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Research article

Long-time asymptotics for the generalized Sasa-Satsuma equation

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Abstract: In this paper, we study the long-time asymptotic behavior of the solution of the Cauchy problem for the generalized Sasa-Satsuma equation. Starting with the 3×3 Lax pair related to the generalized Sasa-Satsuma equation, we construct a Rieman-Hilbert problem, by which the solution of the generalized Sasa-Satsuma equation is converted into the solution of the corresponding Rieman-Hilbert problem. Using the nonlinear steepest decent method for the Riemann-Hilbert problem, we obtain the leading-order asymptotics of the solution of the Cauchy problem for the generalized Sasa-Satsuma equation through several transformations of the Riemann-Hilbert problem and with the aid of the parabolic cylinder function.

Keywords: nonlinear steepest descent method; generalized Sasa-Satsuma equation; long-time asymptotics
Mathematics Subject Classification: 35Q53, 35B40

1. Introduction

The Sasa-Satsuma equation

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u(|u|^2)_x = 0, \tag{1.1}$$

so-called high-order nonlinear Schrödinger equation [1], is relevant to several physical phenomena, for example, in optical fibers [2, 3], in deep water waves [4] and generally in dispersive nonlinear media [5]. Because this equation describes these important nonlinear phenomena, it has received considerable attention and extensive research. The Sasa-Satsuma equation has been discussed by means of various approaches such as the inverse scattering transform [1], the Riemann-Hilbert method [6], the Hirota bilinear method [7], the Darboux transformation [8], and others [9, 10, 11]. The initial-boundary value problem for the Sasa-Satsuma equation on a finite interval was studied by the Fokas method [12], which is also effective for the initial-boundary value problems on the half-line [35, 36, 37]. In Ref. [13], finite genus solutions of the coupled Sasa-Satsuma hierarchy are obtained in the basis of the theory of trigonal curves, the Baker-Akhiezer function and the meromorphic functions [14, 15, 16]. In Ref. [17], the super Sasa-Satsuma hierarchy associated with the 3×3 matrix spectral problem was proposed, and its bi-Hamiltonian structures were derived with the aid of the super trace identity.

The nonlinear steepest descent method [18], also called Deift-Zhou method, for oscillatory Riemann-Hilbert problems is a powerful tool to study the long-time asymptotic behavior of the solution for the soliton equation, by which the long-time asymptotic behaviors for a number of integrable nonlinear evolution equations associated with 2×2 matrix spectral problems have been obtained, for example, the mKdV equation, the KdV equation, the sine-Gordon equation, the modified nonlinear Schrödinger equation, the Camassa-Holm equation, the derivative nonlinear Schrödinger equation and so on [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. However, there is little literature on the long-time asymptotic behavior of solutions for integrable nonlinear evolution equations associated with 3×3 matrix spectral problems [31, 32, 33]. Usually, it is difficult and complicated for the 3×3 case. Recently, the nonlinear steepest descent method was successfully generalized to derive the long-time asymptotics of the initial value problems for the coupled nonlinear Schrödinger equation and the Sasa-Satsuma equation with the complex potentials [33, 34]. The main differences between the 2×2 and 3×3 cases is that the former corresponds to a scalar Riemann-Hilbert problem, while the latter corresponds to a matrix Riemann-Hilbert problem. In general, the solution of the matrix Riemann-Hilbert problem can not be given in explicit form, but the scalar Riemann-Hilbert problem can be solved by the Plemelj formula.

The main aim of this paper is to study the long-time asymptotics of the Cauchy problem for the generalized Sasa-Satsuma equation [38] via nonlinear steepest decent method,

$$\begin{cases} u_t + u_{xxx} - 6a|u|^2u_x - 6bu^2u_x - 3au(|u|^2)_x - 3b^*u^*(|u|^2)_x = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.2}$$

where a is a real constant, b is a complex constant that satisfies $a^2 \neq |b|^2$, the asterisk " $*$ " denotes the complex conjugate. It is easy to see that the generalized Sasa-Satsuma equation (1.2) can be reduced to the Sasa-Satsuma equation (1.1) when $a = -1$ and $b = 0$. Suppose that the initial value $u_0(x)$ lies in Schwartz space $\mathscr{S}(\mathbb{R}) = \{f(x) \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} |x^\alpha \partial^\beta f(x)| < \infty, \forall \alpha, \beta \in \mathbb{N}\}$. The vector function $\gamma(k)$ is determined by the initial data in (2.15) and (2.19), and $\gamma(k)$ satisfies the conditions (P_1) and (P_2), where

$$(P_1): \begin{cases} \gamma^\dagger(k^*)B_1\gamma(k) + \frac{|b|^2}{4}(\gamma^\dagger(k^*)\sigma_3\gamma(k))^2 < 1, \\ \gamma^\dagger(k^*)B_1\gamma(k) + a\gamma^\dagger(k^*)\sigma_3\gamma(k) < 2, \\ 2\gamma^\dagger(k^*)B_1\gamma(k) + |\gamma(k)|^2 + |B_1\gamma(k)|^2 < 4, \end{cases}$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a & b^* \\ b & a \end{pmatrix}; \tag{1.3}$$

(P_2): When $\det B_1 > 0$ and $a > 0$, $(2a - |B_1\gamma(k)|^2)$ and $(2a - \det B_1|\gamma(k)|^2)/(1 - \gamma^\dagger(k)B_1\gamma(k))$ are positive and bounded; otherwise, $(|B_1\gamma(k)|^2 - 2a)$ and $(\det B_1|\gamma(k)|^2 - 2a)/(1 - \gamma^\dagger(k)B_1\gamma(k))$ are positive and bounded.

The main result of this paper is as following:

Theorem 1.1. *Let $u(x,t)$ be the solution of the Cauchy problem for the generalized Sasa-Satsuma equation* (1.2) *with the initial value $u_0 \in \mathscr{S}(\mathbb{R})$. Suppose that the vector function $\gamma(k)$ is defined in* (2.19)*, the hypotheses (P_1) and (P_2) hold. Then, for $x < 0$ and $\sqrt{-\frac{x}{t}} < C$,*

$$u(x,t) = u_a(x,t) + O\left(c(k_0)t^{-1}\log t\right). \tag{1.4}$$

where C is a fixed constant, and

$$\begin{gathered}
u_a(x,t) = \sqrt{\frac{\nu}{12tk_0\gamma^{\dagger}(k_0)B_1\gamma(k_0)}}\left(-|\gamma_1(-k_0)|e^{i(\phi+\arg\gamma_1(-k_0))} + |\gamma_2(-k_0)|e^{-i(\phi+\arg\gamma_2(-k_0))}\right),\\
k_0 = \sqrt{-\frac{x}{12t}}, \quad \nu = -\frac{1}{2\pi}\log(1-\gamma^{\dagger}(k_0)B_1\gamma(k_0)),\\
\phi = \nu\log(196tk_0^3) - 16tk_0^3 + \arg\Gamma(-i\nu) + \frac{1}{\pi}\int_{-k_0}^{k_0}\log|\xi+k_0|\,\mathrm{d}(1-\gamma^{\dagger}(\xi)B_1\gamma(\xi)) - \frac{\pi}{4},
\end{gathered}$$

$c(\cdot)$ is rapidly decreasing, $\Gamma(\cdot)$ is the Gamma function, γ_1 and γ_2 are the first and the second row of $\gamma(k)$, respectively.

Remark 1.1. *The two conditions (P_1) and (P_2) satisfied by $\gamma(k)$ are necessary. The condition (P_1) guarantees the existence and the uniqueness of the solutions of the basic Riemann-Hilbert problem* (2.16) *and the Riemann-Hilbert problem* (3.1)*. The boundedness of the function $\delta(k)$ defined in subsection 3.1 relies on the condition (P_2).*

Remark 1.2. *In the case of $a = -1$ and $b = 0$, the generalized Sasa-Satsuma equation* (1.2) *can be reduced to the Sasa-Satsuma equation. Then it is obvious that the condition (P_1) is true, and the condition (P_2) is reduced to the case that $|\gamma(k)|$ is bounded. Therefore, the conditions (P_1) and (P_2) in this case are equivalent to the condition related to the reflection coefficient in [34], that is, $|\gamma(k)|$ is bounded for the Sasa-Satsuma equation.*

The outline of this paper is as follows. In section 2, we derive a Riemann-Hilbert problem from the scattering relation. The solution of the generalized Sasa-Satsuma equation is changed into the solution of the Riemann-Hilbert problem. In section 3, we deal with the Riemann-Hilbert problem via nonlinear steepest decent method, from which the long-time asymptotics in Theorem 1.1 is obtained at the end.

2. Basic Riemann-Hilbert problem

We begin with the 3×3 Lax pair of the generalized Sasa-Satsuma equation

$$\psi_x = (ik\sigma + U)\psi, \tag{2.1a}$$
$$\psi_t = (4ik^3\sigma + V)\psi, \tag{2.1b}$$

where ψ is a matrix function and k is the spectral parameter, $\sigma = \mathrm{diag}(1,1,-1)$,

$$U = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & u^* \\ au^*+bu & au+b^*u^* & 0 \end{pmatrix} \tag{2.2}$$

$$V = 4k^2U + 2ik(U^2 + U_x)\sigma + [U_x, U] - U_{xx} + 2U^3. \tag{2.3}$$

We introduce a new eigenfunction μ through $\mu = \psi e^{-ik\sigma x - 4ik^3\sigma t}$, where $e^{\sigma} = \mathrm{diag}(e, e, e^{-1})$. Then (2.1a) and (2.1b) become

$$\mu_x = ik[\sigma, \mu] + U\mu, \tag{2.4a}$$
$$\mu_t = 4ik^3[\sigma, \mu] + V\mu, \tag{2.4b}$$

where $[\cdot, \cdot]$ is the commutator, $[\sigma, \mu] = \sigma\mu - \mu\sigma$. From (2.4a), the matrix Jost solution $\mu_\pm$ satisfy the Volterra integral equations

$$\mu_\pm(k; x, t) = I + \int_{\pm\infty}^{x} e^{ik\sigma(x-\xi)} U(\xi, t)\mu_\pm(k; \xi, t) e^{-ik\sigma(x-\xi)}\,\mathrm{d}\xi. \tag{2.5}$$

Set $\mu_{\pm L}$ represent the first two columns of $\mu_\pm$, and $\mu_{\pm R}$ denote the third column, i.e., $\mu_\pm = (\mu_{\pm L}, \mu_{\pm R})$. Furthermore, we can infer that μ_{+R} and μ_{-L} are analytic in the lower complex k-plane $\mathbb{C}_-$, μ_{+L} and μ_{-R} are analytic in the the upper complex k-plane $\mathbb{C}_+$. Then we can introduce sectionally analytic function $P_1(k)$ and $P_2(k)$ by

$$P_1(k) = (\mu_{-L}(k), \mu_{+R}(k)), \quad k \in \mathbb{C}_-,$$
$$P_2(k) = (\mu_{+L}(k), \mu_{-R}(k)), \quad k \in \mathbb{C}_+.$$

One can find that U is traceless from (2.2), so $\det\mu_\pm$ are independent of x. Besides, $\det\mu_\pm = 1$ according to the evolution of $\det\mu_\pm$ at $x = \pm\infty$. Because all the $\mu_\pm e^{ik\sigma x + 4ik^3\sigma t}$ satisfy the differential equations (2.1a) and (2.1b), they are linear related. So there exists a scattering matrix $s(k)$ that satisfies

$$\mu_- = \mu_+ e^{ik\sigma x + 4ik^3\sigma t} s(k) e^{-ik\sigma x - 4ik^3\sigma t}, \quad \det s(k) = 1. \tag{2.6}$$

In this paper, we denote a 3×3 matrix A by the block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a 2×2 matrix and A_{22} is scalar. Let $q = (u, u^*)^T$ and we can rewrite U of (2.2) as

$$U = \begin{pmatrix} \mathbf{0}_{2\times 2} & q \\ q^\dagger B_1 & 0 \end{pmatrix},$$

where "†" is the Hermitian conjugate. In addition, there are two symmetry properties for U,

$$B^{-1}U^\dagger(k^*)B = -U(k), \quad \tau U^*(-k^*)\tau = U(k), \tag{2.7}$$

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.8}$$

where B and τ are represented as block forms. Hence, the Jost solutions $\mu_\pm$ and the scattering matrix $s(k)$ also have the corresponding symmetry properties

$$B^{-1}\mu_\pm^\dagger(k^*)B = \mu_\pm^{-1}(k), \quad \tau\mu_\pm^*(-k^*)\tau = \mu_\pm(k); \tag{2.9}$$

$$B^{-1}s^{\dagger}(k^*)B = s^{-1}(k), \quad \tau s^*(-k^*)\tau = s(k). \tag{2.10}$$

We write $s(k)$ as block form $(s_{ij})_{2\times 2}$ and from the symmetry properties (2.10) we have

$$s_{22}(k) = \det[s_{11}^{\dagger}(k^*)], \quad B_1^{-1}s_{21}^{\dagger}(k^*) = \operatorname{adj}[s_{11}(k)]s_{12}(k), \tag{2.11}$$

where adj$X$ denote the adjoint of matrix X. Then we can write $s(k)$ as

$$s(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{12}^{\dagger}(k^*)\operatorname{adj}[s_{11}^{\dagger}(k^*)]B_1 & \det[s_{11}^{\dagger}(k^*)] \end{pmatrix}, \tag{2.12}$$

where

$$\sigma_1 s_{11}^*(-k^*)\sigma_1 = s_{11}(k), \quad \sigma_1 s_{12}^*(-k^*) = s_{12}(k). \tag{2.13}$$

From the evaluation of (2.6) at $t = 0$, one infers

$$s(k) = \lim_{x\to+\infty} e^{-ikx\sigma}\mu_-(k;x,0)e^{ikx\sigma}, \tag{2.14}$$

which implies that

$$\begin{cases} s_{11}(k) = I + \int_{-\infty}^{+\infty} q(\xi,0)\mu_{-21}(k;\xi,0)\,\mathrm{d}\xi, \\ s_{12}(k) = \int_{-\infty}^{+\infty} e^{-2ik\xi}q(\xi,0)\mu_{-22}(k;\xi,0)\,\mathrm{d}\xi. \end{cases} \tag{2.15}$$

Theorem 2.1. *Let $M(k;x,t)$ be analytic for $k \in \mathbb{C}\backslash\mathbb{R}$ and satisfy the Riemann-Hilbert problem*

$$\begin{cases} M_+(k;x,t) = M_-(k;x,t)J(k;x,t), & k \in \mathbb{R}, \\ M(k;x,t) \to I, & k \to \infty, \end{cases} \tag{2.16}$$

where

$$M_\pm(k;x,t) = \lim_{\epsilon\to 0^+} M(k \mp i\epsilon;x,t), \tag{2.17}$$

$$J(k;x,t) = \begin{pmatrix} I - \gamma(k)\gamma^{\dagger}(k^*)B_1 & -e^{-2it\theta}\gamma(k) \\ e^{2it\theta}\gamma^{\dagger}(k^*)B_1 & 1 \end{pmatrix} \tag{2.18}$$

$$\theta(k;x,t) = -\frac{x}{t}k - 4k^3, \quad \gamma(k) = s_{11}^{-1}(k)s_{12}(k), \tag{2.19}$$

$\gamma(k)$ lies in Schwartz space and satisfies

$$\sigma_1\gamma^*(-k^*) = \gamma(k). \tag{2.20}$$

Then the solution of this Riemann-Hilbert problem exists and is unique, the function

$$q(x,t) = (u(x,t), u^*(x,t))^T = -2i\lim_{k\to\infty}(k(M(k;x,t))_{12}) \tag{2.21}$$

and $u(x,t)$ is the solution of the generalized Sasa-Satsuma equation.

Proof. The matrix $(J(k;x,t)+J^{\dagger}(k;x,t))/2$ is positive definite because of the condition (P_1) that $\gamma(k)$ satisfies, then the solution of the Riemann-Hilbert problem (2.16) is existent and unique according to the Vanishing Lemma [39]. We define $M(k;x,t)$ by

$$M(k;x,t)=\begin{cases}\left(\mu_{-L}(k),\mu_{+R}(k)\det[a^{\dagger}(k^*)]\right), & k\in\mathbb{C}_-,\\ (\mu_{+L}(k)a(k),\mu_{-R}(k)), & k\in\mathbb{C}_+.\end{cases}\tag{2.22}$$

Considering the scattering relation (2.6) and the construction of $M(k;x,t)$, we can obtain the jump condition and the corresponding Riemann-Hilbert problem (2.16) after tedious but straightforward algebraic manipulations. Substituting the large k asymptotic expansion of $M(k;x,t)$ into (2.4a) and compare the coefficients of $O(\frac{1}{k})$, we can get (2.21). □

3. Long-time asymptotic behavior

In this section, we compute the Riemann-Hilbert problem (2.16) by the nonlinear steepest decent method and study the long-time asymptotic behavior of the solution. We make the following basic notations. (i) For any matrix M define $|M|=(\mathrm{tr}M^{\dagger}M)^{\frac{1}{2}}$ and for any matrix function $A(\cdot)$ define $\|A(\cdot)\|_p=\||A(\cdot)|\|_p$. (ii) For two quantities A and B define $A\lesssim B$ if there exists a constant $C>0$ such that $|A|\leqslant CB$. If C depends on the parameter α we shall say that $A\lesssim_{\alpha} B$. (iii) For any oriented contour Σ, we say that the left side is + and the right side is −.

3.1. The first transformation: reorientation

First of all, it is noteworthy that there are two stationary points $\pm k_0$, where $\pm k_0=\pm\sqrt{-\frac{x}{12t}}$ satisfied $\frac{\mathrm{d}\theta}{\mathrm{d}k}\big|_{k=\pm k_0}=0$. The jump matrix $J(k;x,t)$ have a lower-upper triangular factorization and a upper-lower triangular factorization. We can introduce an appropriate Rieman-Hilbert problem to unify these two forms of factorizations. In this process, we have to reorient the contour of the Riemann-Hilbert problem.

The two factorizations of the jump matrix J are

$$J=\begin{cases}\begin{pmatrix} I & -e^{-2it\theta}\gamma(k)\\ 0 & 1\end{pmatrix}\begin{pmatrix} I & 0\\ e^{2it\theta}\gamma^{\dagger}(k^*)B_1 & 1\end{pmatrix},\\ \begin{pmatrix} I & 0\\ \frac{e^{2it\theta}\gamma^{\dagger}(k^*)B_1}{1-\gamma^{\dagger}(k^*)B_1\gamma(k)} & 1\end{pmatrix}\begin{pmatrix} I-\gamma(k)\gamma^{\dagger}(k^*)B_1 & 0\\ 0 & (1-\gamma^{\dagger}(k^*)B_1\gamma(k))^{-1}\end{pmatrix}\begin{pmatrix} I & \frac{-e^{-2it\theta}\gamma(k)}{1-\gamma^{\dagger}(k^*)B_1\gamma(k)}\\ 0 & 1\end{pmatrix}.\end{cases}$$

We introduce a 2×2 matrix function $\delta(k)$ to make the two factorization unified, and $\delta(k)$ satisfies the following Riemann-Hilbert problem

$$\begin{cases}\delta_+(k) &= \delta_-(k)(I-\gamma(k)\gamma^{\dagger}(k^*)B_1), & k\in(-k_0,k_0),\\ &= \delta_-(k), & k\in(-\infty,-k_0)\cup(k_0,+\infty),\\ \delta(k) &\to I, & k\to\infty,\end{cases}\tag{3.1}$$

which implies a scalar Riemann-Hilbert problem

$$\begin{cases}\det\delta_+(k) &= \det\delta_-(k)(1-\gamma^{\dagger}(k^*)B_1\gamma(k)), & k\in(-k_0,k_0),\\ &= \det\delta_-(k), & k\in(-\infty,-k_0)\cup(k_0,+\infty),\\ \det\delta(k) &\to 1, & k\to\infty.\end{cases}\tag{3.2}$$

The jump matrix $I-\gamma(k)\gamma^{\dagger}(k^*)B_1$ of Riemann-Hilbert problem (3.1) is positive definite, so the solution $\delta(k)$ exists and is unique. The scalar Riemann-Hilbert problem (3.2) can be solved by the Plemelj formula,

$$\det\delta(k)=\left(\frac{k-k_0}{k+k_0}\right)^{-i\nu}e^{\chi(k)}, \tag{3.3}$$

where

$$\nu=-\frac{1}{2\pi}\log(1-\gamma^{\dagger}(k_0)B_1\gamma(k_0)),$$
$$\chi(k)=-\frac{1}{2\pi i}\int_{-k_0}^{k_0}\log\left(\frac{1-\gamma^{\dagger}(\xi^*)B_1\gamma(\xi)}{1-\gamma^{\dagger}(k_0^*)B_1\gamma(k_0)}\right)\frac{\mathrm{d}\xi}{\xi-k}.$$

Then we have by uniqueness that

$$\delta(k)=B_1^{-1}(\delta^{\dagger}(k^*))^{-1}B_1,\quad \delta(k)=\sigma_1\delta^*(-k^*)\sigma_1. \tag{3.4}$$

Substituting (3.4) to (3.1), we have

$$\delta_+^{\dagger}(k^*)B_1\delta_+(k)=B_1-B_1\gamma(k)\gamma^{\dagger}(k^*)B_1, \tag{3.5}$$

which means that

$$\mathrm{tr}[\delta_+^{\dagger}(k^*)B_1\delta_+(k)]=2a-|B_1\gamma(k)|^2. \tag{3.6}$$

Actually, the condition (P_2) satisfied by $\gamma(k)$ guarantee the boundedness of $\delta_{\pm}(k)$ and we give a brief proof below. When $\det B_1>0$, we find that the Hermitian matrix B_1 can be decomposition. In other words, there exists a triangular matrix S that satisfies $B_1=aS^{\dagger}S$. So $\mathrm{tr}[\delta_+^{\dagger}B_1\delta_+]=a|S\delta_+|^2$. When $\det B_1<0$ and $|a|>0$, the matrix B_1 has a decomposition $B_1=S^{\dagger}DS$, where S is a triangular matrix and D is a diagonal matrix and the diagonal elements have opposite signs. In the case of $a>0$, B_1 can be decomposed as below,

$$B_1=\begin{pmatrix}-a & b^*\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}a\det B_1 & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}-a & 0\\ b & 1\end{pmatrix}^{-1}. \tag{3.7}$$

We denote $S\delta_+(k)$ by $(G_{ij})_{3\times3}$ and $c_1=2a-|B_1\gamma(k)|^2$ is negative, then

$$a\det B_1(|G_{11}|^2+|G_{21}|^2)+a(|G_{12}|^2+|G_{22}|^2)=c_1. \tag{3.8}$$

Noticing that $\det B_1<0$, we find a negative constant c_2 that satisfies $c_2\leqslant a\det B_1(c_3-1)/(1-\det B_1c_3)$, where c_3 is a constant and $0<c_3<1$, which impies

$$|S\delta_+(k)|^2\leqslant\frac{c_1}{c_2}\lesssim 1. \tag{3.9}$$

The case that $a<0$ is similar. In particular, when $a=0$, then $|b|>0$, it is easy to see that B_1 is not definite. For $|\mathrm{Re}b|>0$, we have the decomposition

$$B_1=\begin{pmatrix}\frac{b}{|b|^2+(b^*)^2} & \frac{b^*}{|b|^2+(b^*)^2}\\ \frac{b^*}{|b|^2+(b^*)^2} & -\frac{b^*}{|b|^2+(b^*)^2}\end{pmatrix}^{-1}\begin{pmatrix}\frac{1}{b+b^*} & 0\\ 0 & -\frac{1}{b+b^*}\end{pmatrix}\begin{pmatrix}\frac{b^*}{|b|^2+b^2} & \frac{b}{|b|^2+b^2}\\ \frac{b}{|b|^2+b^2} & -\frac{b}{|b|^2+b^2}\end{pmatrix}^{-1}. \tag{3.10}$$

For $|\mathrm{Re}b| = 0$, we have

$$B_1 = \begin{pmatrix} \frac{i}{2} & \frac{1-i}{2} \\ -\frac{i}{2} & \frac{1+i}{2} \end{pmatrix}^{-1} \begin{pmatrix} -ib/2 & 0 \\ 0 & ib/2 \end{pmatrix} \begin{pmatrix} -\frac{i}{2} & \frac{i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{pmatrix}^{-1}. \tag{3.11}$$

So we get the boundedness of $|\delta_+(k)|$. The others have the same analysis,

$$\delta_-^\dagger(k^*)B_1\delta_-(k) = (B_1 - \gamma(k)\gamma^\dagger(k^*))^{-1}, \quad \text{as} \quad k \in (-k_0, k_0), \tag{3.12}$$

$$|\delta_+(k)|^2 = |\delta_-(k)|^2 = 2, \quad \text{as} \quad k \in (-\infty, -k_0) \cup (k_0, +\infty), \tag{3.13}$$

$$|\det \delta_+(k)| = \begin{cases} 1 - \gamma^\dagger(k^*)B_1\gamma(k), & k \in (-k_0, k_0), \\ 1, & k \in (-\infty, -k_0) \cup (k_0, +\infty), \end{cases} \tag{3.14}$$

$$|\det \delta_-(k)| = \begin{cases} \frac{1}{1-\gamma^\dagger(k^*)B_1\gamma(k)}, & k \in (-k_0, k_0), \\ 1, & k \in (-\infty, -k_0) \cup (k_0, +\infty). \end{cases} \tag{3.15}$$

Hence, by the maximum principle, we have

$$|\delta(k)| \leqslant \text{const} < \infty, \quad |\det \delta(k)| \leqslant \text{const} < \infty, \tag{3.16}$$

for all $k \in \mathbb{C}$. We define the functions

$$\rho(k) = \begin{cases} -\gamma(k), & k \in (-\infty, -k_0) \cup (k_0, +\infty), \\ \dfrac{\gamma(k)}{1 - \gamma^\dagger(k^*)B_1\gamma(k)}, & k \in (-k_0, k_0), \end{cases} \tag{3.17}$$

$$\Delta(k) = \begin{pmatrix} \delta(k) & 0 \\ 0 & (\det \delta(k))^{-1} \end{pmatrix}. \tag{3.18}$$

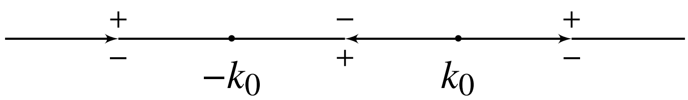

Figure 1. The reoriented contour on $\mathbb{R}$.

We reverse the orientation for $k \in (-\infty, k_0) \cup (k_0, +\infty)$ as in Figure 1, and $M^\Delta(k; x, t) = M(k; x, t)\Delta^{-1}(k)$ satisfies the Riemann-Hilbert problem on the reoriented contour

$$\begin{cases} M_+^\Delta(k; x, t) = M_-^\Delta(k; x, t)J^\Delta(k; x, t), & k \in \mathbb{R}, \\ M^\Delta(k; x, t) \to I, & k \to \infty, \end{cases} \tag{3.19}$$

where the jump matrix $J^\Delta(k; x, t)$ has a decomposition

$$J^\Delta(k; x, t) = (b_-)^{-1}b_+ = \begin{pmatrix} I & 0 \\ \dfrac{e^{2it\theta}(k)\rho^\dagger(k^*)B_1\delta_-^{-1}(k)}{\det \delta_-(k)} & 1 \end{pmatrix} \begin{pmatrix} I & -e^{-2it\theta}\delta_+(k)\rho(k)[\det \delta_+(k)] \\ 0 & 1 \end{pmatrix}. \tag{3.20}$$

3.2. *Extend to the augmented contour*

For the convenience of discussion, we define

$$L=\{k_0+\alpha k_0 e^{-\frac{3\pi i}{4}}:-\infty<\alpha\leqslant\sqrt{2}\}\cup\{-k_0+\alpha k_0 e^{-\frac{\pi i}{4}}:-\infty<\alpha\leqslant\sqrt{2}\},$$
$$L_\epsilon=\{k_0+\alpha k_0 e^{-\frac{3\pi i}{4}}:-\epsilon<\alpha\leqslant\sqrt{2}\}\cup\{-k_0+\alpha k_0 e^{-\frac{\pi i}{4}}:-\epsilon<\alpha\leqslant\sqrt{2}\}.$$

Theorem 3.1. *The vector function $\rho(k)$ has a decomposition*

$$\rho(k)=h_1(k)+h_2(k)+R(k),\quad k\in\mathbb{R},$$

where $R(k)$ is a piecewise-rational function and $h_2(k)$ has a analytic continuation to L. Besides, they admit the following estimates

$$|e^{-2it\theta(k)}h_1(k)|\lesssim\frac{1}{(1+|k|^2)t^l},\quad k\in\mathbb{R},\tag{3.21}$$
$$|e^{-2it\theta(k)}h_2(k)|\lesssim\frac{1}{(1+|k|^2)t^l},\quad k\in L,\tag{3.22}$$
$$|e^{-2it\theta(k)}R(k)|\lesssim e^{-16\epsilon^2k_0^3t},\quad k\in L_\epsilon,\tag{3.23}$$

for an arbitrary positive integer l. Considering the Schwartz conjugate

$$\rho^\dagger(k^*)=R^\dagger(k^*)+h_1^\dagger(k^*)+h_2^\dagger(k^*),$$

we can obtain the same estimate for $e^{2it\theta(k)}h_1^\dagger(k^*)$, $e^{2it\theta(k)}h_2^\dagger(k^*)$ and $e^{2it\theta(k)}R^\dagger(k^*)$ on $\mathbb{R}\cup L^*$.

Proof. It follows from Proposition 4.2 in [18]. □

A direct calculation shows that $b_\pm$ of (3.20) can be decomposed further

$$\begin{aligned}
b_+&=b_+^ob_+^a=(I_{3\times3}+\omega_+^o)(I_{3\times3}+\omega_+^a)\\
&=\begin{pmatrix} I_{2\times2} & -e^{-2it\theta}[\det\delta_+(k)]\delta_+(k)h_1(k)\\ 0 & 1\end{pmatrix}\begin{pmatrix} I_{2\times2} & -e^{-2it\theta}[\det\delta_+(k)]\delta_+(k)[h_2(k)+R(k)]\\ 0 & 1\end{pmatrix},\\
b_-&=b_-^ob_-^a=(I_{3\times3}-\omega_-^o)(I_{3\times3}-\omega_-^a)\\
&=\begin{pmatrix} I_{2\times2} & 0\\ -\dfrac{e^{2it\theta}h_1^\dagger(k^*)B_1\delta_-^{-1}(k)}{\det\delta_-(k)} & 1\end{pmatrix}\begin{pmatrix} I_{2\times2} & 0\\ -\dfrac{e^{2it\theta}[h_2^\dagger(k^*)+R^\dagger(k^*)]B_1\delta_-^{-1}(k)}{\det\delta_-(k)} & 1\end{pmatrix}.
\end{aligned}$$

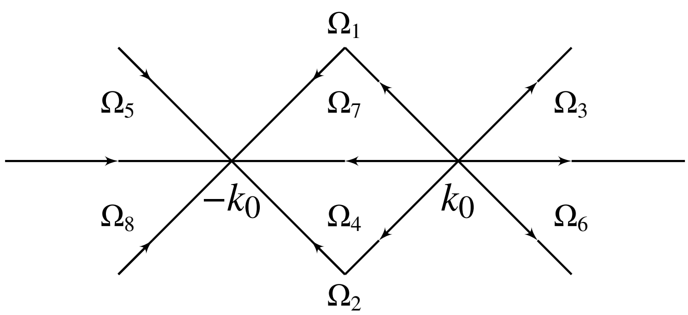

Figure 2. The contour Σ.

Define the oriented contour Σ by $\Sigma = L \cup L^*$ as in Figure 2. Let

$$M^\sharp(k;x,t) = \begin{cases} M^\Delta(k;x,t), & k \in \Omega_1 \cup \Omega_2, \\ M^\Delta(k;x,t)(b_+^a)^{-1}, & k \in \Omega_3 \cup \Omega_4 \cup \Omega_5, \\ M^\Delta(k;x,t)(b_-^a)^{-1}, & k \in \Omega_6 \cup \Omega_7 \cup \Omega_8. \end{cases} \tag{3.24}$$

Lemma 3.1. *$M^\sharp(k;x,t)$ is the solution of the Riemann-Hilbert problem*

$$\begin{cases} M_+^\sharp(k;x,t) = M_-^\sharp(k;x,t)J^\sharp(k;x,t), & k \in \Sigma, \\ M^\sharp(k;x,t) \to I, & k \to \infty, \end{cases} \tag{3.25}$$

where the jump matrix $J^\sharp(k;x,t)$ satisfies

$$J^\sharp(k;x,t) = (b_-^\sharp)^{-1} b_+^\sharp = \begin{cases} I^{-1} b_+^a, & k \in L, \\ (b_-^a)^{-1} I, & k \in L^*, \\ (b_-^o)^{-1} b_+^o, & k \in \mathbb{R}. \end{cases} \tag{3.26}$$

Proof. We can construct the Riemann-Hilbert problem (3.25) based on the Riemann-Hilbert problem (3.19) and the decomposition of $b_\pm$. In the meantime, the asymptotics of $M^\sharp(k;x,t)$ is derived from the convergence of $b_\pm$ as $k \to \infty$. For fixed x and t, we pay attention to the domain Ω_3. Noticing the boundedness of $\delta(k)$ and $\det\delta(k)$ in (3.16), we arrive at

$$|e^{-2it\theta}[\det\delta(k)][h_2(k) + R(k)]\delta(k)| \lesssim |e^{-2it\theta} h_2(k)| + |e^{-2it\theta} R(k)|.$$

Consider the definition of $R(k)$ in this domain,

$$|e^{-2it\theta} h_2(k)| \lesssim \frac{1}{|k+i|^2}, \quad |e^{-2it\theta} R(k)| \lesssim \frac{|\sum_{i=0}^m \mu_i (k-k_0)^i|}{|(k+i)^{m+5}|} \lesssim \frac{1}{|k+i|^5},$$

where m is a positive integer and μ_i is the coefficient of the Taylor series around k_0. Combining with the boundedness of $h_2(k)$ in Theorem 3.1, we obtain that $M^\sharp(k;x,t) \to I$ when $k \in \Omega_3$ and $k \to \infty$. The others are similar to this domain. □

The above Riemann-Hilbert problem (3.25) can be solved as follows. Set

$$\omega_\pm^\sharp = \pm(b_\pm^\sharp - I), \quad \omega^\sharp = \omega_+^\sharp + \omega_-^\sharp.$$

Let

$$(C_\pm f)(k) = \int_\Sigma \frac{f(\xi)}{\xi - k_\pm} \frac{d\xi}{2\pi i}, \quad f \in \mathscr{L}^2(\Sigma) \tag{3.27}$$

denote the Cauchy operator, where $C_+ f$ ($C_- f$) denotes the left (right) boundary value for the oriented contour Σ in Figure 2. Define the operator $C_{\omega^\sharp} : \mathscr{L}^2(\Sigma) + \mathscr{L}^\infty(\Sigma) \to \mathscr{L}^2(\Sigma)$ by

$$C_{\omega^\sharp} f = C_+\left(f\omega_-^\sharp\right) + C_-\left(f\omega_+^\sharp\right) \tag{3.28}$$

for the 3×3 matrix function f.

Lemma 3.2 (Beals-Coifman). *If $\mu^\sharp(k;x,t) \in \mathscr{L}^2(\Sigma) + \mathscr{L}^\infty(\Sigma)$ is the solution of the singular integral equation*

$$\mu^\sharp = I + C^\sharp_{\omega^\sharp}\mu^\sharp.$$

Then

$$M^\sharp(k;x,t) = I + \int_\Sigma \frac{\mu^\sharp(\xi;x,t)\omega^\sharp(\xi;x,t)}{\xi - k}\frac{\mathrm{d}\xi}{2\pi i}$$

is the solution of the Riemann-Hilbert problem (3.25).

Proof. See [18], P. 322 and [40]. □

Theorem 3.2. *The expression of the solution $q(x,t)$ can be written as*

$$q(x,t) = (u(x,t), u^*(x,t))^T = \frac{1}{\pi}\left(\int_\Sigma \left((1 - C_{\omega^\sharp})^{-1}I\right)(\xi)\omega^\sharp(\xi)\,\mathrm{d}\xi\right)_{12}. \tag{3.29}$$

Proof. From (2.21), (3.24) and Lemma 3.2, the solution $q(x,t)$ of the generalized Sasa-Satsuma equation is expressed by

$$\begin{aligned} q(x,t) &= \lim_{k\to\infty} -2i\left[k(M^\sharp(k;x,t))_{12}\right] \\ &= \frac{1}{\pi}\left(\int_\Sigma \mu^\sharp(\xi;x,t)\omega^\sharp(\xi)\,\mathrm{d}\xi\right)_{12} \\ &= \frac{1}{\pi}\left(\int_\Sigma ((1 - C_{\omega^\sharp})^{-1}I)(\xi)\omega^\sharp(\xi)\,\mathrm{d}\xi\right)_{12}. \end{aligned}$$

□

3.3. Contour truncation

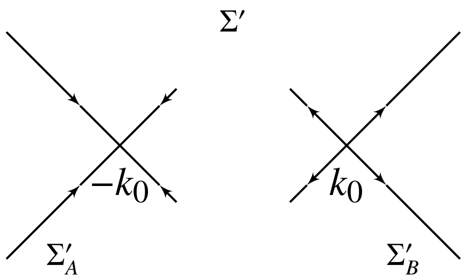

Figure 3. The contour Σ'

Set $\Sigma' = \Sigma\backslash(\mathbb{R} \cup L_\epsilon \cup L_\epsilon^*)$ oriented as in Figure 3. We will convert the Riemann-Hilbert problem on the contour Σ to a Riemann-Hilbert problem on the contour Σ' and estimate the errors between the two Riemann-Hilbert problems. Let $\omega^\sharp = \omega^e + \omega' = \omega^a + \omega^b + \omega^c + \omega'$, where $\omega^a = \omega^\sharp|_\mathbb{R}$ is supported on $\mathbb{R}$ and is composed of terms of type $h_1(k)$ and $h_1^\dagger(k^*)$; ω^b is supported on $L \cup L^*$ and is composed of contribution to $\omega^\sharp$ from terms of type $h_2(k)$ and $h_2^\dagger(k^*)$; ω^c is supported on $L_\epsilon \cup L_\epsilon^*$ and is composed of contribution to $\omega^\sharp$ from terms of type $R(k)$ and $R^\dagger(k^*)$.

Lemma 3.3. *For arbitrary positive integer l, as $t \to \infty$,*

$$\|\omega^a\|_{\mathscr{L}^1(\mathbb{R})\cap\mathscr{L}^2(\mathbb{R})\cap\mathscr{L}^\infty(\mathbb{R})} \lesssim t^{-l}, \tag{3.30}$$

$$\|\omega^b\|_{\mathscr{L}^1(L\cup L^*)\cap\mathscr{L}^2(L\cup L^*)\cap\mathscr{L}^\infty(L\cup L^*)} \lesssim t^{-l}, \tag{3.31}$$

$$\|\omega^c\|_{\mathscr{L}^1(L_\epsilon\cup L_\epsilon^*)\cap\mathscr{L}^2(L_\epsilon\cup L_\epsilon^*)\cap\mathscr{L}^\infty(L_\epsilon\cup L_\epsilon^*)} \lesssim e^{-16\epsilon^2 k_0^3 t}, \tag{3.32}$$

$$\|\omega'\|_{\mathscr{L}^2(\Sigma)} \lesssim (tk_0^3)^{-\frac{1}{4}}, \quad \|\omega'\|_{\mathscr{L}^1(\Sigma)} \lesssim (tk_0^3)^{-\frac{1}{2}} \tag{3.33}$$

Proof. The proof of estimates (3.30), (3.31), (3.32) follows from Theorem 3.1. Afterwards, we consider the definition of $R(k)$ on the contour $\{k = k_0 + \alpha k_0 e^{\frac{-3\pi i}{4}} | -\infty < \alpha < \epsilon\}$,

$$|R(k)| \lesssim (1 + |k|^5)^{-1}.$$

Resorting to $\mathrm{Re}(i\theta) \geqslant 8\alpha^2 k_0^3$ and the boundedness of $\delta(k)$ and $\det\delta(k)$ in (3.16), we can obtain

$$|e^{-2it\theta}[\det\delta(k)]R(k)\delta(k)| \lesssim e^{-16tk_0^3\alpha^2}(1 + |k|^5)^{-1}.$$

Then we obtain (3.33) by simple computations. □

Lemma 3.4. *As $t \to \infty$, $(1 - C_{\omega'})^{-1} : \mathscr{L}^2(\Sigma) \to \mathscr{L}^2(\Sigma)$ exists and is uniformly bounded:*

$$\|(1 - C_{\omega'})^{-1}\|_{\mathscr{L}^2(\Sigma)} \lesssim 1.$$

Furthermore, $\|(1 - C_{\omega^\sharp})^{-1}\|_{\mathscr{L}^2(\Sigma)} \lesssim 1$.

Proof. It follows from Proposition 2.23 and Corollary 2.25 in [18]. □

Lemma 3.5. *As $t \to \infty$,*

$$\int_\Sigma ((1 - C_{\omega^\sharp})^{-1}I)(\xi)\omega^\sharp(\xi)\,\mathrm{d}\xi = \int_\Sigma ((1 - C_{\omega'})^{-1}I)(\xi)\omega'(\xi)\,\mathrm{d}\xi + O((tk_0^3)^{-l}). \tag{3.34}$$

Proof. A simple computation shows that

$$\begin{aligned}((1 - C_{\omega^\sharp})^{-1}\,I)\omega^\sharp =& \left((1 - C_{\omega'})^{-1}I\right)\omega' + \omega^e + \left((1 - C_{\omega'})^{-1}(C_{\omega^e}I)\right)\omega^\sharp \\ &+ ((1 - C_{\omega'})^{-1}(C_{\omega'}I))\omega^e + \left((1 - C_{\omega'})^{-1}C_{\omega^e}(1 - C_{\omega^\sharp})\right)(C_{\omega^\sharp}I)\omega^\sharp.\end{aligned} \tag{3.35}$$

After a series of tedious computations and utilizing the consequence of Lemma 3.4, we arrive at

$$\begin{aligned}
&\|\omega^e\|_{\mathscr{L}^1(\Sigma)} \leqslant \|\omega^a\|_{\mathscr{L}^1(\mathbb{R})} + \|\omega^b\|_{\mathscr{L}^1(L\cup L^*)} + \|\omega^c\|_{\mathscr{L}^1(L_\epsilon\cup L_\epsilon^*)} \lesssim (tk_0^3)^{-l},\\
&\|\left((1 - C_{\omega'})^{-1}(C_{\omega^e}I)\right)\omega^\sharp\|_{\mathscr{L}^1(\Sigma)} \leqslant \|(1 - C_{\omega'})^{-1}\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^e}I\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\lesssim \|\omega^e\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)} \lesssim (tk_0^3)^{-l-\frac{1}{4}},\\
&\|\left((1 - C_{\omega'})^{-1}(C_{\omega'}I)\right)\omega^e\|_{\mathscr{L}^1(\Sigma)} \leqslant \|(1 - C_{\omega'}^{-1})\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega'}I\|_{\mathscr{L}^2(\Sigma)}\|\omega^e\|_{\mathscr{L}^2(\Sigma)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\lesssim \|\omega'\|_{\mathscr{L}^2(\Sigma)}\|\omega^e\|_{\mathscr{L}^2(\Sigma)} \lesssim (tk_0^3)^{-l-\frac{1}{4}},\\
&\|\left((1 - C_{\omega'})^{-1}C_{\omega^e}(1 - C_{\omega^\sharp})\right)(C_{\omega^\sharp}I)\omega^\sharp\|_{\mathscr{L}^1(\Sigma)}\\
&\leqslant \|(1 - C_{\omega'})^{-1}\|_{\mathscr{L}^2(\Sigma)}\|(1 - C_{\omega^\sharp})^{-1}\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^e}\|_{\mathscr{L}^2(\Sigma)}\|C_{\omega^\sharp}I\|_{\mathscr{L}^2(\Sigma)}\|\omega^\sharp\|_{\mathscr{L}^2(\Sigma)}\\
&\lesssim \|\omega^e\|_{\mathscr{L}^\infty(\Sigma)}\|\omega^\sharp\|^2_{\mathscr{L}^2(\Sigma)} \lesssim (tk_0^3)^{-l-\frac{1}{2}}.
\end{aligned}$$

Then the proof is accomplished as long as we substitute the estimates above into (3.35). □

Notice that $\omega'(k) = 0$ when $k \in \Sigma\backslash\Sigma'$, let $C_{\omega'}|_{\mathscr{L}^2(\Sigma')}$ denote the restriction of $C_{\omega'}$ to $\mathscr{L}^2(\Sigma')$. For simplicity, we write $C_{\omega'}|_{\mathscr{L}^2(\Sigma')}$ as $C_{\omega'}$. Then

$$\int_{\Sigma}((1-C_{\omega'})^{-1}I)(\xi)\omega'(\xi)\,\mathrm{d}\xi = \int_{\Sigma'}((1-C_{\omega'})^{-1}I)(\xi)\omega'(\xi)\,\mathrm{d}\xi.$$

Lemma 3.6. *As $t \to \infty$,*

$$q(x,t) = (u(x,t), u^*(x,t))^T = \frac{1}{\pi}\left(\int_{\Sigma'}((1-C_{\omega'})^{-1}I)(\xi)\omega'(\xi)\,\mathrm{d}\xi\right)_{12} + O((tk_0^3)^{-l}). \tag{3.36}$$

Proof. From (3.29) and (3.34), we can obtain the result directly. □

Let $L' = L\backslash L_\epsilon$ and $\mu' = (1-C_{\omega'})^{-1}I$. Then

$$M'(k;x,t) = I + \int_{\Sigma'}\frac{\mu'(k;x,t)\omega'(k;x,t)}{\xi - k}\frac{\mathrm{d}\xi}{2\pi i}$$

solves the Riemann-Hilbert problem

$$\begin{cases} M'_+(k;x,t) = M'_-(k;x,t)J'(k;x,t), & k \in \Sigma', \\ M'(k;x,t) \to I, & k \to \infty, \end{cases}$$

where

$$J' = (b'_-)^{-1}b'_+ = (I-\omega'_-)^{-1}(I+\omega'_+),$$
$$\omega' = \omega'_+ + \omega'_-,$$
$$b'_+ = \begin{pmatrix} I & -e^{-2it\theta}[\det\delta(k)]\delta(k)R(k) \\ 0 & 1 \end{pmatrix}, \quad b'_- = I, \quad \text{on } L',$$
$$b'_+ = I, \quad b'_- = \begin{pmatrix} I & 0 \\ -\dfrac{e^{2it\theta}R^\dagger(k^*)B_1\delta^{-1}(k)}{\det\delta(k)} & 1 \end{pmatrix}, \quad \text{on } (L')^*.$$

3.4. Noninteraction of disconnected contour components

Let the contour $\Sigma' = \Sigma'_A \cup \Sigma'_B$ and $\omega'_\pm = \omega'_{A\pm} + \omega'_{B\pm}$, where

$$\omega'_{A\pm}(k) = \begin{cases} \omega'_\pm(k), & k \in \Sigma'_A, \\ 0, & k \in \Sigma'_B, \end{cases} \quad \omega'_{B\pm}(k) = \begin{cases} 0, & k \in \Sigma'_A, \\ \omega'_\pm(k), & k \in \Sigma'_B. \end{cases} \tag{3.37}$$

Define the operators $C_{\omega'_A}$ and $C_{\omega'_B}$: $\mathscr{L}^2(\Sigma') + \mathscr{L}^\infty(\Sigma') \to \mathscr{L}^2(\Sigma')$ as in definition (3.28).

Lemma 3.7.

$$\|C_{\omega'_B}C_{\omega'_A}\|_{\mathscr{L}^2(\Sigma')} = \|C_{\omega'_A}C_{\omega'_B}\|_{\mathscr{L}^2(\Sigma')} \lesssim_{k_0} (tk_0^3)^{-\frac{1}{2}},$$
$$\|C_{\omega'_B}C_{\omega'_A}\|_{\mathscr{L}^\infty(\Sigma')\to\mathscr{L}^2(\Sigma')}, \|C_{\omega'_A}C_{\omega'_B}\|_{\mathscr{L}^\infty(\Sigma')\to\mathscr{L}^2(\Sigma')} \lesssim_{k_0} (tk_0^3)^{-\frac{3}{4}}.$$

Proof. See Lemma 3.5 in [18]. □

Lemma 3.8. *As $t \to \infty$,*

$$\begin{aligned}\int_{\Sigma'}((1-C_{\omega'})^{-1}I)(\xi)\omega'(\xi)\,\mathrm{d}\xi =& \int_{\Sigma'_A}((1-C_{\omega'_A})^{-1}I)(\xi)\omega'_A(\xi)\,\mathrm{d}\xi \\ &+ \int_{\Sigma'_B}((1-C_{\omega'_B})^{-1}I)(\xi)\omega'_B(\xi)\,\mathrm{d}\xi + O(\frac{c(k_0)}{t}).\end{aligned} \tag{3.38}$$

Proof. From identity

$$\begin{aligned}&(1-C_{\omega'_A}-C_{\omega'_B})(1+C_{\omega'_A}(1-C_{\omega'_A})^{-1}+C_{\omega'_B}(1-C_{\omega'_B})^{-1}) \\ &\quad = 1-C_{\omega'_B}C_{\omega'_A}(1-C_{\omega'_A})^{-1}-C_{\omega'_A}C_{\omega'_B}(1-C_{\omega'_B})^{-1},\end{aligned}$$

we have

$$\begin{aligned}(1-C_{\omega'})^{-1} =& 1+C_{\omega'_A}(1-C_{\omega'_A})^{-1}+C_{\omega'_B}(1-C_{\omega'_B})^{-1} \\ &+ [1+C_{\omega'_A}(1-C_{\omega'_A})^{-1}+C_{\omega'_B}(1-C_{\omega'_B})^{-1}][1-C_{\omega'_B}C_{\omega'_A}(1-C_{\omega'_A})^{-1} \\ &- C_{\omega'_A}C_{\omega'_B}(1-C_{\omega'_B})^{-1}]^{-1}[C_{\omega'_B}C_{\omega'_A}(1-C_{\omega'_A})^{-1}+C_{\omega'_A}C_{\omega'_B}(1-C_{\omega'_B})^{-1}].\end{aligned}$$

Based on Lemma (3.7) and Lemma (3.4) , we arrive at (3.38). □

For the sake of convenience, we write the restriction $C_{\omega'_A}\,|_{\mathscr{L}^2(\Sigma'_A)}$ as $C_{\omega'_A}$, similar for $C_{\omega'_B}$. From the consequences of Lemma 3.6 and Lemma 3.8, as $t \to \infty$, we have

$$\begin{aligned}q(x,t) =& -\left(\int_{\Sigma'_A}((1-C_{\omega'_A})^{-1}I)(\xi)\omega'_A(\xi)\,\frac{\mathrm{d}\xi}{\pi}\right)_{12} \\ &- \left(\int_{\Sigma'_B}((1-C_{\omega'_B})^{-1}I)(\xi)\omega'_B(\xi)\,\frac{\mathrm{d}\xi}{\pi}\right)_{12} + O(\frac{c(k_0)}{t}).\end{aligned} \tag{3.39}$$

3.5. Rescaling and further reduction of the Riemann-Hilbert problems

Extend the contours Σ'_A and Σ'_B to the contours

$$\hat{\Sigma}'_A = \left\{k = -k_0 + k_0\alpha e^{\pm\frac{\pi i}{4}} : \alpha \in \mathbb{R}\right\}, \tag{3.40}$$

$$\hat{\Sigma}'_B = \left\{k = k_0 + k_0\alpha e^{\pm\frac{3\pi i}{4}} : \alpha \in \mathbb{R}\right\}, \tag{3.41}$$

respectively. We introduce $\hat{\omega}'_A$ and $\hat{\omega}'_B$ on $\hat{\Sigma}'_A$ and $\hat{\Sigma}'_B$, respectively, by

$$\hat{\omega}'_{A\pm} = \begin{cases}\omega'_{A\pm}(k), & k \in \Sigma'_A, \\ 0, & k \in \hat{\Sigma}'_A \backslash \Sigma'_A,\end{cases} \quad \hat{\omega}'_{B\pm} = \begin{cases}\omega'_{B\pm}(k), & k \in \Sigma'_B, \\ 0, & k \in \hat{\Sigma}'_B \backslash \Sigma'_B.\end{cases} \tag{3.42}$$

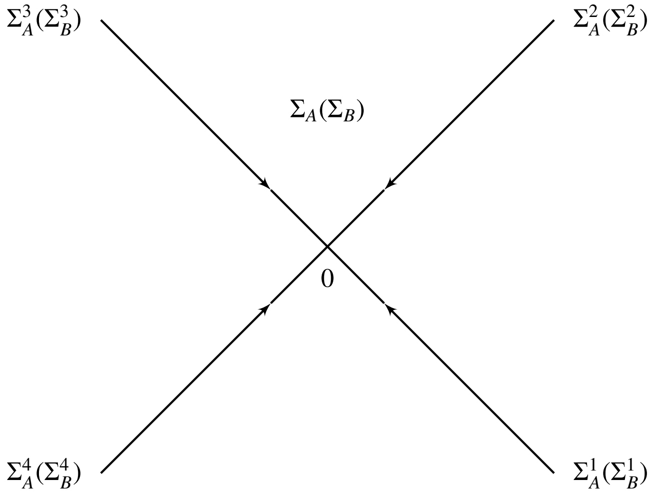

Figure 4. The contour $\Sigma_A(\Sigma_B)$.

Let Σ_A and Σ_B denote the contours $\{k = k_0\alpha e^{\pm\frac{\pi i}{4}} : \alpha \in \mathbb{R}\}$ oriented inward as in Σ'_A, $\hat{\Sigma}'_A$, and outward as in Σ'_B, $\hat{\Sigma}'_B$, respectively. Define the scaling operators

$$\begin{aligned} &N_A : \mathscr{L}^2(\hat{\Sigma}'_A) \to \mathscr{L}^2(\Sigma_A), \\ &\qquad f(k) \to (N_A f)(k) = f(\frac{k}{\sqrt{48tk_0}} - k_0), \end{aligned} \tag{3.43}$$

$$\begin{aligned} &N_B : \mathscr{L}^2(\hat{\Sigma}'_B) \to \mathscr{L}^2(\Sigma_B), \\ &\qquad f(k) \to (N_B f)(k) = f(\frac{k}{\sqrt{48tk_0}} + k_0), \end{aligned} \tag{3.44}$$

and set

$$\omega_A = N_A\hat{\omega}'_A, \quad \omega_B = N_B\hat{\omega}'_B.$$

A simple change-of-variable arguments shows that

$$C_{\hat{\omega}'_A} = N_A^{-1}C_{\omega_A}N_A, \quad C_{\hat{\omega}'_B} = N_B^{-1}C_{\omega_B}N_B,$$

where the operator C_{ω_A} (C_{ω_B}) is a bounded map from $\mathscr{L}^2(\Sigma_A)$ ($\mathscr{L}^2(\Sigma_B)$) into $\mathscr{L}^2(\Sigma_A)$ ($\mathscr{L}^2(\Sigma_B)$). On the part

$$L_A = \left\{k = \alpha k_0\sqrt{48tk_0}e^{\frac{3\pi i}{4}} : -\epsilon < \alpha < +\infty\right\}$$

of Σ_A, we have

$$\omega_A = \omega_{A+} = \begin{pmatrix} 0 & (N_A s_1)(k) \\ 0 & 0 \end{pmatrix},$$

on L_A^* we have

$$\omega_A = \omega_{A-} = \begin{pmatrix} 0 & 0 \\ (N_A s_2)(k) & 0 \end{pmatrix},$$

where

$$s_1(k) = -e^{-2it\theta(k)}[\det\delta(k)]\delta(k)R(k), \quad s_2(k) = \frac{e^{2it\theta}R^\dagger(k^*)B_1\delta^{-1}(k)}{\det\delta(k)}.$$

Lemma 3.9. *As $t \to \infty$, and $k \in L_A$, then*

$$\left|(N_A\tilde{\delta})(k)\right| \lesssim t^{-l}, \tag{3.45}$$

where $\tilde{\delta}(k) = e^{-2it\theta(k)}[\delta(k)R(k) - (\det\delta(k))R(k)]$.

Proof. It follows from (3.1) and (3.2) that $\tilde{\delta}$ satisfies the following Riemann-Hilbert problem:

$$\begin{cases} \tilde{\delta}_+(k) = \tilde{\delta}_-(k)(1 - \gamma^\dagger(k^*)B_1\gamma(k)) + e^{-2it\theta}f(k), & k \in (-k_0, k_0), \\ \tilde{\delta}(k) \to 0, & k \to \infty. \end{cases} \tag{3.46}$$

where $f(k) = \delta_-(k)[\gamma^\dagger(k^*)B_1\gamma(k)I - \gamma(k)\gamma^\dagger(k^*)B_1]R(k)$. The solution for the above Riemann-Hilbert problem can be expressed by

$$\tilde{\delta}(k) = X(k)\int_{k_0}^{-k_0} \frac{e^{-2it\theta(\xi)}f(\xi)}{X_+(\xi)(\xi - k)}\frac{\mathrm{d}\xi}{2\pi i},$$
$$X(k) = \exp\left\{\frac{1}{2\pi i}\int_{k_0}^{-k_0}\frac{\log(1 - |\gamma(\xi)|^2)}{\xi - k}\mathrm{d}\xi\right\}.$$

Observing that

$$\begin{aligned}(\gamma^\dagger(k^*)B_1\gamma(k)I - \gamma(k)\gamma^\dagger(k^*)B_1)R(k) &= (\gamma^\dagger(k^*)B_1\gamma(k)I - \gamma(k)\gamma^\dagger(k^*)B_1)(R(k) - \rho(k)) \\ &= \mathrm{adj}[B_1]\mathrm{adj}[\gamma(k)\gamma^\dagger(k^*)](h_1(k) + h_2(k)),\end{aligned}$$

we obtain $f(k) = O((k^2 - k_0^2)^l)$. Similar to the Lemma 3.1, $f(k)$ can be decomposed into two parts: $f(k) = f_1(k) + f_2(k)$, and

$$\left|e^{-2it\theta(k)}f_1(k)\right| \lesssim \frac{1}{(1 + |k|^2)t^l}, \quad k \in \mathbb{R}, \tag{3.47}$$

$$\left|e^{-2it\theta(k)}f_2(k)\right| \lesssim \frac{1}{(1 + |k|^2)t^l}, \quad k \in L_t, \tag{3.48}$$

where $f_2(k)$ has an analytic continuation to L_t, l is a positive integer and $l \geqslant 2$,

$$\begin{aligned} L_t = &\left\{k = k_0 + k_0\alpha e^{-\frac{3\pi i}{4}} : 0 \leqslant \alpha \leqslant \sqrt{2}(1 - \frac{1}{2t})\right\} \\ &\cup \left\{k = \frac{k_0}{t} - k_0 + k_0\alpha e^{\frac{-\pi i}{4}} : 0 \leqslant \alpha \leqslant \sqrt{2}(1 - \frac{1}{2t})\right\},\end{aligned}$$

(see Figure 5).

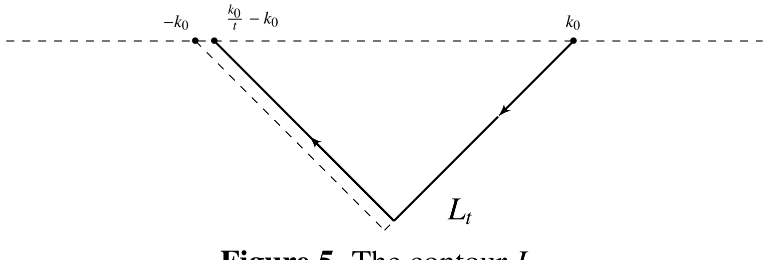

Figure 5. The contour L_t.

As $k \in L_A$, we obtain

$$\begin{aligned}(N_A\tilde{\delta})(k) =&X(\frac{k}{\sqrt{48tk_0}}-k_0)\int_{\frac{k_0}{t}-k_0}^{-k_0}\frac{e^{-2it\theta(\xi)}f(\xi)}{X_+(\xi)(\xi+k_0-\frac{k}{\sqrt{48tk_0}})}\frac{\mathrm{d}\xi}{2\pi i}\\&+X(\frac{k}{\sqrt{48tk_0}}-k_0)\int_{k_0}^{\frac{k_0}{t}-k_0}\frac{e^{-2it\theta(\xi)}f_1(\xi)}{X_+(\xi)(\xi+k_0-\frac{k}{\sqrt{48tk_0}})}\frac{\mathrm{d}\xi}{2\pi i}\\&+X(\frac{k}{\sqrt{48tk_0}}-k_0)\int_{k_0}^{\frac{k_0}{t}-k_0}\frac{e^{-2it\theta(\xi)}f_2(\xi)}{X_+(\xi)(\xi+k_0-\frac{k}{\sqrt{48tk_0}})}\frac{\mathrm{d}\xi}{2\pi i}\\=&I_1+I_2+I_3.\end{aligned}$$

$$|I_1|\lesssim\int_{-k_0}^{\frac{k_0}{t}-k_0}\frac{|f(\xi)|}{|\xi+k_0-\frac{k}{\sqrt{48tk_0}}|}\,\mathrm{d}\xi\lesssim t^{-l-1},$$

$$|I_2|\lesssim\int_{\frac{k_0}{t}-k_0}^{k_0}\frac{\left|e^{-2it\theta(\xi)}f_1(\xi)\right|}{|\xi+k_0-\frac{k}{\sqrt{48tk_0}}|}\,\mathrm{d}\xi\leqslant t^{-l}\frac{\sqrt{2}t}{k_0}(2k_0-\frac{k_0}{t})\lesssim t^{-l+1}.$$

As a consequence of Cauchy's Theorem, we can evaluate I_3 along the contour L_t instead of the interval $(\frac{k_0}{t}-k_0,k_0)$ and obtain $|I_3|\lesssim t^{-l+1}$. Therefore, (3.45) holds. □

Corollary 3.1. *As $t\to\infty$, and $k\in L_A^*$, then*

$$\left|(N_A\hat{\delta})(k)\right|\lesssim t^{-l},\quad t\to\infty,\quad k\in L_A^*,\tag{3.49}$$

where $\hat{\delta}(k)=e^{2it\theta(k)}R^\dagger(k^*)B_1[\delta^{-1}(k)-(\det\delta(k))^{-1}I]$.

Let $J^{A^0}=(I-\omega_{A^0-})^{-1}(I+\omega_{A^0+})$, where

$$\omega_{A^0}=\omega_{A^0+}=\begin{cases}\begin{pmatrix}0&-(\delta_A^0)^2(-k)^{2i\nu}e^{-\frac{ik^2}{2}}\frac{\gamma(-k_0)}{1-\gamma^\dagger(-k_0)B_1\gamma(-k_0)}\\0&0\end{pmatrix},&k\in\Sigma_A^1,\\\begin{pmatrix}0&(\delta_A^0)^2(-k)^{2i\nu}e^{-\frac{ik^2}{2}}\gamma(-k_0)\\0&0\end{pmatrix},&k\in\Sigma_A^3,\end{cases}\tag{3.50}$$

$$\delta_A^0=(196tk_0^3)^{-\frac{i\nu}{2}}e^{8itk_0^3}e^{\chi(-k_0)}\tag{3.51}$$

$$\omega_{A^0}=\omega_{A^0-}=\begin{cases}\begin{pmatrix}0&0\\(\delta_A^0)^{-2}(-k)^{-2i\nu}e^{\frac{ik^2}{2}}\frac{\gamma^\dagger(-k_0)B_1}{1-\gamma^\dagger(-k_0)B_1\gamma(-k_0)}&0\end{pmatrix},&k\in\Sigma_A^2,\\\begin{pmatrix}0&0\\-(\delta_A^0)^{-2}(-k)^{-2i\nu}e^{\frac{ik^2}{2}}\gamma^\dagger(-k_0)B_1&0\end{pmatrix},&k\in\Sigma_A^4.\end{cases}\tag{3.52}$$

It follows from (3.78) in [18] that

$$\|\omega_A-\omega_{A^0}\|_{\mathscr{L}^1(\Sigma_A)\cap\mathscr{L}^2(\Sigma_A)\cap\mathscr{L}^\infty(\Sigma_A)}\lesssim_{k_0}\frac{\log t}{\sqrt{tk_0^3}}.\tag{3.53}$$

There are similar consequences for $k \in \Sigma_B$. Let $J^{B^0} = (I - \omega_{B^0-})^{-1}(I + \omega_{B^0+})$, where

$$\omega_{B^0} = \omega_{B^0+} = \begin{cases} \begin{pmatrix} 0 & (\delta_B^0)^2 k^{-2i\nu} e^{\frac{ik^2}{2}} \gamma(k_0) \\ 0 & 0 \end{pmatrix}, & k \in \Sigma_B^2, \\ \begin{pmatrix} 0 & -(\delta_B^0)^2 k^{-2i\nu} e^{\frac{ik^2}{2}} \frac{\gamma(k_0)}{1-\gamma^\dagger(k_0) B_1 \gamma(k_0)} \\ 0 & 0 \end{pmatrix}, & k \in \Sigma_B^4, \end{cases} \tag{3.54}$$

$$\delta_B^0 = (196tk_0^3)^{\frac{i\nu}{2}} e^{-8itk_0^3} e^{\chi(k_0)} \tag{3.55}$$

$$\omega_{B^0} = \omega_{B^0-} = \begin{cases} \begin{pmatrix} 0 & 0 \\ -(\delta_B^0)^{-2} k^{2i\nu} e^{-\frac{ik^2}{2}} \gamma^\dagger(k_0) B_1 & 0 \end{pmatrix}, & k \in \Sigma_B^1, \\ \begin{pmatrix} 0 & 0 \\ (\delta_B^0)^{-2} k^{2i\nu} e^{-\frac{ik^2}{2}} \frac{\gamma^\dagger(k_0) B_1}{1-\gamma^\dagger(k_0) B_1 \gamma(k_0)} & 0 \end{pmatrix}, & k \in \Sigma_B^3. \end{cases} \tag{3.56}$$

Theorem 3.3. *As $t \to \infty$,*

$$\begin{aligned} q(x,t) =& (u(x,t), u^*(x,t))^T \\ =& \frac{1}{\pi\sqrt{48tk_0}} \left(\int_{\Sigma_A} \left((1 - C_{\omega_{A^0}})^{-1} I \right)(\xi) \omega_{A^0}(\xi)\, \mathrm{d}\xi \right)_{12} \\ &+ \frac{1}{\pi\sqrt{48tk_0}} \left(\int_{\Sigma_B} \left((1 - C_{\omega_{B^0}})^{-1} I \right)(\xi) \omega_{B^0}(\xi)\, \mathrm{d}\xi \right)_{12} + O\left(\frac{c(k_0)\log t}{t} \right). \end{aligned} \tag{3.57}$$

Proof. Notice that

$$\begin{aligned} \left((1 - C_{\omega_A})^{-1} I \right) \omega_A =& \left((1 - C_{\omega_{A^0}})^{-1} I \right) \omega_{A^0} + \left((1 - C_{\omega_A})^{-1} I \right) (\omega_A - \omega_{A^0}) \\ &+ (1 - C_{\omega_A})^{-1} (C_{\omega_A} - C_{\omega_{A^0}})(1 - C_{\omega_{A^0}}) I \omega_{A^0}. \end{aligned}$$

Utilizing the triangle inequality and the boundedness in (3.53), we have

$$\int_{\Sigma_A} \left((1 - C_{\omega_A})^{-1} I \right)(\xi) \omega_A(\xi)\, \mathrm{d}\xi = \int_{\Sigma_A} \left((1 - C_{\omega_{A^0}})^{-1} I \right)(\xi) \omega_{A^0}(\xi)\, \mathrm{d}\xi + O\left(\frac{\log t}{\sqrt{t}} \right).$$

According to (3.5) and a simple change-of-variable argument, we have

$$\begin{aligned} & \frac{1}{\pi} \left(\int_{\Sigma'} \left((1 - C_{\omega'_A})^{-1} I \right)(\xi) \omega'_A(\xi)\, \mathrm{d}\xi \right)_{12} \\ =& \frac{1}{\pi} \left(\int_{\hat{\Sigma}'_A} \left(N_A^{-1} (1 - C_{\omega_A})^{-1} I \right)(\xi) \omega'_A(\xi)\, \mathrm{d}\xi \right)_{12} \\ =& \frac{1}{\pi} \left(\int_{\hat{\Sigma}'_A} \left((1 - C_{\omega_A})^{-1} I \right) \left((\xi + k_0)\sqrt{48tk_0} \right) (N_A \omega'_A) \left((\xi + k_0)\sqrt{48tk_0} \right) \mathrm{d}\xi \right)_{12} \\ =& \frac{1}{\pi\sqrt{48tk_0}} \left(\int_{\Sigma_A} \left((1 - C_{\omega_A})^{-1} I \right)(\xi) \omega_A(\xi)\, \mathrm{d}\xi \right)_{12} \\ =& \frac{1}{\pi\sqrt{48tk_0}} \left(\int_{\Sigma_A} \left((1 - C_{\omega_{A^0}})^{-1} I \right)(\xi) \omega_{A^0}(\xi)\, \mathrm{d}\xi \right)_{12} + O\left(\frac{c(k_0)\log t}{t} \right). \end{aligned}$$

There are similar computations for the other case. Together with (3.39), one can obtain (3.57). □

For $k \in \mathbb{C}\backslash\Sigma_A$, set

$$M^{A^0}(k;x,t) = I + \int_{\Sigma_A} \frac{\left((1 - C_{\omega_{A^0}})^{-1} I\right)(\xi)\omega_{A^0}(\xi)}{\xi - k} \frac{\mathrm{d}\xi}{2\pi i}. \tag{3.58}$$

Then $M^{A^0}(k;x,t)$ is the solution of the Riemann-Hilbert problem

$$\begin{cases} M_+^{A^0}(k;x,t) = M_-^{A^0}(k;x,t)J^{A^0}(k;x,t), & k \in \Sigma_A, \\ M^{A^0}(k;x,t) \to I, & k \to \infty. \end{cases} \tag{3.59}$$

In particular

$$M^{A^0}(k) = I + \frac{M_1^{A^0}}{k} + O(k^{-2}), \quad k \to \infty, \tag{3.60}$$

then

$$M_1^{A^0} = -\int_{\Sigma_A} \left((1 - C_{\omega_{A^0}})^{-1} I\right)(\xi)\omega_{A^0}(\xi) \frac{\mathrm{d}\xi}{2\pi i}. \tag{3.61}$$

There is a analogous Riemann-Hilbert problem on Σ_B,

$$\begin{cases} M_+^{B^0}(k;x,t) = M_-^{B^0}(k;x,t)J^{B^0}(k;x,t), & k \in \Sigma_B, \\ M^{B^0}(k;x,t) \to I, & k \to \infty, \end{cases} \tag{3.62}$$

where $J^{B^0}(k;x,t)$ is defined in (3.54) and (3.56). In the meantime, we have

$$M^{B^0}(k) = I + \frac{M_1^{B^0}}{k} + O(k^{-2}), \quad k \to \infty. \tag{3.63}$$

Next, we consider the relation between $M_1^{A^0}$ and $M_1^{B^0}$. From the expression (3.50), (3.52), (3.54) and (3.56), we have the symmetry relation

$$J^{A^0}(k) = \tau(J^{B^0}(-k^*))^*\tau.$$

By the uniqueness of the Riemann-Hilbert problem,

$$M^{A^0}(k) = \tau(M^{B^0}(-k^*))^*\tau.$$

Combining with the expansion (3.60) and (3.63), one can verify that

$$M_1^{A^0} = -\tau(M_1^{B^0})^*\tau, \quad (M_1^{A^0})_{12} = -\sigma_1(M_1^{B^0})^*_{12}.$$

Therefore, from (3.57) and (3.61), we have

$$\begin{aligned} q(x,t) =& (u(x,t), u^*(x,t))^T \\ =& \frac{-2i}{\sqrt{48tk_0}}\left(M_1^{A^0} + M_1^{B^0}\right)_{12} + O\left(\frac{c(k_0)\log t}{t}\right) \\ =& -\frac{i}{\sqrt{12tk_0}}\left((M_1^{A^0})_{12} - \sigma_1(M_1^{A^0})^*_{12}\right) + O\left(\frac{c(k_0)\log t}{t}\right). \end{aligned} \tag{3.64}$$

3.6. Solving the model problem

In this subsection, we compute $(M_1^{A^0})_{12}$ explicitly. It is important to set

$$\Psi(k) = H(k)(-k)^{i\nu\sigma}e^{-\frac{1}{4}ik^2\sigma}, \quad H(k) = (\delta_A^0)^{-\sigma}M^{A^0}(k)(\delta_A^0)^{\sigma}. \tag{3.65}$$

Then it follows from (3.59) that

$$\Psi_+(k) = \Psi_-(k)v(-k_0), \quad v = e^{\frac{1}{4}ik^2\sigma}(-k)^{-i\nu\sigma}(\delta_A^0)^{-\sigma}J^{A^0}(k)(\delta_A^0)^{\sigma}(-k)^{i\nu\sigma}e^{-\frac{1}{4}ik^2\sigma}. \tag{3.66}$$

The jump matrix is the constant one on the four rays Σ_A^1, Σ_A^2, Σ_A^3, Σ_A^4, so

$$\frac{\mathrm{d}\Psi_+(k)}{\mathrm{d}k} = \frac{\mathrm{d}\Psi_-(k)}{\mathrm{d}k}v(-k_0). \tag{3.67}$$

Then it follows that $(\mathrm{d}\Psi/\mathrm{d}k + ik\sigma\Psi)\Psi^{-1}$ has no jump discontinuity along any of the four rays. Besides, from the relation between $\Psi(k)$ and $H(k)$, we have

$$\begin{aligned}\frac{\mathrm{d}\Psi(k)}{\mathrm{d}k}\Psi^{-1}(k) =& \frac{\mathrm{d}H(k)}{\mathrm{d}k}H^{-1}(k) - \frac{ik}{2}H(k)\sigma H^{-1}(k) + \frac{i\nu}{k}H(k)\sigma H^{-1}(k)\\ =& O(k^{-1}) - \frac{ik\sigma}{2} + \frac{i}{2}(\delta_A^0)^{\sigma}[\sigma, M_1^{A^0}](\delta_A^0)^{-\sigma}.\end{aligned}$$

It follows by the Liouville's Theorem that

$$\frac{\mathrm{d}\Psi(k)}{\mathrm{d}k} + \frac{ik}{2}\sigma\Psi(k) = \beta\Psi(k), \tag{3.68}$$

where

$$\beta = \frac{i}{2}(\delta_A^0)^{\sigma}[\sigma, M_1^{A^0}](\delta_A^0)^{-\sigma} = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix}.$$

Moreover,

$$(M_1^{A^0})_{12} = -i(\delta_A^0)^{-2}\beta_{12}. \tag{3.69}$$

Set

$$\Psi(k) = \begin{pmatrix} \Psi_{11}(k) & \Psi_{12}(k) \\ \Psi_{21}(k) & \Psi_{22}(k) \end{pmatrix}.$$

From (3.68) and its differential, we obtain

$$\begin{aligned}&\frac{\mathrm{d}^2\beta_{21}\Psi_{11}(k)}{\mathrm{d}k^2} + \left(\frac{i}{2} + \frac{k^2}{4} - \beta_{21}\beta_{12}\right)\beta_{21}\Psi_{11}(k) = 0,\\ &\Psi_{21}(k) = \frac{1}{\beta_{21}\beta_{12}}\left(\frac{\mathrm{d}\beta_{21}\Psi_{11}(k)}{\mathrm{d}k} + \frac{ik}{2}\beta_{21}\Psi_{11}(k)\right),\\ &\frac{\mathrm{d}^2\Psi_{22}(k)}{\mathrm{d}k^2} + \left(-\frac{i}{2} + \frac{k^2}{4} - \beta_{21}\beta_{12}\right)\Psi_{22}(k) = 0,\\ &\beta_{21}\Psi_{12}(k) = \left(\frac{\mathrm{d}\Psi_{22}(k)}{\mathrm{d}k} - \frac{ik}{2}\Psi_{22}(k)\right).\end{aligned}$$

As is well known, the Weber's equation

$$\frac{\mathrm{d}^2 g(\zeta)}{\mathrm{d}\zeta^2} + \left(\varrho + \frac{1}{2} - \frac{\zeta^2}{4}\right) g(\zeta) = 0$$

has the solution

$$g(\zeta) = c_1 D_\varrho(\zeta) + c_2 D_\varrho(-\zeta),$$

where $D_\varrho(\cdot)$ denotes the standard parabolic-cylinder function, and c_1, c_2 are constants. The parabolic-cylinder function satisfies [41]

$$\frac{\mathrm{d}D_\varrho(\zeta)}{\mathrm{d}\zeta} + \frac{\zeta}{2} D_\varrho(\zeta) - \varrho D_{\varrho-1}(\zeta) = 0, \tag{3.70}$$

$$D_\varrho(\pm\zeta) = \frac{\Gamma(\varrho+1)e^{\frac{i\pi\varrho}{2}}}{\sqrt{2\pi}} D_{-\varrho-1}(\pm i\zeta) + \frac{\Gamma(\varrho+1)e^{-\frac{i\pi\varrho}{2}}}{\sqrt{2\pi}} D_{-\varrho-1}(\mp i\zeta). \tag{3.71}$$

As $\zeta \to \infty$, from [42], we have

$$D_\varrho(\zeta) = \begin{cases} \zeta^\varrho e^{-\frac{\zeta^2}{4}}(1 + O(\zeta^{-2})), & \left|\arg\zeta\right| < \frac{3\pi}{4}, \\ \zeta^\varrho e^{-\frac{\zeta^2}{4}}(1 + O(\zeta^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-\varrho)} e^{\varrho\pi i + \frac{\zeta^2}{4}} \zeta^{-\varrho-1}(1 + O(\zeta^{-2})), & \frac{\pi}{4} < \arg\zeta < \frac{5\pi}{4}, \\ \zeta^\varrho e^{-\frac{\zeta^2}{4}}(1 + O(\zeta^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-\varrho)} e^{-\varrho\pi i + \frac{\zeta^2}{4}} \zeta^{-\varrho-1}(1 + O(\zeta^{-2})), & -\frac{5\pi}{4} < \arg\zeta < -\frac{\pi}{4}, \end{cases} \tag{3.72}$$

where $\Gamma(\cdot)$ is the Gamma function. Set $\varrho = i\beta_{21}\beta_{12}$,

$$\beta_{21}\Psi_{11}(k) = c_1 D_\varrho\left(e^{\frac{\pi i}{4}} k\right) + c_2 D_\varrho\left(e^{\frac{-3\pi i}{4}} k\right), \tag{3.73}$$

$$\Psi_{22}(k) = c_3 D_{-\varrho}\left(e^{\frac{-\pi i}{4}} k\right) + c_4 D_{-\varrho}\left(e^{\frac{3\pi i}{4}} k\right), \tag{3.74}$$

where a_1, a_2, a_3, a_4 are constants. As $\arg k \in (-\pi, -\frac{3\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$ and $k \to \infty$, we arrive at

$$\Psi_{11}(k)(-k)^{-i\nu} e^{\frac{ik^2}{4}} \to I, \quad \Psi_{22}(k)(-k)^{i\nu} e^{-\frac{ik^2}{4}} \to 1,$$

then

$$\beta_{21}\Psi_{11}(k) = \beta_{21} e^{\frac{\pi\nu}{4}} D_\varrho\left(e^{-\frac{3\pi i}{4}} k\right), \quad \nu = \beta_{21}\beta_{12},$$
$$\Psi_{22}(k) = e^{\frac{\pi\nu}{4}} D_{-\varrho}\left(e^{\frac{3\pi i}{4}} k\right).$$

Consequently,

$$\Psi_{21}(k) = \beta_{21} e^{\frac{\pi\nu}{4}} e^{-\frac{\pi i}{4}} D_{\varrho-1}\left(e^{-\frac{3\pi i}{4}} k\right),$$
$$\beta_{21}\Psi_{12}(k) = \varrho e^{\frac{\pi\nu}{4}} e^{-\frac{\pi i}{4}} D_{-\varrho-1}\left(e^{\frac{3\pi i}{4}} k\right).$$

For $\arg k \in (-\frac{3\pi}{4}, -\frac{\pi}{4})$ and $k \to \infty$, we have

$$\Psi_{11}(k)(-k)^{-i\nu} e^{\frac{ik^2}{4}} \to I, \quad \Psi_{22}(k)(-k)^{i\nu} e^{-\frac{ik^2}{4}} \to 1,$$

then

$$\beta_{21}\Psi_{11}(k) = \beta_{21}e^{-\frac{3\pi\nu}{4}}D_{\varrho}\left(e^{\frac{\pi i}{4}}k\right),$$
$$\Psi_{22}(k) = e^{\frac{\pi\nu}{4}}D_{-\varrho}\left(e^{\frac{3\pi i}{4}}k\right).$$

Consequently,

$$\Psi_{21}(k) = \beta_{21}e^{-\frac{3\pi\nu}{4}}e^{\frac{3\pi i}{4}}D_{\varrho-1}\left(e^{\frac{\pi i}{4}}k\right),$$
$$\beta_{21}\Psi_{12}(k) = \varrho e^{\frac{\pi\nu}{4}}e^{-\frac{\pi i}{4}}D_{-\varrho-1}\left(e^{\frac{3\pi i}{4}}k\right).$$

Along the ray $\arg k = -\frac{3\pi}{4}$,

$$\Psi_{+}(k) = \Psi_{-}(k)\begin{pmatrix} I & 0 \\ -\gamma^{\dagger}(-k_0)B_1 & 1 \end{pmatrix}. \tag{3.75}$$

Notice the (2, 1) entry of the Riemann-Hilbert problem,

$$\begin{aligned}&\beta_{21}e^{\frac{\pi(\nu-i)}{4}}D_{\varrho-1}(e^{-\frac{3\pi i}{4}}k)\\ =&\beta_{21}e^{\frac{\pi(3i-3\nu)}{4}}D_{\varrho-1}(e^{\frac{\pi i}{4}}k) - e^{\frac{\pi\nu}{4}}D_{-\varrho}(e^{\frac{3\pi i}{4}}k)\gamma^{\dagger}(-k_0)B_1.\end{aligned}$$

It follows from (3.71) that

$$D_{-\varrho}(e^{\frac{3\pi i}{4}}k) = \frac{\Gamma(-\varrho+1)e^{\frac{\pi\nu}{2}}}{\sqrt{2\pi}}D_{\varrho-1}(e^{-\frac{3\pi i}{4}}k) + \frac{\Gamma(-\varrho+1)e^{-\frac{\pi\nu}{2}}}{\sqrt{2\pi}}D_{\varrho-1}(e^{\frac{\pi i}{4}}k).$$

Then we separate the coefficients of the two independent functions and obtain

$$\beta_{21} = e^{-\frac{3\pi i}{4}}e^{\frac{\pi\nu}{2}}\frac{\Gamma(-\varrho+1)}{\sqrt{2\pi}}\gamma^{\dagger}(-k_0)B_1. \tag{3.76}$$

Noting that $B^{-1}(J^{A^0}(k^*))^{\dagger}B = (J^{A^0}(k))^{-1}$, we have $\beta_{12} = -B_1^{-1}\beta_{21}^{\dagger}$, which means that

$$\beta_{12} = -B_1^{-1}B_1^{\dagger}\gamma(-k_0)e^{\frac{3\pi i}{4}}e^{\frac{\pi\nu}{2}}\frac{\Gamma(-\varrho+1)}{\sqrt{2\pi}} = e^{-\frac{\pi i}{4}}e^{\frac{\pi\nu}{2}}\nu\frac{\Gamma(-i\nu)}{\sqrt{2\pi}}\gamma(-k_0). \tag{3.77}$$

Finally, we can obtain (1.4) from (3.64), (3.69) and (3.77).

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Conflict of interest

The authors declare no conflict of interest.

References

1. N. Sasa, J. Satsuma, *New-type of soliton solutions for a higher-order nonlinear Schrödinger equation,* J. Phys. Soc. Jpn, **60** (1991), 409–417.
2. Y. Kivshar, G. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals,* Waltham: Academic Press, 2003.
3. K. Porsezian, *Soliton models in resonant and nonresonant optical fibers,* Pramana, **57** (2001), 1003–1039.
4. A. V. Slunyaev, *A high-order nonlinear envelope equation for gravity waves in finite-depth water,* J. Exp. Theor. Phys., **101** (2005), 926–941.
5. M. Trippenbach, Y. B. Band, *Effects of self-steepening and self-frequency shifting on short-pulse splitting in disper-sive nonlinear media,* Phys. Rev. A, **57** (1991), 4791–4803.
6. J. K. Yang, D. J. Kaup, *Squared eigenfunctions for the Sasa-Satsuma equation,* J. Math. Phys., **50** (2009), 023504.
7. C. Gilson, J. Hietarinta, J. Nimmo, et al. *Sasa-Satsuma higher-order nonlinear Schrödinger equation and its bilinearization and multisoliton solutions,* Phys. Rev. E, **68** (2003), 016614.
8. J. J. C. Nimmo, H. Yilmaz, *Binary Darboux transformation for the Sasa-Satsuma equation,* J. Phys. A, **48** (2015), 425202.
9. X. G. Geng, R. M. Li, B. Xue, *A vector general nonlinear Schrödinger equation with* $(m + n)$ *components,* J. Nonlinear Sci., **30** (2020), 991–1013.
10. R. M. Li, X. G. Geng, *On a vector long wave-short wave-type model,* Stud, Appl. Math., **144** (2020), 164–184.
11. R. M. Li, X. G. Geng, *Rogue periodic waves of the sine-Gordon equation,* Appl. Math. Lett., **102** (2020), 106147.
12. J. Xu, Q. Z. Zhu, E. G. Fan, *The initial-boundary value problem for the Sasa-Satsuma equation on a finite interval via the Fokas method,* J. Math. Phys., **59** (2018), 073508.
13. Y. Y. Zhai, X. G. Geng, *The coupled Sasa-Satsuma hierarchy: trigonal curve and finite genus solutions,* Anal. Appl., **15** (2017), 667–697.
14. X. G. Geng, L. H. Wu, G. L. He, *Quasi-periodic solutions of the Kaup-Kupershmidt hierarchy,* J. Nonlinear Sci., **23** (2013), 527–555.
15. X. G. Geng, Y. Y. Zhai, H. H. Dai, *Algebro-geometric solutions of the coupled modified Korteweg-de Vries hierarchy,* Adv. Math., **263** (2014), 123–153.
16. J. Wei, X. G. Geng, X. Zeng, *The Riemann theta function solutions for the hierarchy of Bogoyavlensky lattices,* Trans. Amer. Math. Soc., **371** (2019), 1483–1507.
17. J. Wei, X. G. Geng, *A super Sasa-Satsuma hierarchy and bi-Hamiltonian structures,* Appl. Math. Lett., **83** (2018), 46–52.
18. P. Deift, X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation,* Ann. of Math., **137** (1993), 295–368.

19. K. Grunert, G. Teschl, *Long-time asymptotics for the Korteweg-de Vries equation via nonlinear steepest descent,* Math. Phys. Anal. Geom., **12** (2009), 287–324.

20. P. J. Cheng, S. Venakides, X. Zhou, *Long-time asymptotics for the pure radiation solution of the sine-Gordon equation,* Comm. Partial Differential Equations, **24** (1999), 1195–1262.

21. A. V. Kitaev, A. H. Vartanian, *Leading-order temporal asymptotics of the modified nonlinear Schrödinger equation: Solitonless sector,* Inverse Problem, **13** (1997), 1311–1339.

22. A. V. Kitaev, A. H. Vartanian, *Asymptotics of solutions to the modified nonlinear Schrödinger equation: Solution on a nonvanishing continuous background,* SIAM J. Math. Anal., **30** (1999), 787–832.

23. A. H. Vartanian, *Higher order asymptotics of the modified nonlinear Schrödinger equation,* Comm. Partial Differential Equations, **25** (2000), 1043–1098.

24. A. Boutet de Monvel, A. Kostenko, D. Shepelsky, G. Teschl, *Long-time asymptotics for the Camassa-Holm equation,* SIAM J. Math. Anal., **41** (2009), 1559–1588.

25. L. K. Arruda, J. Lenells, *Long-time asymptotics for the derivative nonlinear Schrödinger equation on the half-line,* Nonlinearity, **30** (2017), 4141–4172.

26. A. Boutet de Monvel, A. Its, V. Kotlyarov, *Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition on the half-line,* Comm. Math. Phys., **290** (2009), 479–522.

27. P. Deift, J. Park, *Long-time asymptotics for solutions of the NLS equation with a delta potential and even initial data,* Int. Math. Res. Not. IMRN, **24** (2011), 5505–5624.

28. I. Egorova, J. Michor, G. Teschl, *Rarefaction waves for the Toda equation via nonlinear steepest descent,* Discrete Contin. Dyn. Syst., **38** (2018), 2007–2028.

29. H. Yamane, *Long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation,* J. Math. Soc. Japan, **66** (2014), 765–803.

30. J. Lenells, *The nonlinear steepest descent method: asymptotics for initial-boundary value problems,* SIAM J. Math. Anal., **48** (2016), 2076–2118.

31. A. Boutet de Monvel, D. Shepelsky, *A Riemann-Hilbert approach for the Degasperis-Procesi equation,* Nonlinearity, **26** (2013), 2081–2107.

32. A. Boutet de Monvel, J. Lenells, D. Shepelsky, *Long-time asymptotics for the Degasperis-Procesi equation on the half-line,* Ann. Inst. Fourier, **69** (2019), 171–230.

33. X. G. Geng, H. Liu, *The nonlinear steepest descent method to long-time asymptotics of the coupled nonlinear Schrödinger equation,* J. Nonlinear Sci., **28** (2018), 739–763.

34. H. Liu, X. G. Geng, B. Xue, *The Deift-Zhou steepest descent method to long-time asymptotics for the Sasa-Satsuma equation,* J. Differential Equations, **265** (2018), 5984–6008.

35. B. B. Hu, T. C. Xia, W. X. Ma, *Riemann-Hilbert approach for an initial-boundary value problem of the two-component modified Korteweg-de Vries equation on the half-line,* Appl. Math. Comput., **332** (2018), 148–159.

36. B. B. Hu, T. C. Xia, N. Zhang, et al. *Initial-boundary value problems for the coupled higher-order nonlinear Schrödinger equations on the half-line,* Int. J. Nonlinear Sci. Numer. Simul., **19** (2018), 83–92.

37. N. Zhang, T. C. Xia, E. G. Fan, *A Riemann-Hilbert approach to the Chen-Lee-Liu equation on the half line,* Acta Math. Appl. Sin. Engl. Ser., **34** (2018), 493–515.

38. X. G. Geng, J. P. Wu, *Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation,* Wave Motion, **60** (2016), 62–72.

39. M. J. Ablowitz, A. S. Fokas, *Complex variables: Introduction and applications,* Cambridge: Cambridge University Press, 2003.

40. R. Beals, R. R. Coifman, *Scattering and inverse scattering for first order systems,* Comm. Pure Appl. Math., **37** (1984), 39–90.

41. R. Beals, R. Wong, *Special functions and orthogonal polynomials,* Cambridge: Cambridge University Press, 2016.

42. E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis* 4^{th} Ed, Cambridge: Cambridge University Press, 1927.

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