_{1} z(t) & =\mathcal{D}^{2 \alpha}(w(t)-v(t))+M \mathcal{D}^{\alpha}(w(t)-v(t))+N(w(t)-v(t)) \\
& \geq\left(f\left(t, w, \mathcal{D}^{\alpha} w\right)-f\left(t, v, \mathcal{D}^{\alpha} v\right)\right)+M \mathcal{D}^{\alpha}(w(t)-v(t))+N(w(t)-v(t)) \geq \theta,
\end{aligned}
$$

and $z(0)-z(1) \geq \theta$. By (2.5) of Lemma 2.10, $x(t) \geq \theta$ if $d \geq \theta$ and $\sigma(t) \geq \theta$. Thus we obtain that $z(t) \geq \theta$, this implies $D_{1}(t) \leq \mathcal{D}^{\alpha} v(t) \leq D_{2}(t)$ and $D_{1}(t) \leq \mathcal{D}^{\alpha} w(t) \leq D_{2}(t)$.

Set $\Omega=\left\{\eta \in[v, w] \cap C_{\alpha}(J, E): D_{1}(t) \leq \mathcal{D}^{\alpha} \eta(t) \leq D_{2}(t)\right\}$. Obviously, $\Omega$ is well defined by Remark 3.2 when the condition (H1) holds. Now we are in the position to state our main results.

Theorem 3.3. Assume that $f \in C([0,1] \times E \times E), v$ and $w$ are lower and upper solutions of BPVP (1.1) and the conditions (H1), (H2) and (H3) are valid. Then there exist $p(t), q(t) \in C_{\alpha}(J, E)$ such that $p(t), q(t)$ are minimal and maximal solutions on the ordered interval $[v, w]$ for BPVP (1.1), respectively, that is, for any solution $x(t)$ of $\operatorname{BPVP}(1.1)$ such that $x \in[v, w]$, we have $v(t) \leq p(t) \leq x(t) \leq q(t) \leq$ $w(t), t \in J$.

Proof. Let $\sigma(\eta)(t)=f\left(t, \eta(t), \mathcal{D}^{\alpha} \eta(t)\right)+M \mathcal{D}^{\alpha} \eta(t)+N \eta(t)$. For any $\eta \in \Omega$, consider the linear PBVP

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} x(t)+M \mathcal{D}^{\alpha} x(t)+N x(t)=\sigma(\eta)(t), t \in(0,1]  \tag{3.3}\\
x(0)=x(1), \mathcal{D}^{\alpha} x(0)=\mathcal{D}^{\alpha} x(1)
\end{array}\right.
$$

By Lemma 2.12, (3.3) has exactly one solution $x(t)$ given by

$$
\begin{equation*}
x(t)=\int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int_{0}^{1} G_{\lambda_{1}}(s, \tau) \sigma(\eta)(\tau) d \tau\right) d s \tag{3.4}
\end{equation*}
$$

For any $\eta \in \Omega$, define the operator $T$ as $T \eta(t)=x(t)$, then the fixed points of $T$ are exactly the solutions of PBVP (1.1). For clarity, we divide the proof into several steps.

Step 1: Firstly, $T: \Omega \rightarrow C_{\alpha}(J, E)$ is well defined. Indeed, for any $\eta \in \Omega$, by (3.4),

$$
\begin{align*}
T \eta(t) & =\int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int_{0}^{1} G_{\lambda_{1}}(s, \tau) \sigma(\eta)(\tau) d \tau\right) d s  \tag{3.5}\\
& =A_{\eta} e^{\lambda_{2} \frac{\alpha^{\alpha}}{\alpha}}+B_{\eta} e^{\lambda_{2} \frac{\sigma^{\alpha}}{\alpha}} \mathcal{I}^{\alpha} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\alpha^{\alpha}}{\alpha}}+e^{\lambda_{2} \frac{\alpha^{\alpha}}{\alpha}} \mathcal{I}^{\alpha}\left(e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\alpha}{\alpha}} \mathcal{I}^{\alpha} e^{-\lambda_{1} \frac{\alpha^{\alpha}}{\alpha}} \sigma(\eta)(t)\right),
\end{align*}
$$

where

$$
\begin{aligned}
A_{\eta}= & \left.\left.\frac{e^{\frac{\lambda_{1}+\lambda_{2}}{\alpha}}}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)}\left(I^{\alpha} e^{-\lambda_{1} \frac{\alpha^{\alpha}}{\alpha}} \sigma(\eta)(t)\right)\right|_{t=1}\left(I^{\alpha} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\alpha}{\alpha}}\right)\right|_{t=1} \\
& +\left.\frac{e^{\frac{\lambda_{2}}{\alpha}}}{1-e^{\frac{\lambda_{2}}{\alpha}}} I^{\alpha}\left(e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\sigma^{\alpha}}{\alpha}} I^{\alpha} e^{-\lambda_{1} \frac{\sigma^{\alpha}}{\alpha}} \sigma(\eta)(t)\right)\right|_{t=1}
\end{aligned}
$$

and

$$
B_{\eta}=\left.\frac{e^{\frac{\lambda_{1}}{\alpha}}}{1-e^{\frac{1_{1}}{\alpha}}}\left(I^{\alpha} e^{-\lambda_{1} \frac{\alpha^{\alpha}}{\alpha}} \sigma(\eta)(t)\right)\right|_{t=1}
$$

Furthermore, Property 2.6, Property 2.7 and Remark 2.8 indicate for $t>0$,

$$
\begin{align*}
\mathcal{D}^{\alpha} T \eta(t)= & A_{\eta} \lambda_{2} e^{\lambda_{2} \frac{\sigma^{\alpha}}{\alpha}}+B_{\eta} \lambda_{2} e^{\lambda_{2} \frac{\sigma^{\alpha}}{\alpha}} \mathcal{I}^{\alpha} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\sigma^{\alpha}}{\alpha}}+\lambda_{2} e^{\lambda_{2} \frac{\alpha}{\alpha}} \mathcal{I}^{\alpha}\left(e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\alpha}{\alpha}} \mathcal{I}^{\alpha} e^{-\lambda_{1} \frac{\alpha}{\alpha}} \sigma(\eta)(t)\right) \\
& +B_{\eta} e^{\lambda_{1} \frac{\alpha}{\alpha}}+e^{\lambda_{1} \frac{\alpha}{\alpha}} \mathcal{I}^{\alpha} e^{-\lambda_{1} \frac{\alpha}{\alpha}} \sigma(\eta)(t)  \tag{3.6}\\
& =\lambda_{2} \int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int_{0}^{1} G_{\lambda_{1}}(s, \tau) \sigma(\eta)(\tau) d \tau\right) d s+\int_{0}^{1} G_{\lambda_{1}}(t, s) \sigma(\eta)(s) d s .
\end{align*}
$$

The continuity of $f$ together with Property 2.7 ensures that the right side of (3.6) is continuous on $J$. Hence, (3.6) is also valid for $t=0$. By Property 2.7, (3.5) and (3.6) we have $T \eta(t) \in C_{\alpha}(J, E)$.

Step 2: For any $\eta \in \Omega,\{T \eta\} \subset C_{\alpha}(J, E)$ is bounded. In fact, from condition (H2) we can derive

$$
f\left(t, v, \mathcal{D}^{\alpha} v\right)+M \mathcal{D}^{\alpha} v+N v \leq f\left(t, \eta, \mathcal{D}^{\alpha} \eta\right)+M \mathcal{D}^{\alpha} \eta+N \eta \leq f\left(t, w, \mathcal{D}^{\alpha} w\right)+M \mathcal{D}^{\alpha} w+N w .
$$

In view of the normality of the cone $P$, there exists $\bar{L}>0$ such that $\left\|f\left(t, \eta, \mathcal{D}^{\alpha} \eta\right)+M \mathcal{D}^{\alpha} \eta+N \eta\right\| \leq \bar{L}$. Since $\lambda_{1}<\lambda_{2}<0$, by Remark 2.11, (3.5) and (3.6) we can find

$$
\|T \eta(t)\| \leq \frac{\bar{L}}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)},
$$

and

$$
\left\|\mathcal{D}^{\alpha} T \eta(t)\right\| \leq \frac{\left|\lambda_{2}\right| \bar{L}}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)}+\frac{\bar{L}}{\alpha} \frac{1}{1-e^{\frac{\lambda_{1}}{\alpha}}} .
$$

Hence, $\{T \eta\} \subset C_{\alpha}(J, E)$ is bounded.

Step 3: For any $\eta \in \Omega,\{T \eta\} \subset C_{\alpha}(J, E)$ is equicontinuous. Noticing that

$$
\begin{align*}
& \left|e^{\lambda_{2} \frac{r_{1}^{\alpha}}{\alpha}} \int_{0}^{t_{1}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{s^{\alpha}}{\alpha}} d s-e^{\lambda_{2} \frac{\sigma_{2}^{\alpha}}{\alpha}} \int_{0}^{t_{2}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2} \frac{s}{\alpha}\right.} d s\right| \\
& \leq\left|e^{\lambda_{2} \frac{r_{1}^{\alpha}}{\alpha}} \int_{0}^{t_{1}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\alpha^{\alpha}}{\alpha}} d s-e^{\lambda_{2} \frac{\alpha_{2}^{\alpha}}{\alpha}} \int_{0}^{t_{1}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{s^{\alpha}}{\alpha}} d s\right| \\
& +\left|e^{\lambda_{2} \frac{\sigma^{\alpha}}{\alpha}} \int_{0}^{t_{1}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\sigma^{\alpha}}{\alpha}} d s-e^{\lambda_{2} \frac{f_{2}^{\alpha}}{\alpha}} \int_{0}^{t_{2}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{s^{\alpha}}{\alpha}} d s\right|  \tag{3.7}\\
& \leq \frac{1}{\alpha}\left|e^{\lambda_{2} \frac{r_{1}^{\alpha}}{\alpha}}-e^{\lambda_{2} \frac{\tau_{2}^{\alpha}}{\alpha}}\right|+\left|\int_{t_{1}}^{t_{2}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\alpha^{\alpha}}{\alpha}} d s\right| \leq \frac{1}{\alpha}\left|e^{\lambda_{2} \frac{r_{1}^{\alpha}}{\alpha}}-e^{\lambda_{2} \frac{\alpha_{2}^{\alpha}}{\alpha}}\right|+\frac{1}{\alpha}\left|t_{2}^{\alpha}-t_{1}^{\alpha}\right|, \\
& \| e^{\lambda_{2} \frac{q_{1}^{\alpha}}{\alpha}} \int_{0}^{t_{1}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\alpha^{\alpha}}{\alpha}}\left(\int_{0}^{s} \tau^{\alpha-1} e^{-\lambda_{1} \frac{\tau^{\frac{\tau^{\alpha}}{\alpha}}}{\alpha}} \sigma(\eta)(\tau) d \tau\right) d s \\
& -e^{\lambda_{2} \frac{\sigma^{\alpha}}{\alpha}} \int_{0}^{t_{2}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{\alpha^{\alpha}}{\alpha}}\left(\int_{0}^{s} \tau^{\alpha-1} e^{-\lambda_{1} \frac{\tau^{\alpha}}{\alpha}} \sigma(\eta)(\tau) d \tau\right) d s \| \\
& \left.\leq\left|e^{\lambda_{2} \frac{t_{1}^{\alpha}}{\alpha}}-e^{\lambda_{2} \frac{\tau_{2}^{\alpha}}{\alpha}}\right| \| \int_{0}^{t_{1}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{s^{\alpha}}{\alpha}}\left(\int_{0}^{s} \tau^{\alpha-1} e^{-\lambda_{1} \frac{\tau^{\alpha}}{\alpha}} \sigma(\eta)(\tau) d \tau\right) d s \right\rvert\,  \tag{3.8}\\
& +e^{\lambda_{2} \frac{h_{2}}{\alpha}}\left\|\int_{t_{1}}^{t_{2}} s^{\alpha-1} e^{\left(\lambda_{1}-\lambda_{2}\right) \frac{s^{\alpha}}{\alpha}}\left(\int_{0}^{s} \tau^{\alpha-1} e^{-\lambda_{1} \frac{\tau^{\alpha}}{\alpha}} \sigma(\eta)(\tau) d \tau\right) d s\right\| \\
& \leq \frac{\bar{L}}{\alpha^{2}} e^{-\frac{\lambda_{1}}{\alpha}}\left|e^{\lambda_{2} \frac{t_{1}^{\alpha}}{\alpha}}-e^{\lambda_{2} \frac{\alpha_{2}^{\alpha}}{\alpha}}\right|+\frac{\bar{L}}{\alpha^{2}} e^{-\frac{\lambda_{1}}{\alpha}}\left|t_{1}^{\alpha}-t_{2}^{\alpha}\right|,
\end{align*}
$$

and

$$
\begin{align*}
& \left\|e^{\lambda_{1} \frac{t_{\alpha}^{\alpha}}{\alpha}} \int_{0}^{t_{1}} s^{\alpha-1} e^{-\lambda_{1} \frac{\alpha^{\alpha}}{\alpha}} \sigma(\eta)(s) d s-e^{\lambda_{1} \frac{\sigma^{\alpha}}{\alpha}} \int_{0}^{t_{2}} s^{\alpha-1} e^{-\lambda_{1} \frac{s^{\frac{\alpha}{\alpha}}}{\alpha}} \sigma(\eta)(s) d s\right\| \\
& \quad \leq \left\lvert\, e^{\lambda_{1} \frac{f_{1}^{\alpha}}{\alpha}}-e^{\lambda_{1} \frac{\rho_{2}^{\alpha}}{\alpha}}\| \| \int_{0}^{t_{1}} s^{\alpha-1} e^{-\lambda_{1} \frac{s^{\alpha}}{\alpha}} \sigma(\eta)(s) d s\left\|+e^{\lambda_{1} \frac{f^{\alpha}}{\alpha}}\right\| \int_{t_{1}}^{t_{2}} s^{\alpha-1} e^{-\lambda_{1} \frac{s^{\frac{\alpha}{\alpha}}}{\alpha}} \sigma(\eta)(s) d s\right. \|  \tag{3.9}\\
& \quad \leq \frac{\bar{L}}{\alpha} e^{-\frac{\lambda_{1}}{\alpha}}\left|e^{\lambda_{1} \frac{r_{1}^{\alpha}}{\alpha}}-e^{\lambda_{1} \frac{\alpha_{2}}{\alpha}}\right|+\frac{\bar{L}}{\alpha} e^{-\frac{\lambda_{1}}{\alpha}}\left|t_{1}^{\alpha}-t_{2}^{\alpha}\right|
\end{align*}
$$

we can arrive at from (3.5)-(3.9) that $\{T \eta\} \subset C_{\alpha}(J, E)$ is equicontinuous.
Step 4: Now we prove $v(t) \leq T v(t), w(t) \geq T w(t)$. Set $v_{1}=T v$, then we have

$$
\left\{\begin{align*}
& \mathcal{D}^{2 \alpha} v_{1}(t)+M \mathcal{D}^{\alpha} v_{1}(t)+N v_{1}(t)= f\left(t, v(t), \mathcal{D}^{\alpha} v(t)\right)  \tag{3.10}\\
&+M \mathcal{D}^{\alpha} v(t)+N v(t), t \in(0,1] \\
& v_{1}(0)=v_{1}(1), \mathcal{D}^{\alpha} v_{1}(0)=\mathcal{D}^{\alpha} v_{1}(1)
\end{align*}\right.
$$

Set $z(t)=v_{1}(t)-v(t)$, then by (3.1) and (3.10) we can know

$$
\left\{\begin{array}{l}
\mathcal{D}^{2 \alpha} z(t)+M \mathcal{D}^{\alpha} z(t)+N z(t) \geq \theta, t \in(0,1] \\
z(0) \geq z(1), \mathcal{D}^{\alpha} z(0) \geq \mathcal{D}^{\alpha} z(1)
\end{array}\right.
$$

Thus, $z(t) \geq \theta$ by Lemma 2.13 and this means $v(t) \leq T v(t)$. Similarly, $w(t) \geq T w(t)$.

Next, let $\eta_{1}, \eta_{2} \in \Omega$ be such that $\eta_{1}(t) \leq \eta_{2}(t)$. (H2) and (3.5) show that $\sigma\left(\eta_{1}\right)(t) \leq \sigma\left(\eta_{2}\right)(t)$, $T \eta_{1}(t) \leq T \eta_{2}(t)$. For each $\eta \in \Omega$, we get by the similar method to Remark 3.2 that $\mathcal{D}^{\alpha}(T \eta(t)-v(t))-$ $\lambda_{2}(T \eta(t)-v(t)) \geq \theta, t \in J$. This implies

$$
D_{1}(t) \leq \mathcal{D}^{\alpha} v(t)+\lambda_{2}(T \eta(t)-v(t)) \leq \mathcal{D}^{\alpha} T \eta(t) .
$$

Similarly, $\mathcal{D}^{\alpha}(w(t)-T \eta(t))-\lambda_{2}(w(t)-T \eta(t)) \geq \theta, t \in J$, and hence

$$
\mathcal{D}^{\alpha} T \eta(t) \leq \mathcal{D}^{\alpha} w(t)-\lambda_{2}(w(t)-T \eta(t)) \leq D_{2}(t) .
$$

Therefore, $T(\Omega) \subset \Omega$ and $T$ is a monotone operator on $[v, w]$. Consequently, let $v_{n}=T v_{n-1}, w_{n}=$ $T w_{n-1}(n=1,2, \ldots)$, we obtain

$$
v=v_{0} \leq v_{1} \leq \ldots \leq v_{n} \leq \ldots \leq w_{n} \ldots \leq w_{1} \leq w_{0}=w
$$

and

$$
D_{1}(t) \leq \mathcal{D}^{\alpha} T v_{n}(t), \mathcal{D}^{\alpha} T w_{n}(t) \leq D_{2}(t), t \in J
$$

Step 5: Let $V=\left\{v_{n}: n=1,2, \ldots\right\}$. In the following, we will show that $V$ is a relatively compact set in $C_{\alpha}(J, E)$. Note that $\mu_{c}(V)=\mu_{c}\left(V \cup\left\{v_{0}\right\}\right)$ and $\mu_{c}\left(\mathcal{D}^{\alpha} V\right)=\mu_{c}\left(\mathcal{D}^{\alpha} V \cup\left\{\mathcal{D}^{\alpha} v_{0}\right\}\right)$ by Property 2.2. In view of Lemma 2.5, Property 2.2, condition (H3), Remark 2.11 and Lemma 2.3, it follows that

$$
\begin{align*}
\mu\left(\left\{\int_{0}^{1} G_{\lambda_{1}}(t, s) \sigma\left(v_{n-1}\right)(s) d s\right\}\right) & \leq 2 \int_{0}^{1} G_{\lambda_{1}}(t, s) \mu\left(\left\{\sigma\left(v_{n-1}\right)(s)\right\}\right) d s \\
& \leq 2 K \int_{0}^{1} G_{\lambda_{1}}(t, s)\left(\mu\left(\left\{v_{n-1}(s)\right\}\right)+\mu\left(\left\{\mathcal{D}^{\alpha} v_{n-1}(s)\right\}\right)\right) d s  \tag{3.11}\\
& \leq \frac{2 K}{\alpha} \frac{1}{1-e^{\frac{\Lambda_{1}}{\alpha}}}\left(\mu_{c}(V)+\mu_{c}\left(\mathcal{D}^{\alpha} V\right)\right) .
\end{align*}
$$

Furthermore,

$$
\begin{align*}
\mu(V(t)) & =\mu\left(\left\{\int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int_{0}^{1} G_{\lambda_{1}}(s, \tau) \sigma\left(v_{n-1}\right)(\tau) d \tau\right) d s\right\}\right) \\
& \leq 2 \int_{0}^{1} G_{\lambda_{2}}(t, s) \mu\left(\left\{\int_{0}^{1} G_{\lambda_{1}}(s, \tau) \sigma\left(v_{n-1}\right)(\tau) d \tau\right\}\right) d s  \tag{3.12}\\
& \leq \frac{4 K}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)}\left(\mu_{c}(V)+\mu_{c}\left(\mathcal{D}^{\alpha} V\right)\right) .
\end{align*}
$$

Hence, we obtain from Lemma 2.4

$$
\begin{equation*}
\mu(V(t)) \leq \frac{8 K}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)} \mu_{c_{\alpha}}(V) \tag{3.13}
\end{equation*}
$$

At the same time, by (3.6), (3.11), (3.12) and (3.13), we get

$$
\begin{align*}
\mu\left(\mathcal{D}^{\alpha} V(t)\right) & \leq\left|\lambda_{2}\right| \frac{8 K}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)} \mu_{c_{\alpha}}(V)+\frac{4 K}{\alpha} \frac{1}{1-e^{\frac{\lambda_{1}}{\alpha}}} \mu_{c_{\alpha}}(V) \\
& \leq \frac{8 K\left(1-\lambda_{2}\right)}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)} \mu_{c_{\alpha}}(V) . \tag{3.14}
\end{align*}
$$

Combining Lemma 2.4, (3.13) and (3.14), we have

$$
\mu_{c_{\alpha}}(V) \leq \frac{8 K\left(1-\lambda_{2}\right)}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)} \mu_{c_{\alpha}}(V) .
$$

Therefore, $\mu_{c_{\alpha}}(V)=0$ by (H3), then Property 2.2 shows $\left\{v_{n}\right\}$ is a relatively compact set of $C_{\alpha}(J, E)$, and thus there exists subsequence converging uniformly to $p \in C_{\alpha}(J, E)$. By the monotone property of $\left\{v_{n}\right\}$ and the assumption (H2) of function $f$ which implies $\left\{\sigma\left(v_{n}\right)\right\}$ is monotone, we obtain that $\left\{v_{n}\right\}$ converges uniformly to $p \in C_{\alpha}(J, E)$ and $\left\{\sigma\left(v_{n}\right)\right\}$ converges to $\sigma(p)$. Similarly, we can show that $\left\{w_{n}\right\}$ converges uniformly to $q \in C_{\alpha}(J, E)$ and $\left\{\sigma\left(w_{n}\right)\right\}$ converges to $\sigma(q)$. Moreover, the limits $p, q$ satisfy

$$
v=v_{0} \leq v_{1} \leq \ldots \leq v_{n} \leq p \leq q \leq w_{n} \ldots \leq w_{1} \leq w_{0}=w,
$$

and

$$
D_{1}(t) \leq \mathcal{D}^{\alpha} p(t), \mathcal{D}^{\alpha} q(t) \leq D_{2}(t), t \in J .
$$

Step 6: Let $n \rightarrow \infty$ in the relation

$$
v_{n}(t)=T v_{n-1}(t)=\int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int_{0}^{1} G_{\lambda_{1}}(s, \tau) \sigma\left(v_{n-1}\right)(\tau) d \tau\right) d s
$$

Applying the dominated convergence theorem, we have $p$ satisfies the equation

$$
p(t)=\int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int_{0}^{1} G_{\lambda_{1}}(s, \tau) \sigma(p)(\tau) d \tau\right) d s, t \in J
$$

which implies that $p(t)$ is an integral representation of the solution to the problem (3.3), and thus by the definition of function $\sigma, p$ is a solution of PBVP (1.1). Similarly, we can prove that $q$ is a solution of PBVP (1.1).

Finally, we prove $p$ and $q$ are extremal solutions of $\operatorname{PBVP}$ (1.1). Assume that $\bar{x}$ is a fixed point of $T$ in $\Omega$, then by the monotonicity of $T$ proved in Step 4, it is easy to see that $T v \leq T \bar{x} \leq T w$, that is, $v_{1} \leq \bar{x} \leq w_{1}$. Furthermore, we have $v_{n} \leq \bar{x} \leq w_{n}$ for $n=1,2, \ldots$. Let $n \rightarrow \infty$ we get $p \leq \bar{x} \leq q$. Therefore, $p$ and $q$ are the minimal and maximal solutions of PBVP (1.1) in $[v, w]$, respectively.

This completes the proof of Theorem 3.3.
Theorem 3.4. Assume that $f \in C([0,1] \times E \times E), v$ and $w$ are lower and upper solutions of BPVP (1.1) and the conditions (H1), (H2) and (H4) are valid. Then the BPVP (1.1) has a unique solution $x \in[v, w]$.

Proof. First of all, we prove that (H2) and (H4) imply (H3). Let $\left\{x_{n}\right\} \subset[v, w]$ be increasing sequence and $\left\{y_{n}\right\}$ be such that $\left\{y_{n}\right\} \subset\left[D_{1}(t), D_{2}(t)\right], t \in J$. For $m, n \in \mathbb{N}$ with $m>n$, in view of (H2) and (H4), we have

$$
\begin{align*}
\theta & \leq f\left(t, x_{m}, y_{m}\right)-f\left(t, x_{n}, y_{n}\right)+M\left(y_{m}(t)-y_{n}(t)\right)+N\left(x_{m}(t)-x_{n}(t)\right) \\
& \leq(\bar{M}+M)\left(y_{m}(t)-y_{n}(t)\right)+(\bar{N}+N)\left(x_{m}(t)-x_{n}(t)\right) . \tag{3.15}
\end{align*}
$$

By (3.15) and the normality of cone $P$, we can derive that

$$
\begin{align*}
\| f\left(t, x_{m}, y_{m}\right) & -f\left(t, x_{n}, y_{n}\right)+M\left(y_{m}(t)-y_{n}(t)\right)+N\left(x_{m}(t)-x_{n}(t)\right) \| \\
& \leq L\left\|(\bar{M}+M)\left(y_{m}(t)-y_{n}(t)\right)+(\bar{N}+N)\left(x_{m}(t)-x_{n}(t)\right)\right\| \\
& \leq L(\bar{M}+M)\left\|y_{m}(t)-y_{n}(t)\right\|+L(\bar{N}+N)\left\|x_{m}(t)-x_{n}(t)\right\|  \tag{3.16}\\
& \leq \bar{K}\left(\left\|y_{m}(t)-y_{n}(t)\right\|+\left\|x_{m}(t)-x_{n}(t)\right\|\right) .
\end{align*}
$$

From (3.16) and the definition of Kuratowski measure of noncompactness, it follows that

$$
\mu\left(\left\{f\left(t, x_{n}, y_{n}\right)+N x_{n}(t)+M y_{n}(t)\right\}\right) \leq \bar{K}\left(\mu\left(\left\{x_{n}(t)\right\}\right)+\mu\left(\left\{y_{n}(t)\right\}\right)\right), \quad t \in J .
$$

If $\left\{x_{n}\right\} \subset[v, w]$ is a decreasing sequence and $\left\{y_{n}\right\} \subset\left[D_{1}(t), D_{2}(t)\right]$, the above inequality is also valid. Hence (H3) is satisfied. Therefore, Theorem 3.3 asserts that the PBVP (1.1) has minimal and maximal solutions $p$ and $q$ between $v$ and $w$. In the following we show that $p=q$.

Since by (H2) and (H4),

$$
\begin{align*}
\theta \leq q(t)-p(t) & =\int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int _ { 0 } ^ { 1 } G _ { \lambda _ { 1 } } ( s , \tau ) \left[f\left(\tau, q(\tau), \mathcal{D}^{\alpha} q(\tau)\right)-f\left(\tau, p(\tau), \mathcal{D}^{\alpha} p(\tau)\right)\right.\right. \\
& \left.\left.+M\left(\mathcal{D}^{\alpha} q(\tau)-\mathcal{D}^{\alpha} p(\tau)\right)+N(q(\tau)-p(\tau))\right] d \tau\right) d s \\
& \leq \int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int _ { 0 } ^ { 1 } G _ { \lambda _ { 1 } } ( s , \tau ) \left[(\bar{M}+M)\left(\mathcal{D}^{\alpha} q(\tau)-\mathcal{D}^{\alpha} p(\tau)\right)\right.\right.  \tag{3.17}\\
& +(\bar{N}+N)(q(\tau)-p(\tau))] d \tau) d s
\end{align*}
$$

(3.17) together with the normality of cone $P$ ensures

$$
\begin{align*}
\|q(t)-p(t)\| & \leq \int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int _ { 0 } ^ { 1 } G _ { \lambda _ { 1 } } ( s , \tau ) \left[L(\bar{M}+M)\left\|\mathcal{D}^{\alpha} q(\tau)-\mathcal{D}^{\alpha} p(\tau)\right\|\right.\right. \\
& +L(\bar{N}+N)\|q(\tau)-p(\tau)\|] d \tau) d s  \tag{3.18}\\
& \leq[L(\bar{M}+M)+L(\bar{N}+N)] \frac{1}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)}\|q-p\|_{C_{\alpha}} .
\end{align*}
$$

Furthermore, if $\mathcal{D}^{\alpha} q(t) \geq \mathcal{D}^{\alpha} p(t)$, again by (H2) and (H4), we have

$$
\begin{align*}
\theta & \leq \mathcal{D}^{\alpha} q(t)-\mathcal{D}^{\alpha} p(t) \\
& =\lambda_{2} \int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int _ { 0 } ^ { 1 } G _ { \lambda _ { 1 } } ( s , \tau ) \left[f\left(\tau, q(\tau), \mathcal{D}^{\alpha} q(\tau)\right)-f\left(\tau, p(\tau), \mathcal{D}^{\alpha} p(\tau)\right)\right.\right. \\
& \left.\left.+M\left(\mathcal{D}^{\alpha} q(\tau)-\mathcal{D}^{\alpha} p(\tau)\right)+N(q(\tau)-p(\tau))\right] d \tau\right) d s \\
& +\int_{0}^{1} G_{\lambda_{1}}(t, s)\left[f\left(s, q(s), \mathcal{D}^{\alpha} q(s)\right)-f\left(s, p(s), \mathcal{D}^{\alpha} p(s)\right)\right.  \tag{3.19}\\
& \left.+M\left(\mathcal{D}^{\alpha} q(s)-\mathcal{D}^{\alpha} p(s)\right)+N(q(s)-p(s))\right] d s \\
& \left.\leq \int_{0}^{1} G_{\lambda_{1}}(t, s)[\bar{M}+M)\left(\mathcal{D}^{\alpha} q(s)-\mathcal{D}^{\alpha} p(s)\right)+(\bar{N}+N)(q(s)-p(s))\right] d s .
\end{align*}
$$

From (3.19) and the normality of cone $P$ we get

$$
\begin{align*}
\| \mathcal{D}^{\alpha} q(t) & -\mathcal{D}^{\alpha} p(t) \| \\
& \leq \int_{0}^{1} G_{\lambda_{1}}(t, s)\left[L(\bar{M}+M)\left\|\mathcal{D}^{\alpha} q(s)-\mathcal{D}^{\alpha} p(s)\right\|+L(\bar{N}+N)\|q(s)-p(s)\|\right] d s  \tag{3.20}\\
& \leq[L(\bar{M}+M)+L(\bar{N}+N)] \frac{1}{\alpha} \frac{1}{1-e^{\frac{\lambda_{1}}{\alpha}}}\|q-p\|_{C_{\alpha}} .
\end{align*}
$$

On the other hand, if $\mathcal{D}^{\alpha} q(t) \leq \mathcal{D}^{\alpha} p(t)$, also by (H2) and (H4) we get

$$
\begin{align*}
\theta & \leq \mathcal{D}^{\alpha} p(t)-\mathcal{D}^{\alpha} q(t) \\
& =\lambda_{2} \int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int _ { 0 } ^ { 1 } G _ { \lambda _ { 1 } } ( s , \tau ) \left[f\left(\tau, p(\tau), \mathcal{D}^{\alpha} p(\tau)\right)-f\left(\tau, q(\tau), \mathcal{D}^{\alpha} q(\tau)\right)\right.\right. \\
& \left.\left.+M\left(\mathcal{D}^{\alpha} p(\tau)-\mathcal{D}^{\alpha} q(\tau)\right)+N(p(\tau)-q(\tau))\right] d \tau\right) d s \\
& +\int_{0}^{1} G_{\lambda_{1}}(t, s)\left[f\left(s, p(s), \mathcal{D}^{\alpha} p(s)\right)-f\left(s, q(s), \mathcal{D}^{\alpha} q(s)\right)\right.  \tag{3.21}\\
& \left.+M\left(\mathcal{D}^{\alpha} p(s)-\mathcal{D}^{\alpha} q(s)\right)+N(p(s)-q(s))\right] d s \\
& \leq \lambda_{2} \int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int_{0}^{1} G_{\lambda_{1}}(s, \tau)[\bar{M}+M)\left(\mathcal{D}^{\alpha} p(\tau)-\mathcal{D}^{\alpha} q(\tau)\right)\right. \\
& +(\bar{N}+N)(p(\tau)-q(\tau))] d \tau) d s .
\end{align*}
$$

From (3.21) and the normality of cone $P$ we know

$$
\begin{align*}
\| \mathcal{D}^{\alpha} p(t) & -\mathcal{D}^{\alpha} q(t) \| \\
& \leq\left|\lambda_{2}\right| \int_{0}^{1} G_{\lambda_{2}}(t, s)\left(\int _ { 0 } ^ { 1 } G _ { \lambda _ { 1 } } ( s , \tau ) \left[L(\bar{M}+M)\left\|\mathcal{D}^{\alpha} p(\tau)-\mathcal{D}^{\alpha} q(\tau)\right\|\right.\right. \\
& +L(\bar{N}+N)\|p(\tau)-q(\tau)\|] d \tau) d s  \tag{3.22}\\
& \leq\left|\lambda_{2}\right|[L(\bar{M}+M)+L(\bar{N}+N)] \frac{1}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)}\|p-q\|_{C_{\alpha}} .
\end{align*}
$$

Consequently, from (3.18), (3.20) and (3.22) we can conclude

$$
\|p-q\|_{C_{\alpha}} \leq \frac{2 \bar{K}\left(1-\lambda_{2}\right)}{\alpha^{2}} \frac{1}{\left(1-e^{\frac{\lambda_{1}}{\alpha}}\right)\left(1-e^{\frac{\lambda_{2}}{\alpha}}\right)}\|p-q\|_{C_{\alpha}} .
$$

Thus, $p=q$ by (H4), which means that there exists a unique solution of PBVP (1.1) in $[v, w]$.
This completes the proof of Theorem 3.4.
Remark 3.5. Using the methods of our main results Theorem 3.3 and Theorem 3.4, we can easy to obtain the existence of solutions of the following PBVP for fractional differential equation

$$
\left\{\begin{array}{l}
\mathcal{D}^{\alpha} x(t)=f(t, x(t)), t \in(0,1], \quad 0<\alpha \leq 1  \tag{3.23}\\
x(0)=x(1)
\end{array}\right.
$$

Let $v, w \in C(J, E)$. We say that the function $v$ is a lower solution of problem (3.23) if

$$
\left\{\begin{array}{l}
\mathcal{D}^{\alpha} v(t) \leq f(t, v(t)), t \in(0,1] \\
v(0) \leq v(1)
\end{array}\right.
$$

Analogously, $w$ is an upper solution for problem (3.23) if it verifies similar conditions for the inequalities reversed.
(G1) There exists a constant $N>0$ such that $f\left(t, x_{2}\right)-f\left(t, x_{1}\right) \geq-N\left(x_{2}-x_{1}\right), t \in J$, where $v \leq x_{1} \leq$ $x_{2} \leq w$.
(G2) There exists a constant $K \geq 0$ such that $\mu\left(\left\{f\left(t, x_{n}\right)+N x_{n}(t)\right\}\right) \leq K \mu\left(\left\{x_{n}(t)\right\}\right), t \in J$ for any monotonic sequence $\left\{x_{n}\right\} \subset[v, w]$. Moreover, $\frac{2 K}{\alpha} \frac{1}{1-e^{\frac{-N}{\alpha}}}<1$.
(G3) There exists a constant $\bar{N}>0$ such that $f\left(t, x_{2}\right)-f\left(t, x_{1}\right) \leq \bar{N}\left(x_{2}-x_{1}\right), t \in J$, where $v \leq x_{1} \leq$ $x_{2} \leq w$. Moreover, $\frac{2 \bar{K}}{\alpha} \frac{1}{1-e^{-\frac{N}{\alpha}}}<1$, where $\bar{K}=L(\bar{N}+N)$ and $L$ is the normal constant of cone $P$.

Theorem 3.6. Assume that $f \in C(J \times E), v, w$ are lower and upper solutions of BPVP (3.23) and $v \leq w$. The conditions (G1) and (G2) are valid. Then there exist $p(t), q(t) \in C(J, E)$ such that $p(t), q(t)$ are minimal and maximal solutions on the ordered interval $[v, w]$ for BPVP (3.23), respectively, that is, for any solution $x(t)$ of BPVP (3.23) such that $x \in[v, w]$, we have $v(t) \leq p(t) \leq x(t) \leq q(t) \leq w(t), t \in J$.

Theorem 3.7. Assume that $f \in C(J \times E), v, w$ are lower and upper solutions of $\operatorname{BPVP}(3.23), v \leq w$, and the conditions (G1) and (G3) hold. Then the BPVP (3.23) has a unique solution $x \in[v, w]$.

## 4. An example

Let $E=\left\{x: x=\left(x_{1}, x_{2}, \ldots, x_{n}, \ldots\right), x_{n} \rightarrow 0\right\}$ with the norm $\|x\|=\sup _{n}\left|x_{n}\right|$ and $P=\left\{x \in E: x_{n} \geq\right.$ $0, n=1,2,3, \ldots\}$. Then $P$ is a normal cone in $E$. Consider the PBVP of infinite system for differential equations in $E$

$$
\left\{\begin{align*}
\mathcal{D}^{2 \alpha} x_{n}(t)=\frac{1}{8 e^{n t}} & {\left[\left(1-x_{n}(t)\right)^{2}-\frac{2 n^{2}-1}{2 n^{2}}\right] }  \tag{4.1}\\
& \quad+\frac{(n-1)(2 n+1)^{3}}{4 n^{2} e^{n t}} \sin ^{3} x_{2 n+1}(t)-\mathcal{D}^{\alpha} x_{n}(t), t \in(0,1], \\
x_{n}(0)=x_{n}(1), & \mathcal{D}^{\alpha} x_{n}(0)=\mathcal{D}^{\alpha} x_{n}(1) .
\end{align*}\right.
$$

Evidently, (4.1) can be regarded as a PBVP of the form (1.1) in $E$. In this situation, $x=\left(x_{1}, x_{2}, \ldots, x_{n}, \ldots\right), y=\left(y_{1}, y_{2}, \ldots, y_{n}, \ldots\right)$ and $f=\left(f_{1}, f_{2}, \ldots, f_{n}, \ldots\right)$, in which

$$
f_{n}(t, x, y)=\frac{1}{8 e^{n t}}\left[\left(1-x_{n}(t)\right)^{2}-\frac{2 n^{2}-1}{2 n^{2}}\right]+\frac{(n-1)(2 n+1)^{3}}{4 n^{2} e^{n t}} \sin ^{3} x_{2 n+1}(t)-y_{n}(t) .
$$

It is clear $f \in C(J \times E \times E, E)$. Let $v=(0,0, \ldots, 0, \ldots)$ and $w=\left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right)$. Then $v, w \in$ $C_{\alpha}(J, E), \nu(t) \leq w(t), t \in J$, and

$$
v_{n}(0)=v_{n}(1)=\mathcal{D}^{\alpha} v_{n}(0)=\mathcal{D}^{\alpha} v_{n}(1)=0, w_{n}(0)=w_{n}(1)=\frac{1}{n}, \mathcal{D}^{\alpha} w_{n}(0)=\mathcal{D}^{\alpha} w_{n}(1)=0
$$

Moreover,

$$
f_{n}\left(t, v(t), \mathcal{D}^{\alpha} v(t)\right)=\frac{1}{8 e^{n t}}\left(1-\frac{2 n^{2}-1}{2 n^{2}}\right)=\frac{1}{8 e^{n t}} \frac{1}{2 n^{2}}>0,
$$

and

$$
\begin{aligned}
f_{n}\left(t, w(t), \mathcal{D}^{\alpha} w(t)\right) & =\frac{1}{8 e^{n t}}\left[\left(1-\frac{1}{n}\right)^{2}-\frac{2 n^{2}-1}{2 n^{2}}\right]+\frac{(n-1)(2 n+1)^{3}}{4 n^{2} e^{n t}} \sin ^{3} \frac{1}{2 n+1} \\
& \leq \frac{1}{8 e^{n t}}\left[\left(1-\frac{1}{n}\right)^{2}-\frac{2 n^{2}-1}{2 n^{2}}\right]+\frac{1}{8 e^{n t}} \frac{2 n-2}{n^{2}}=-\frac{1}{8 e^{n t}} \frac{1}{2 n^{2}}<0 .
\end{aligned}
$$

Hence, $v$ and $w$ are lower and upper solutions of (4.1).
The conditions (H1) and (H2) are satisfied with $M=1, N=\frac{1}{4}$. Let us first verify the condition (H2) . Noticing that $\lambda_{2}=-\frac{1}{2}, D_{1}(t)=-\frac{w(t)}{2}$ and $D_{2}(t)=\frac{w(t)}{2}$, for $v(t) \leq x^{(1)}(t) \leq x^{(2)}(t) \leq w(t)$ and $D_{1}(t) \leq y^{(i)}(t) \leq D_{2}(t), i=1,2$, we have

$$
0 \leq x_{n}^{(1)}(t) \leq x_{n}^{(2)}(t) \leq \frac{1}{n},-\frac{1}{2 n} \leq y_{n}^{(i)}(t) \leq \frac{1}{2 n}, i=1,2, n=1,2,3, \ldots
$$

Therefore,

$$
\begin{aligned}
f_{n}\left(t, x^{(2)}, y^{(2)}\right) & -f_{n}\left(t, x^{(1)}, y^{(1)}\right)=\frac{1}{8 e^{n t}}\left[\left(1-x_{n}^{(2)}(t)\right)^{2}-\left(1-x_{n}^{(1)}(t)\right)^{2}\right]-\left(y_{n}^{(2)}(t)-y_{n}^{(1)}(t)\right) \\
& +\frac{(n-1)(2 n+1)^{3}}{4 n^{2} e^{n t}}\left[\sin ^{3} x_{2 n+1}^{(2)}(t)-\sin ^{3} x_{2 n+1}^{(1)}(t)\right] \\
& \left.\geq \frac{1}{8 e^{n t}}\left[1-x_{n}^{(2)}(t)\right)^{2}-\left(1-x_{n}^{(1)}(t)\right)^{2}\right]-\left(y_{n}^{(2)}(t)-y_{n}^{(1)}(t)\right) \\
& =-\frac{1}{8 e^{n t}}\left(2-x_{n}^{(1)}(t)-x_{n}^{(2)}(t)\right)\left(x_{n}^{(2)}(t)-x_{n}^{(1)}(t)\right)-\left(y_{n}^{(2)}(t)-y_{n}^{(1)}(t)\right) \\
& \geq-\frac{1}{4 e^{n t}}\left(x_{n}^{(2)}(t)-x_{n}^{(1)}(t)\right)-\left(y_{n}^{(2)}(t)-y_{n}^{(1)}(t)\right) \\
& \geq-\frac{1}{4}\left(x_{n}^{(2)}(t)-x_{n}^{(1)}(t)\right)-\left(y_{n}^{(2)}(t)-y_{n}^{(1)}(t)\right) .
\end{aligned}
$$

This implies that (H2) is satisfied. Obviously, by the same method we can verify (H1).
Finally, we check condition (H3). Let the sequences $\left\{x^{(m)}: x^{(m)}=\left(x_{1}^{(m)}, x_{2}^{(m)}, \ldots, x_{n}^{(m)}, \ldots\right)\right\}$ and $\left\{y^{(m)}: y^{(m)}=\left(y_{1}^{(m)}, y_{2}^{(m)}, \ldots, y_{n}^{(m)}, \ldots\right)\right\}$ be given such that $\left\{x^{(m)}\right\} \subset[v, w]$ is monotonous and $-\frac{1}{2 n} \leq$ $y_{n}^{(m)}(t) \leq \frac{1}{2 n}, n=1,2,3, \ldots$. Let $z_{n}^{(m)}(t)=f_{n}\left(t, x^{(m)}, y^{(m)}\right)+N x_{n}^{(m)}(t)+M y_{n}^{(m)}(t)$. In view of

$$
\left|z_{n}^{(m)}(t)\right| \leq \frac{1}{8} \frac{4 n-3}{2 n^{2}}+\frac{n-1}{4 n^{2}}+\frac{1}{2 n}+\frac{N}{n}+\frac{M}{2 n}, t \in J, n, m=1,2,3, \ldots,
$$

it follows that $\left\{z_{n}^{(m)}(t)\right\}$ is bounded, so we can choose a subsequence $\left\{m_{i}\right\} \subset\{m\}$ such that $z_{n}^{\left(m_{i}\right)}(t) \rightarrow z_{n}(t)$ as $i \rightarrow \infty, n=1,2,3, \ldots$ and

$$
\left|z_{n}(t)\right| \leq \frac{1}{8} \frac{4 n-3}{2 n^{2}}+\frac{n-1}{4 n^{2}}+\frac{1}{2 n}+\frac{N}{n}+\frac{M}{2 n}, t \in J, \quad n=1,2,3, \ldots
$$

Hence, $z(t)=\left(z_{1}(t), z_{2}(t), z_{3}(t), \ldots\right) \in E$ for any $t \in J$, and it is easy to see that

$$
\left\|f\left(t, x^{\left(m_{i}\right)}, y^{\left(m_{i}\right)}\right)+N x^{\left(m_{i}\right)}(t)+M y^{\left(m_{i}\right)}(t)-z(t)\right\|=\sup _{n}\left|z_{n}^{\left(m_{i}\right)}(t)-z_{n}(t)\right| \rightarrow 0, i \rightarrow \infty
$$

Consequently, we conclude that condition (H3) is satisfied for $K=0$. Therefore, Theorem 3.3 ensures that PBVP (4.1) has extremal solutions in [ $v, w$ ], which can be obtained by taking limits from the iterative sequences $\left\{v^{(m)}: v^{(m)}=\left(v_{1}^{(m)}, v_{2}^{(m)}, \ldots, v_{n}^{(m)}, \ldots\right)\right\}$ and $\left\{w^{(m)}: w^{(m)}=\left(w_{1}^{(m)}, w_{2}^{(m)}, \ldots, w_{n}^{(m)}, \ldots\right)\right\}$, here

$$
v^{(0)}=v, \quad w^{(0)}=w,
$$

$$
\begin{aligned}
v_{n}^{(m)}(t)= & \int_{0}^{1} G_{\lambda_{2}}(t, s)\left\{\int_{0}^{1} G_{\lambda_{1}}(s, \tau)\left[f_{n}\left(\tau, v^{(m-1)}(\tau), \mathcal{D}^{\alpha} v^{(m-1)}(\tau)\right)+M \mathcal{D}^{\alpha} v_{n}^{(m-1)}(\tau)+N v_{n}^{(m-1)}(\tau)\right] d \tau\right\} d s \\
= & \int_{0}^{1} G_{-\frac{1}{2}}(t, s)\left\{\int_{0}^{1} G_{-\frac{1}{2}}(s, \tau)\right. \\
& {\left.\left[\frac{1}{8 e^{n \tau}}\left(\left(1-v_{n}^{(m-1)}(\tau)\right)^{2}-\frac{2 n^{2}-1}{2 n^{2}}\right)+\frac{(n-1)(2 n+1)^{3}}{4 n^{2} e^{n \tau}} \sin ^{3} v_{2 n+1}^{(m-1)}(\tau)+\frac{1}{4} v_{n}^{(m-1)}(\tau)\right] d \tau\right\} d s, }
\end{aligned}
$$

and

$$
\begin{aligned}
w_{n}^{(m)}(t)= & \int_{0}^{1} G_{\lambda_{2}}(t, s)\left\{\int_{0}^{1} G_{\lambda_{1}}(s, \tau)\left[f_{n}\left(\tau, w^{(m-1)}(\tau), \mathcal{D}^{\alpha} w^{(m-1)}(\tau)\right)+M \mathcal{D}^{\alpha} w_{n}^{(m-1)}(\tau)+N w_{n}^{(m-1)}(\tau)\right] d \tau\right\} d s \\
= & \int_{0}^{1} G_{-\frac{1}{2}}(t, s)\left\{\int_{0}^{1} G_{-\frac{1}{2}}(s, \tau)\right. \\
& {\left.\left[\frac{1}{8 e^{n \tau}}\left(\left(1-w_{n}^{(m-1)}(\tau)\right)^{2}-\frac{2 n^{2}-1}{2 n^{2}}\right)+\frac{(n-1)(2 n+1)^{3}}{4 n^{2} e^{n \tau}} \sin ^{3} w_{2 n+1}^{(m-1)}(\tau)+\frac{1}{4} w_{n}^{(m-1)}(\tau)\right] d \tau\right\} d s . }
\end{aligned}
$$

## 5. Conclusions

This paper explores periodic solutions of some nonlinear fractional differential equations. The problem discussed involves sequential conformable fractional derivative. Under suitable monotonicity conditions and noncompactness measure conditions, the existence and uniqueness of solutions are derived from monotone iterative technique and upper and lower solutions method. Further, it is analyzed that the similar methods are well suited for investigating the existence and uniqueness of periodic solutions of the non-sequential fractional differential equation. In particular, for $\alpha=1$, the classical results corresponding to ordinary differential equations of integer order are yielded. An example is given to illustrate an application of our theoretical work. To the best of our knowledge, the results obtained throughout this article are not recorded in any published literature. It is worth to be pointed out that the techniques applied in the main results of this paper can be used to investigate initial value problems or the differential equations with Riemann-Liouville sequential fractional derivatives.

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## Conflict of interest

The authors declare that they have no competing interests.

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