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Research article

Locally finiteness and convolution products in groupoids

In Ho Hwang[1], **Hee Sik Kim**[2,*] **and Joseph Neggers**[3]

[1] Department of Mathematics, Incheon National University, Incheon, 22012, Korea

[2] Research Institute of Natural Sciences, Department of Mathematics, Hanyang University, Seoul, 04763, Korea

[3] Department of Mathematics, University of Alabama, Tuscaloosa, AL, 35487-0350, USA

* **Correspondence:** Email: heekim@hanyang.ac.kr; Tel: +82222200899.

Abstract: In this paper, we introduce a version of the Moebius function and other special functions on a particular class of intervals for groupoids, and study them to obtain results analogous to those obtained in the usual lattice, combinatorics and number theory setting, but of course much more general due to the viewpoint taken in this paper.

Keywords: groupoid; below, above; locally finite; transitive interval property; convolution product; interval value function; zeta function; Moebius function
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1. Introduction

The well-known book, *A survey of binary systems*, was written by Bruck [1], and he discussed the theory of groupoids, loops and quasigroups, and several algebraic structures. Borůvka [2] discussed the theory of decompositions of sets and its application to binary systems. Nebeský [3] introduced the notion of a travel groupoid by adding two axioms to a groupoid, and he described an algebraic interpretation of the graph theory. Recently, several researchers investigated groupoids, and obtained some interesting results [4–8]. Kim et al. [8] introduced the notions of "below", "above" and "between" in groupoids, and applied these notions to semigroups and $Bin(X)$. The locally finiteness and Moebius functions were discussed in partially ordered sets and combinatorics [9, 10]. For general reference on partially ordered sets, we refer to [11].

In this paper, we apply the notions of "below" and "above" to the theory of groupoids, and discuss the notion of the locally finiteness and convolution products in groupoids.

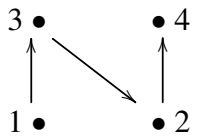
2. Preliminaries

Let $(X, *)$ be a groupoid, i.e., X is a non-empty set and "$*$" is a binary operation defined on X [12], and let $x, y, z \in X$. An element x is said to be *below* y, denoted by $x\beta y$, if $x * y = y$; an element x is said to be *above* y, denoted by $x\alpha y$, if $x * y = x$.

Example 1. [8] Let $D = (V, E)$ be a digraph and let $(V, *)$ be its associated groupoid, i.e., $*$ is a binary operation on V defined by

$$x * y := \begin{cases} x & \text{if } x \to y \notin E, \\ y & \text{otherwise} \end{cases}$$

Let $D = (V, E)$ be a digraph with the following graph:



Then its associated groupoid $(V, *)$ has the following table:

$*$	1	2	3	4
1	1	1	3	1
2	2	2	2	4
3	3	2	3	3
4	4	4	4	4

It is easy to see that there are no elements $x, y \in V$ such that both $x\alpha y$ and $x\beta y$ hold simultaneously. Note that the relations α and β need not be transitive. In fact, $1 \to 3, 3 \to 2$ in E, but not $1 \to 2$ in E imply that $1\beta 3, 3\beta 2$, but not $1\beta 2$. Similarly, $1\alpha 4, 4\alpha 3$, but not $1\alpha 3$.

Proposition 1. [8] *Let $(X, *)$ be a groupoid. Then for any $x, y, z \in X$,*

(i) *if $x\beta y, x\alpha y$, then $x = y$;*
(ii) *if $(X, *)$ is commutative, i.e., $x * y = y * x$, then $x\beta y \iff y\alpha x$;*
(iii) *if $x\beta y, y\alpha x$, then $x * y = y * x = y$.*

Let $(X, *)$ be a groupoid and let $x, y \in X$. Define a binary relation "$\leq$" on X by $x \leq y \iff x\beta y, y\alpha x$. Then it is easy to see that $\leq$ is anti-symmetric.

Proposition 2. [8] *If α, β are transitive, then $\leq$ is transitive.*

Let $(X, *)$ be a groupoid and let $x, y \in X$. We define an *interval* (or a *segment*) as follows:

$$[x, y] := \{q \in X \mid x \leq q,\ q \leq y\}.$$

Note that the interval (segment) $[x, y]$ in groupoids is different from the intervals in (linear) ordered sets.

3. Locally finiteness

Given a groupoid $(X, *)$, the interval $[x, y]$, $x, y \in X$, consists of all elements $q \in X$ such that $x \leq q \leq y$. Since $x \leq y$ if and only if $x\beta y, y\alpha x$ if and only if $x * y = y = y * x$, we may put the interval $[x, y]$ as follows:

$$[x, y] = \{q \in X \mid x * q = q = q * x, \ \ q * y = y = y * q\}.$$

Proposition 3. *Let $(X, *)$ be a groupoid and let $x \in X$. Then $x * x = x$ if and only if $x \in [x, x]$ if and only if $[x, x] = \{x\}$.*

Proof. Straightforward. □

Proposition 3 shows that $x * x \neq x$ if and only if $x \notin [x, x]$.

Example 2. Consider a set $X := \{0, a, b, c\}$ with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	a	b	a
b	b	b	b	b
c	c	c	c	b

It is easy to see that $a, b \in [a, b], a \in [a, a], b \in [b, b], 0 \in [0, 0], [b, c] = [a, c] = [0, b] = \emptyset$. Since $c * c = b \neq c$, we have $[c, c] = \emptyset$ and $c \notin [c, c]$.

A groupoid $(X, *)$ is said to be an *idempotent* if $x * x = x$ for all $x \in X$.

In Example 2, $(X, *)$ is not an idempotent groupoid, since $c * c = b \neq c$, but $X_1 := \{0, a, b\}$ is an idempotent subgroupoid under "$*$".

A groupoid $(X, *)$ is said to be *locally finite* if for all $x, y \in X$, the interval $[x, y]$ is finite. The set of all intervals on $(X, *)$ is denoted by $I(X, *)$, and the set of all finite intervals on $(X, *)$ is denoted by $I_f(X, *)$. Hence a groupoid $(X, *)$ is locally finite if and only if $I(X, *) = I_f(X, *)$. The set of all non-empty locally finite intervals on a groupoid $(X, *)$ is denoted by $I_f^P(X, *)$.

Example 3. Let X be the set of all non-negative integers and let "+" be the usual addition on integers. Given $x, y \in X$, we have

$$\begin{aligned} [x, y] &= \{q \in X \mid x \leq q \leq y\} \\ &= \{q \in X \mid x + q = q + x = q, \ \ q + y = y + q = y\}. \end{aligned}$$

If $x \neq 0$, then $[x, y] = \emptyset$, and if $x = 0$, then $[x, y] = [0, y] = \{0\}$ for all $y \in X$. Hence $(X, +)$ is locally finite.

Example 4. Let X be the set of all rational numbers and let $x * y := \frac{1}{2}(x + y)$ for all $x, y \in X$. Assume that $x, y \in X$ such that $[x, y] \neq \emptyset$. Then there exists an element $q \in [x, y]$. It follows that $x * q = q * x = q, q * y = y * q = y$, i.e., $\frac{1}{2}(x + q) = \frac{1}{2}(q + x) = q, \frac{1}{2}(q + y) = \frac{1}{2}(y + q) = y$, proving that $x = q, y = q$. Hence $[x, y] = \{x\}$. Hence $(X, *)$ is locally finite.

Example 5. (a). Let X be the set of all integers. Define $x * y := \max\{x, y\}$ on X. Assume $x, y \in X$ such that $[x, y] \neq \emptyset$. Then there exists $q \in X$ such that $x * q = q * x = q, q * y = y * q = y$. It follows that

$\max\{x, q\} = q, \max\{q, y\} = y$, i.e., $x \le q \le y$ where $\le$ is the usual order relation on the integers. Hence if $y \ge x$, then $|[x, y]| = y - x + 1$. Otherwise, $[x, y] = \emptyset$. Thus $(X, *)$ is locally finite.
(b). Let X be the set of all rational numbers and let $x * y := \max\{x, y\}$ for all $x, y \in X$. If $x \le q \le y$ where $\le$ is the usual order relation, then $[x, y]$ is not finite unless $x = y$, i.e., $[x, y] = \{x\} = \{y\}$. Hence $(X, *)$ is not locally finite and $I_f^P(X, *) = \{\{x\} \mid x \in X\}$.

Proposition 4. *Let $(X, *)$ be a leftoid for f, i.e., $x * y = f(x)$, $\forall x, y \in X$, where $f : X \to X$ is a map. Then $(X, *)$ is locally finite.*

Proof. Given $x, y \in X$, if $[x, y] \neq \emptyset$, then we have

$$\begin{aligned} q \in [x, y] &\Leftrightarrow x \le q \le y \\ &\Leftrightarrow x * q = q * x = q, \;\; q * y = y * q = y \\ &\Leftrightarrow f(x) = f(q) = q, \;\; f(q) = f(y) = y \\ &\Leftrightarrow f(x) = q = f(q) = f(y) = y \\ &\Leftrightarrow [x, y] = \{y\}. \end{aligned}$$

This proves the proposition. □

Corollary 1. *Let $(X, *)$ be a rightoid for f, i.e., $x * y = f(y)$, $\forall x, y \in X$, where $f : X \to X$ is a map. Then $(X, *)$ is locally finite.*

Proof. The proof is similar to Proposition 4. □

4. Convolution products

A groupoid $(X, *)$ is said to have a *transitive interval property* if $[x, y], [y, z] \in I_f(X, *)$, then $[x, z] \in I_f(X, *)$. Every locally finite groupoid $(X, *)$ has the transitive interval property, but the converse does not hold in general.

Example 6. Let $(X, \le)$ be a poset where $X = \{x\} \oplus Y \oplus \{z\}$ is an ordinal sum of two chains $\{x\}, \{z\}$ and an anti-chain $Y := \{y_n | n = 1, 2, 3, \cdots\}$. If we define $x * y := \max\{x, y\}$ for all $x, y \in X$, then $[x, y_i] = \{x, y_i\}$, $[y_i, z] = \{y_i, z\}$ $(i = 1, 2, 3, \cdots)$, and $[x, z] = X$. Clearly, $[x, y_i], [y_i, z] \in I_f(X, *)$, but $[x, z] \notin I_f(X, *)$.

Assume that $(X, *) \in Bin(X)$ and $\emptyset \in I_f(X, *)$. We define a *convolution product* "$\odot$" on $I_f(X, *)$ by

$$[x, y] \odot [y', z] := \begin{cases} [x, z], & \text{if } y = y', \\ \emptyset, & \text{if } y \neq y'. \end{cases}$$

Let K be a field (usually a complex field $\mathbb{C}$). We define a map $f : I(X, *) \to K$ by

$$[x, y] \mapsto \begin{cases} k, & \text{if } [x, y] \in I_f^P(X, *), \\ 0, & \text{otherwise.} \end{cases}$$

for some $k \in K \setminus \{0\}$, i.e., $[x, y] = \emptyset$ or $[x, y] \notin I_f^P(X, *)$ implies $f([x, y]) = 0$, and $f([x, y]) = k$ for some $k \in K \setminus \{0\}$ otherwise. We call such a function f an *interval value function*. Define a *convolution product* "$\otimes$" of interval value functions f and g by

$$(f \otimes g)([x, y]) := \sum_{z \in [x, y]} f([x, z]) g([z, y]).$$

Note that if $f([x,z])g([z,y]) \neq 0$, then $f([x,z]) \neq 0 \neq g([z,y])$ and hence $[x,z] \neq \emptyset \neq [z,y]$, i.e., $[x,z],[z,y] \in I_f^P(X,*)$.

Define a map $\delta : I(X,*) \to K$ by

$$[x,y] \mapsto \begin{cases} 1, & \text{if } x = y,\ x * x = x, \\ 0, & \text{otherwise.} \end{cases}$$

Such a map δ is said to be a *Riemann function* on a groupoid $(X,*)$.

Remark. The condition $x * x = x$ is necessary to define the Riemann function on a groupoid $(X,*)$. As in Example 2, we see that $[c,c] = \emptyset$ and $c \notin [c,c]$. If $[x,x] \neq \emptyset$, then there exists $y \in [x,x]$. It follows that $x \leq y \leq x$, and hence $x * y = y = y * x, x * y = x = y * x$. This shows that $x = y$ and $x * x = x$. Clearly, if $x * x = x$, by Proposition 3, we have $x \in [x,x]$ and hence $[x,x] \neq \emptyset$.

By Proposition 3, the map δ is the characteristic function of $\mathbb{U}$, where $\mathbb{U} := \{x \in X \mid x * x = x\}$.

Proposition 5. *If $f : I(X,*) \to K$ is an interval value function, then*

$$(f \otimes \delta)([x,y]) = \begin{cases} f([x,y]), & \textit{if } y \in \mathbb{U}, \\ 0, & \textit{otherwise,} \end{cases}$$

and

$$(\delta \otimes f)([x,y]) = \begin{cases} f([x,y]), & \textit{if } x \in \mathbb{U}, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Given $[x,y] \in I(X,*)$, we have

$$\begin{aligned} (f \otimes \delta)([x,y]) &= \sum_{z \in [x,y]} f([x,z])\delta([z,y]) \\ &= \sum_{z \in [x,y] \cap \mathbb{U}} f([x,z])\delta([z,y]) + \sum_{z \notin [x,y] \cap \mathbb{U}} f([x,z])\delta([z,y]) \\ &= f([x,y])\delta([y,y]) \\ &= f([x,y]) \end{aligned}$$

if $y \in \mathbb{U}$. Otherwise, $(f \otimes \delta)([x,y]) = 0$. Similarly,

$$\begin{aligned} (\delta \otimes f)([x,y]) &= \sum_{z \in [x,y]} \delta([x,z])f([z,y]) \\ &= \sum_{z \in [x,y] \cap \mathbb{U}} \delta([x,z])f([z,y]) + \sum_{z \notin [x,y] \cap \mathbb{U}} \delta([x,z])f([z,y]) \\ &= \delta([x,y])f([y,y]) \\ &= f([x,y]) \end{aligned}$$

if $x \in \mathbb{U}$. Otherwise, $(f \otimes \delta)([x,y]) = 0$. □

Note that if $\mathbb{U} = \emptyset$, then δ is the zero map on $I(X, *)$. In fact, for any $x, y \in X$, if $x \neq y$, then $\delta([x, y]) = 0$. If $x = y$, since $\mathbb{U} = \emptyset$, $x * x \neq x$ and hence $\delta([x, y]) = \delta([x, x]) = 0$. In this case, $f \otimes \delta = \delta \otimes f = 0$.

Theorem 1. *If $(X, *)$ is a locally finite groupoid, then $\delta \otimes \delta = \delta$.*

Proof. Given $[x, y] \in I(X, *)$, we have

$$\begin{aligned}(\delta \otimes \delta)([x, y]) &= \sum_{z \in [x,y]} \delta([x, z])\delta([z, y]) \\ &= \sum_{z \in [x,y] \cap \mathbb{U}} \delta([x, z])\delta([z, y]) + \sum_{z \notin [x,y] \cap \mathbb{U}} \delta([x, z])\delta([z, y]) \\ &= \begin{cases} \delta([x, y]), & \text{if } x \in \mathbb{U}, \\ 0, & \text{otherwise} \end{cases} \\ &= \delta([x, y]),\end{aligned}$$

proving the theorem. □

A map $g : I(X, *) \to K$ is called an *inverse* of a mapping $f : I(X, *) \to K$ if, for all $[x, y] \in I(X, *)$, $(f \otimes g)([x, y]) = \delta([x, y])$, i.e., $\sum_{z \in [x,y]} f([x, z])g([z, y]) = \delta([x, y])$.

We define a map $\zeta : I(X, *) \to K$ by

$$[x, y] \mapsto \begin{cases} 1, & \text{if } [x, y] \in I_f^P(X, *), \\ 0, & \text{otherwise.} \end{cases}$$

We call such a map ζ a *zeta function*. It follows that, for all $[x, y] \in I(X, *)$,

$$\begin{aligned}(\zeta \otimes \zeta)([x, y]) &= \sum_{z \in [x,y]} \zeta([x, z])\zeta([z, y]) \\ &= |\{z \in X \mid [x, z], [z, y] \in I_f^P(X, *)\}|.\end{aligned}$$

Next, we introduce the *Moebius function* μ_1 on a groupoid $(X, *)$ as follows: if $x = y$, then we define

$$\mu_1([x, x]) := \begin{cases} 1, & \text{if } [x, x] \in I_f^P(X, *), \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $x \neq y$, then we define

$$\mu_1([x, y]) := - \sum_{\substack{z \in [x,y] \\ y \neq z \\ [x,z] \in I_f^P(X,*)}} \mu_1([x, z]) \tag{1}$$

or $\mu_1([x, y]) := 0$ if no such z exists.

Theorem 2. *Let $(X, *)$ be a locally finite groupoid. If $(X, *)$ is an idempotent groupoid, then*

$$\mu_1 \otimes \zeta = \delta.$$

Proof. Since $(X, *)$ is idempotent, by Proposition 3, $[x, x] \neq \emptyset$ and hence $\zeta([x, x]) = 1$ for all $x \in X$. Given $x, y \in X$, we consider the case $x \neq y$. If $[x, y] \neq \emptyset$, then

$$\begin{aligned}
(\mu_1 \otimes \zeta)([x, y]) &= \sum_{z \in [x,y]} \mu_1([x, z])\zeta([z, y]) \\
&= \sum_{\substack{z \in [x,y] \\ z \neq y}} \mu_1([x, z])\zeta([z, y]) + \mu_1([x, y])\zeta([y, y]) \\
&= \sum_{\substack{z \in [x,y] \\ z \neq y}} \mu_1([x, z])\zeta([z, y]) + \mu_1([x, y]) \\
&= \sum_{\substack{z \in [x,y] \\ z \neq y \\ \zeta([z,y]) \neq 0}} \mu_1([x, z])\zeta([z, y]) + \mu_1([x, y]) \\
&= \sum_{\substack{z \in [x,y] \\ z \neq y \\ [z,y] \in I_f^P(X,*)}} \mu_1([x, z])\zeta([z, y]) + \mu_1([x, y]) \\
&= -\mu_1([x, y]) + \mu_1([x, y]) \\
&= 0.
\end{aligned}$$

If $[x, y] = \emptyset$, then there is no $z \in [x, y]$, and hence there is no $[z, y] \in I_f^P(X, *)$. It follows that $\zeta([z, y]) \neq 1$. This shows that

$$\begin{aligned}
(\mu_1 \otimes \zeta)([x, y]) &= \sum_{z \in [x,y]} \mu_1([x, z])\zeta([z, y]) \\
&= 0.
\end{aligned}$$

Consider the case $x = y$. By Proposition 3, we have $[x, x] = \{x\}$. It follows that

$$\begin{aligned}
(\mu_1 \otimes \zeta)([x, x]) &= \sum_{z \in [x,x]} \mu_1([x, z])\zeta([z, y]) \\
&= \mu_1([x, x])\zeta([x, x]) \\
&= 1.
\end{aligned}$$

This proves the theorem. □

Furthermore, we redefine the Moebius function as follows: when $x \neq y$,

$$\mu_2([x, y]) := -\sum_{\substack{z \in [x,y] \\ z \neq x \\ [z,y] \in I_f^P(X,*)}} \mu_2([z, y]) \tag{2}$$

or $\mu_2([x, y]) := 0$ if no such z exists. We obtain an exact analog of Theorem 2 as below:

Theorem 2′. *Let $(X, *)$ be a locally finite groupoid. If $(X, *)$ is an idempotent groupoid, then*

$$\zeta \otimes \mu_2 = \delta.$$

Proof. The proof is similar to Theorem 2. □

Note that if two definitions (1) and (2) of the Moebius function μ for the case $x \neq y$ are the same, i.e., $\mu_1 = \mu_2 (= \mu)$, then we obtain $\mu \otimes \zeta = \zeta \otimes \mu = \delta$.

Example 7. Let $X := \{a, b, 1, 2\}$ be a set with the following table:

$*$	a	b	1	2
a	1	b	1	b
b	a	2	a	2
1	1	b	1	b
2	a	2	a	2

It is easy to compute that the non-empty intervals are $[1, 1] = \{1\}, [2, 2] = \{2\}, [a, 1] = \{1\}, [b, 2] = \{2\}$. Hence $\mu([1, 1]) = \mu([2, 2]) = 1, \mu([a, 1]) = \mu([b, 2]) = 0$. It follows that $(\mu \otimes \zeta)([1, 1]) = (\mu \otimes \zeta)([2, 2]) = 1, (\mu \otimes \zeta)([a, a]) = (\mu \otimes \zeta)([b, b]) = 0$ and $(\mu \otimes \zeta)([a, 1]) = \mu([a, 1])\zeta([1, 1]) = 0 \cdot 1 = 0$.

5. Conclusion

In the usual setting of number theory, the Moebius function will have its ordinary meaning and properties. We have used the rather strong version of the relation $x \leq y$ on the groupoid $(X, *)$ and constructed all our functions μ, ζ and δ which were used in the theory of combinatorics and partially ordered sets. There is nothing in the way of following this same pattern with respect to β and α-betweenness for intervals instead of the intervals $[x, y]$ over groupoids $(X, *)$. Clearly there remains much to be done for a more complete theory. Nevertheless, the outline of a "theory of order" on groupoids $(X, *)$ are discernible.

6. Future research

In sequel we will develop the idea of Moebius functions for arbitrary d/BCK-algebras and we demonstrate the existence of a general Moebius inversion process. If $(X, *, 0)$ is a locally finte d-algebra, and if δ, μ and ζ are the Riemann, Moebius and zeta functions respectively, then we show that $(\mu \odot \delta) \otimes \zeta = \delta$. Moreover, we will define a notion of a dual Moebius function μ^d, and show that $\zeta \otimes (\mu^d \odot \delta) = \delta$.

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Conflict of interest

The authors hereby declare that there are no conflicts of interest regarding the publication of this paper.

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