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Research article

A Fuglede-Putnam property for N-class $A(k)$ operators

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Abstract: This paper studies the Fuglede-Putnam's property for N-class $A(k)$ operators and $\mathcal{Y}$ class operators. Some range-kernel orthogonality results of the generalized derivation induced by the above classes of operators are given.

Keywords: range-kernel orthogonality; hyponormal operator; $\mathcal{Y}$ class operator; N-class $A(k)$-operator
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1. Introduction

For complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}, \mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})$ and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operator on $\mathcal{H}$, on $\mathcal{K}$ and from $\mathcal{H}$ to $\mathcal{K}$ respectively. A bounded operator $A \in \mathcal{B}(\mathcal{H})$ is called normal if $A^*A = AA^*$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be a class $\mathcal{Y}_\kappa$ for $\kappa \leq 1$ if there exists a positive number k_κ such that

$$|AA^* - A^*A|^\kappa \leq k_\kappa^2 (A - \lambda)^*(A - \lambda) \quad \text{for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\kappa \subset \mathcal{Y}_\eta$ if $1 \leq \kappa \leq \eta$. Let $\mathcal{Y} = \cup_{1\leq\kappa}\mathcal{Y}_\kappa$ [2].

The familiar Putnam-Fuglede's theorem asserts that if $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are normal operators and $AX = XB$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$ (see [7]). Many authors have extended this theorem for several classes operators, recently S. Mecheri et al [6] proved that the Fuglede-Putnam theorem holds for p-hyponormal or class $\mathcal{Y}$, A. Bachir et al [1] proved that the theorem holds for w-hyponormal or class $\mathcal{Y}$ operators. We say that the pair (A, B) satisfy Fuglede-Putnam theorem if $AX = XB$ implies $A^*X = XB^*$ for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

2. Materials and method

Definition 1. An operator $A \in \mathcal{B}(\mathcal{H})$ is N-class $A(k)$ if

$$|A|^2 \le N(A^*|A|^{2k}A)^{\frac{1}{k+1}}$$

for a fixed integer N and a positive number k.

Definition 2. We say that $A \in \mathcal{B}(\mathcal{H})$ has the single valued extension property at λ (SVEP for short) if for every neighbourhood U of λ, the only analytic function $f : U \to \mathcal{H}$ which satisfies the equation $(A-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. We say that $A \in \mathcal{B}(\mathcal{H})$ satisfies the SVEP property if A has the single valued extension property at every $\lambda \in \mathbb{C}$.

We will prove and recall any known results which will be used in the sequel.

Lemma 3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a N-$A(k)$ class operator and $\mathcal{M} \subset \mathcal{H}$ an invariant subspace of T. Then $T|_{\mathcal{M}}$ is N-$A(k)$ class operator as well.*

Proof. Let $A = T|_{\mathcal{M}}$ and P be the orthogonal projection on $\mathcal{M}$.
Since $\mathcal{M}$ is an invariant for T, we get

$$TP = PTP = A \oplus 0 \quad \text{on} \quad \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$$

Therefore

$$|A|^2 \oplus 0 = PT^*TP = P|T|^2P \tag{2.1}$$

Since $T \in N$-$A(k)$ class, then

$$P|T|^2P \le NP\Big(T^*|T|^{2k}T\Big)^{\frac{1}{k+1}}P$$

and so

$$|A|^2 \oplus 0 \le NP\Big(T^*|T|^{2k}T\Big)^{\frac{1}{k+1}}P.$$

From Hansen's inequality, we get

$$\begin{aligned} |A|^2 &\le N\Big(PT^*|T|^{2k}TP\Big)^{\frac{1}{k+1}} \\ &= N\Big(PT^*P|T|^{2k}PTP\Big)^{\frac{1}{k+1}} \\ &= N\Big((A^*\oplus 0)P|T|^{2k}P(A\oplus 0)\Big)^{\frac{1}{k+1}} \\ &= N\Big(A^*P|T|^{2k}PA\Big)^{\frac{1}{k+1}} \end{aligned} \tag{2.2}$$

It follows from (2.1) and Hansen's inequality that $|A|^{2k} \ge P|T|^{2k}P$ and so

$$A^*|A|^{2k}A \ge A^*\Big(P|T|^{2k}P\Big)A.$$

By Lowner-Heinz inequality, we get

$$\left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}} \geq \left(A^*P|T|^{2k}PA\right)^{\frac{1}{k+1}}. \tag{2.3}$$

Therefore, from (2.2) and (2.3), we get

$$|A|^2 \leq N\left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}},$$

which means that $A \in N\text{-}A(k)$ class. □

We will need one more lemmas.

Lemma 4. *[9] If A is N-A(k) class operator and $A = U|A|$, then the Aluthge transformation $\tilde{A} = |A|^{1/2}U|A|^{1/2}$ of A is hyponormal.*

Lemma 5. *[1] If A is hyponormal, then A has SVEP.*

Lemma 6. *[13] Let A be a class ($\mathcal{Y}$) and $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for A. If $A\mid_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces A.*

Lemma 7. *[10] Let A be a N-A(k) class operator and $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for A. If $A\mid_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces A.*

Theorem 8. *[12] Let $A_1 \in \mathcal{B}(\mathcal{H})$) and $A_2 \in \mathcal{B}(\mathcal{K})$. Then the following assertions are equivalent*

1. *The pair (A_1, A_2) satisfies Fuglede-Putnam theorem.*
2. *If $A_1X = XA_2$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $\overline{\operatorname{ran} X}$ reduces A_1, $(\ker X)^\perp$ reduces A_2, and $A_1\mid_{\overline{\operatorname{ran} X}}$, $A_2\mid_{(\ker X)^\perp}$ are normal operators.*

3. Results

Theorem 9. *Let $A \in \mathcal{B}(\mathcal{H})$ be an injective N-A(k) class operator and $B^* \in \mathcal{B}(\mathcal{K})$ be a class ($\mathcal{Y}$). If $AX = XB$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$.*

Proof. Since B^* is of class ($\mathcal{Y}$), there exist positive numbers κ and p_κ such that

$$|BB^* - B^*B|^\kappa \leq p_\kappa^2(B-\lambda)(B-\lambda)^* \text{ for all } \lambda \in \mathbb{C}.$$

Hence by [5], for all $x \in |BB^* - B^*B|^{\kappa/2}\mathcal{K}$ there exists a bounded function $g : \mathbb{C} \to \mathcal{K}$ such that

$$(B-\lambda)g(\lambda) = x \text{ for all } \lambda \in \mathbb{C}$$

Let $A = U|A|$ be the polar decomposition of A and defines its Aluthge transform by $\tilde{A} = |A|^{1/2}U|A|^{1/2}$. Then $\tilde{A}$ is hyponormal by lemma 4 and

$$\begin{aligned}(\tilde{A}-\lambda)|A|^{1/2}Xg(\lambda) &= |A|^{1/2}(A-\lambda)Xg(\lambda) \\ &= |A|^{1/2}X(B-\lambda)g(\lambda) \\ &= |A|^{1/2}Xx, \quad \forall\lambda \in \mathbb{C}.\end{aligned}$$

We assert that $|A|^{1/2}Xx = 0$. Otherwise, if $|A|^{1/2}Xx \neq 0$, then from lemma 4 and by [11] the function $g : \mathbb{C} \to \mathcal{H}$ is bounded entire function and hence it is constant by Liouville theorem. Therefore, it follows from

$$g(\lambda) = (\tilde{A} - \lambda)|A|^{1/2}Xx \to 0 \text{ as } \lambda \to \infty,$$

that $g(\lambda) = 0$ and hence $|A|^{1/2}Xx = 0$. This is a contradiction.

Then

$$|A|^{1/2}X|BB^* - B^*B|^{\kappa/2}\mathcal{K} = \{0\}.$$

Since $\ker A = \ker |A| = \{0\}$, we get

$$X(BB^* - B^*B) = 0.$$

Since $\overline{\operatorname{ran}(X)}$ is invariant under A and $(\ker X)^\perp$ is invariant under B^*, we can write

$$\begin{aligned} A &= \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\operatorname{ran}(X)} \oplus \operatorname{ran}(X)^\perp, \\ B &= \begin{pmatrix} B_1 & 0 \\ B_3 & B_2 \end{pmatrix} \text{ on } \mathcal{K} = (\ker X)^\perp \oplus \ker X, \\ X &= \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ on } (\ker X)^\perp \oplus (\ker X) \to \overline{\operatorname{ran}(X)} \oplus \operatorname{ran}(X)^\perp \end{aligned}$$

implying

$$\begin{aligned} 0 &= X(BB^* - B^*B) \\ &= \begin{pmatrix} X_1(B_1B_1^* - B_1^*B_1 - B_3^*B_3) & X_1(B_1B_3^* - B_3^*B_2) \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$X_1(B_1B_1^* - B_1^*B_1 - B_3^*B_3) = 0.$$

Since X_1 is injective and has dense range,

$$B_1B_1^* - B_1^*B_1 = B_3^*B_3 \geq 0.$$

This implies that the operator B_1^* is hyponormal. Now, from the equality $AX = XB$, we get

$$A_1X_1 = X_1B_1, \tag{3.1}$$

where A_1 is N-$A(k)$ by Lemma 3 and B_1^* is hyponormal. Let $A_1 = U|A_1|$ be the polar decomposition of A_1, and multiply in left both sides of (3.1) by $|A_1|^{1/2}$ to obtain

$$\begin{aligned} |A_1|^{1/2}U|A_1|^{1/2}|A_1|^{1/2}X_1 &= |A_1|^{1/2}B_1X_1 \\ \tilde{A}_1|A_1|^{1/2}X_1 &= |A_1|^{1/2}X_1B_1 \\ \tilde{A}_1Y &= YB_1. \end{aligned}$$

where $\tilde{A}_1$ and B_1^* are hyponormal operators. By Fuglede-Putnam Theorem [8] it yields

$$\tilde{A_1}^* Y = YB_1^* \tag{3.2}$$
$$|A_1|^{1/2} U^* |A_1| X_1 = |A_1|^{1/2} X_1 B_1^*. \tag{3.3}$$

Hence
$$|A_1| U^* |A_1| X_1 = |A_1| X_1 B_1^*.$$

And
$$|A_1|(A_1^* X_1 - X_1 B_1^*) = 0.$$

Since A_1 is injective, then
$$A_1^* X_1 = X_1 B_1^*.$$

Hence, A_1 and B_1 are normal by theorem 8 implying $A_2 = 0$ by lemma 7 and $B_2 = 0$ by lemma 6. Consequently
$$A^* X = \begin{pmatrix} A_1^* X_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_1 B_1^* & 0 \\ 0 & 0 \end{pmatrix} = XB^*.$$

□

Theorem 10. *Let $A \in \mathcal{B}(\mathcal{H})$ be N-$A(k)$ class operator and $B^* \in \mathcal{B}(\mathcal{K})$ be a class $\mathcal{Y}$. If $AX = XB$, for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^* X = XB^*$.*

Proof. Decompose A into normal part A_1 and pure part A_2 as

$$A = A_1 \oplus A_2 \quad \text{on} \quad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

and let
$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} : \mathcal{K} \to \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Since A_2 is an injective pure N-$A(k)$ class operator. $AX = XB$ implies

$$\begin{pmatrix} A_1 X_1 \\ A_2 X_2 \end{pmatrix} = \begin{pmatrix} X_1 B_1 \\ X_2 B_2 \end{pmatrix}.$$

Hence
$$A^* X = \begin{pmatrix} A_1^* X_1 \\ A_2^* X_2 \end{pmatrix} = \begin{pmatrix} X_1 B_1^* \\ X_2 B_2^* \end{pmatrix} = XB^*.$$

by applying theorem 9. □

Theorem 11. *Let $A \in \mathcal{B}(\mathcal{H})$ be class $\mathcal{Y}$ and $B^* \in \mathcal{B}(\mathcal{K})$ be N-$A(k)$ class operator. If $AX = XB$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^* X = XB^*$.*

Proof. **Case 1.** If B^* is injective. Suppose that $AX = XB$ for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Since $\overline{\operatorname{ran}(X)}$ is invariant by A and $(\ker X)^\perp$ is invariant by B^*, we consider the following decomposition:

$$\mathcal{H} = \overline{\operatorname{ran}(X)} \oplus (\operatorname{ran}(X))^\perp, \quad \mathcal{K} = (\ker X)^\perp \oplus (\ker X).$$

Then it yields

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}$$

and

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : (\ker X)^{\perp} \oplus (\ker X) \longrightarrow \overline{\operatorname{ran}(X)} \oplus (\operatorname{ran}(X))^{\perp}.$$

From $AX = XB$, we get

$$A_1 X_1 = X_1 B_1 \tag{3.4}$$

Let $B_1^* = U^*|B_1^*|$ be the polar decomposition of B_1^*. Multiply both sides of (3.4) in the right by $|B_1^*|^{1/2}$, we obtain

$$\begin{aligned} A_1 X_1 |B_1^*|^{1/2} &= X_1 B_1 |B_1^*|^{1/2} \\ &= X_1 |B_1^*|^{1/2} (\tilde{B_1^*})^*. \end{aligned}$$

Since A_1 is class $\mathcal{Y}$ and $(\tilde{B_1^*})^*$ is co-hyponormal, then $(A_1, (\tilde{B_1^*}))$ satisfies (FP) property. Therefore $A \mid_{\overline{\operatorname{ran}(X_1|B_1^*|^{1/2})}}$ and $\tilde{B_1^*} \mid_{(\ker X_1|B_1^*|^{1/2})^{\perp}}$ are normal operators by [12]. Since X_1 is injective with dense range and $|B_1^*|^{1/2}$ is injective, then

$$\overline{\operatorname{ran}(X_1|B_1^*|^{1/2})} = \overline{\operatorname{ran}(X_1)} = \overline{\operatorname{ran}(X)}$$

and

$$\ker(X_1|B_1^*|^{1/2}) = \ker(X_1) = \ker(X).$$

It follows that $\tilde{B_1^*} \mid_{(\ker X)^{\perp}}$ is normal and $(\ker X)^{\perp}$ reduces B^*, also $\widetilde{\operatorname{ran}}(X)$ reduces A. Thus $A_2 = B_2 = 0$. Since $A_1 X_1 = X_1 B_1$ with A_1 and B_1 being normal, then $A_1^* X_1 = X_1 B_1^*$. Consequently, $A^* X = X B^*$.

Case 2. Decompose B^* into normal part B_1^* and pure part B_2^* as $B^* = B_1^* \oplus B_2^*$ on $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where B_2^* is an injective N-$A(k)$ class operator. Let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} : \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \to \mathcal{H}.$$

Since B_1^* is an injective pure N-$A(k)$ class operator. $AX = XB$ implies

$$\begin{pmatrix} A_1 X_1 \\ A_2 X_2 \end{pmatrix} = \begin{pmatrix} X_1 B_1 \\ X_2 B_2 \end{pmatrix}.$$

Hence

$$A^* X = \begin{pmatrix} A_1^* X_1 \\ A_2^* X_2 \end{pmatrix} = \begin{pmatrix} X_1 B_1^* \\ X_2 B_2^* \end{pmatrix} = X B^*.$$

by Case 1. □

Theorem 12. *Let $A \in \mathcal{B}(\mathcal{H})$ be an injective N_1-$A(k_1)$ class operator and $B^* \in \mathcal{B}(\mathcal{K})$ be an injective N_2-$A(k_2)$ class operator. If $AX = XB$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^* X = X B^*$.*

Proof. Since $\overline{\text{ran}(X)}$ is invariant by A and $(\ker X)^{\perp}$ is invariant by B^*, if we consider the decomposition

$$\mathcal{H} = \overline{\text{ran}(X)} \oplus \text{ran}(X)^{\perp}, \quad \mathcal{K} = (\ker X)^{\perp} \oplus \ker X,$$

then A, B and X can be written as

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix} \quad X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From $AX = XB$, we get

$$A_1X_1 = X_1B_1. \tag{3.5}$$

Let $A_1 = U_1|A_1|$ and $B_1^* = V^*|B_1^*|$ be the polar decomposition of A_1 and B_1^* respectively. Multiply the both sides of (3.5) in left by $|A_1|^{1/2}$ and in the right by $|B_1^*|^{1/2}$ and uses the polar decomposition, we obtain

$$\begin{aligned} |A_1|^{1/2}U_1|A_1|^{1/2}(|A_1|^{1/2}X|B_1^*|^{1/2}) &= (|A_1|^{1/2}X|B_1^*|^{1/2})|B_1^*|^{1/2}V_1^*|B_1^*|^{1/2} \\ \tilde{A_1}Y &= Y\tilde{B_1^*}^* \\ &= Y\tilde{B}_1, \end{aligned}$$

where $Y = |A_1|^{1/2}X|B_1^*|^{1/2}$. The last equality follows from the fact that $\tilde{T^*} = (\tilde{T})^*$. From the hyponormality of $\tilde{A_1}$ and $\tilde{B_1^*}$, we deduce that the pair $(\tilde{A_1}, \tilde{B_1^*})$ satisfies the Fuglede-Putnam, implying

$$\tilde{A_1}^*Y = Y\tilde{B_1}^*.$$

Hence $\tilde{A_1}\ |_{\overline{\text{ran}(Y)}}$ and $\tilde{B_1}\ |_{(\ker Y)^{\perp}}$ are normal operators by [12].
Since A_1, B_1^* and X_1 are injective, then Y is injective i.e.,

$$\ker Y = \ker(|A_1|^{1/2}X|B_1^*|^{1/2}) = \{0\}.$$

It follows that $\tilde{B_1^*}$ is normal imlying (B_1^* is normal), hence $(\ker X)^{\perp}$ reduces B^*. Therefore $B_2 = 0$. (We use the fact that if the Aluthge tranform of an operator T is normal, then T is normal). Also, since

$$\begin{aligned} \overline{\text{ran}(Y)} &= [\ker(|B_1^*|^{1/2}X^*|A_1|^{1/2})^{\perp} \\ &= 0^{\perp} \\ &= \overline{\text{ran}(X_1)} \\ &= \overline{\text{ran}(X)}. \end{aligned}$$

By the same argument as before, we get $A_2 = 0$. Finally , we obtain $A_1^*X_1 = X_1B_1^*$, and therefore

$$A^*X = XB^*.$$

This completes the proof. □

Corollary 13. *Let $A \in \mathcal{B}(\mathcal{H})$ be N_1-$A(k_1)$ class operator and $B^* \in \mathcal{B}(\mathcal{K})$ be N_2-$A(k_2)$ class operator. If $AX = XB$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$.*

Proof. Decompose A (resp. B^*) into normal part A_1 (resp. B_1^*) and pure part A_2 (resp. B_2^*) as

$$\begin{array}{rcll} A &=& A_1 \oplus A_2 & \text{on } \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \\ B &=& B_1 \oplus B_2 & \text{on } \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2, \end{array}$$

and let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} : \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \to \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Here A_1, B_1 are normal, A_2 is an injective N_1-$A(k_1)$ class operator and B_2^* is an injective N_2-$A(k_2)$ class operator. From $AX = XB$, we get

$$\begin{pmatrix} A_1X_1 \\ A_2X_2 \end{pmatrix} = \begin{pmatrix} X_1B_1 \\ X_2B_2 \end{pmatrix}.$$

Hence

$$A^*X = \begin{pmatrix} A_1^*X_1 \\ A_2^*X_2 \end{pmatrix} = \begin{pmatrix} X_1B_1^* \\ X_2B_2^* \end{pmatrix} = XB^*.$$

by applying theorem 12. □

4. Conclusions

The following Putnam-Fuglede theorem is very well known:

Theorem 14. *(Putnam-Fuglede Theorem) [7]*
Assume that $A, B \in B(\mathcal{H})$ are normal operators. If $AX = XB$ for some $X \in B(\mathcal{H})$, then $A^*X = XB^*$

These are many extensions of this theorem to several classes of operators. In 1978, S.K Berberian [4] showed that the Putnam-Fuglede theorem holds when A and B^* are hyponormal and X a Hilbert-Schmidt operator. Radjapalipour [8] proved that the Putnam-Fuglede theorem remains valid for hyponormal operators. In 2002, Uchiyama and Tanahashi [14] proved that the theorem still holds for p-hyponormal and log-hyponormal operators. Bachir and Lombarkia [1] gave an extension of Putnam-Fuglede theorem for w-hyponormal and class $\mathcal{Y}$. Recently, Bachir and Segres[3] extended this theorem to class (n, k)-quasi-*-paranormal operators.

The novelty to this contribution is to extend the famous Putnam-Fuglede thorem to the N-$A(k)$ class operators which is a superclass containing the normal operators and in other hand, generalize the results obtained in [4, 8].

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Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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