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Research article

Properties of the power-mean and their applications

Jing-Feng Tian[1,2,*], **Ming-Hu Ha**[3] **and Hong-Jie Xing**[4]

[1] School of Management, Hebei University, Wusi Road 180, Baoding 071002, P. R. China
[2] Department of Mathematics and Physics, North China Electric Power University, Yonghua Street 619, Baoding 071003, P. R. China
[3] School of Science, Hebei University of Engineering, Taiji Road 19, Handan 056038, P. R. China
[4] College of Mathematics and Information Science, Hebei University, Wusi Road 180, Baoding 071002, P. R. China

* **Correspondence:** Email: tianjf@ncepu.edu.cn; Tel: +8603127525072.

Abstract: Suppose $w, v > 0$, $w \neq v$ and $A_u(w, v)$ is the u-order power mean (PM) of w and v. In this paper, we completely describe the convexity of $u \mapsto A_u(w, v)$ on $\mathbb{R}$ and $s \mapsto A_{u(s)}(w, v)$ with $u(s) = (\ln 2) / \ln(1/s)$ on $(0, \infty)$. These yield some new inequalities for PMs, and give an answer to an open problem.

Keywords: power mean; power-type mean; convexity; inequality
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1. Introduction

A function $M : \mathbb{R}_+^2 \mapsto \mathbb{R}$ is called a bivariate mean (BM) if for all $w, v > 0$

$$\min(w, v) \leq M(w, v) \leq \max(w, v)$$

is valid. A BM is symmetric if for all $w, v > 0$

$$M(w, v) = M(v, w)$$

is valid. It is said to be homogeneous (of degree one) if for all $\lambda, w, v > 0$

$$M(\lambda w, \lambda v) = \lambda M(w, v)$$

is valid. If a BM M is differentiable on $\mathbb{R}_+^2$, then the function $M_u : \mathbb{R}_+^2 \mapsto \mathbb{R}$ defined by

$$M_u(w, v) = M^{1/u}(w^u, v^u) \text{ if } u \neq 0 \text{ and } M_0(w, v) = w^{M_x(1,1)} v^{M_y(1,1)}, \tag{1.1}$$

is called "u-order M mean", where $M_x(x,y)$, $M_y(x,y)$ are the first-order partial derivatives in regard to the first and second components of $M(x,y)$, respectively (see [1]). For example, the arithmetic mean (AM), logarithmic mean (LM) and identric mean (IM) are given by

$$A(w,v)=\frac{w+v}{2},\quad L(w,v)=\frac{w-v}{\ln w-\ln v},\quad I(w,v)=e^{-1}\left(\frac{v^v}{w^w}\right)^{1/(v-w)},$$

respectively, then

$$A_u(w,v) = \left(\frac{w^u+v^u}{2}\right)^{1/u} \text{ if } u\neq 0 \text{ and } A_0(w,v)=\sqrt{wv}, \tag{1.2}$$

$$L_u(w,v) = \left(\frac{w^u-v^u}{u(\ln w-\ln v)}\right)^{1/p} \text{ if } u\neq 0 \text{ and } L_0(w,v)=\sqrt{wv}, \tag{1.3}$$

$$I_u(w,v) = e^{-1/u}\left(\frac{v^{v^u}}{w^{w^u}}\right)^{1/(v^u-w^u)} \text{ if } u\neq 0 \text{ and } I_0(w,v)=\sqrt{wv} \tag{1.4}$$

are u-order AM, u-order LM and u-order IM, respectively. As usual, the u-order AM is still called u-order PM. Correspondingly, since the form of M_u is similar to PM A_u, it is also known simply as "power-type mean". More general means than power-type mean including Stolarsky means, Gini means, and two-parameters functions, etc., which can be seen in [2–7].

For those means with parameters, there are many nice properties including monotonicity, (log-) convexity, comparability, additivity, stability and inequalities, which can be found in [8–17].

In this paper, we are interested in the properties of the PM A_u. As is well-known that $u\mapsto A_u(w,v)$ is increasing on $\mathbb{R}$ (see [5]). The log-convexity of $u\mapsto A_u(w,v)$, $L_u(w,v)$ and $I_u(w,v)$ is a direct consequence of [9, Conclusion 1. 1)] when $q=0$, that is,

Theorem 1. *The functions $u\mapsto A_u(w,v)$, $L_u(w,v)$ and $I_u(w,v)$ are log-convex on $(-\infty,0)$ and log-concave on $(0,\infty)$.*

The log-convexity of the function $u\mapsto A_u(w,v)$ was reproved in [19] by Begea, Bukor and Tóhb. The authors proposed an open problem on the convexity of the function $u\mapsto A_u(w,v)$:

Problem 1. *Prove that*

$$\inf_{w,v>0}\{u : A_u(w,v) \text{ is concave for variable } u\in\mathbb{R}\} = \frac{1}{2}\ln 2,$$
$$\sup_{w,v>0}\{u : A_u(w,v) \text{ is convex for variable } u\in\mathbb{R}\} = \frac{1}{2}.$$

Problem 1 was proven by Matejíčka in [20]. In 2016, Raïsouli and Sándor [16, Problem 1] proposed the following problem.

Problem 2. *Let $p,q,r\in\mathbb{R}$ with $q>r>p$. Are there $0<\beta,\alpha<1$ with $\beta>\alpha$, such that the double inequality*

$$(1-\alpha)A_p+\alpha A_q<A_r<(1-\beta)A_p+\beta A_q$$

holds? If it is positive, what are the best β and α?

Clearly, this problem is partly related to the convexity of $u \mapsto A_u(w,v)$. Motivated by Problem 2, the main purpose of this paper is to investigate completely the convexity of $u \mapsto A_u(w,v)$ on $\mathbb{R}$ and $s \mapsto A_{u(s)}(w,v)$ with $u(s) = (\ln 2)/\ln(1/s)$ on $(0,\infty)$. As applications, some new inequalities for power means are established, and an answer to Problem 2 is given. Final, three problems on the convexity of certain power-type means and inequalities are proposed.

It should be noted that a homogeneous BM can be represented by the exponential functions. If $M(x,y)$ is a HM of positive arguments x and y, then $M(x,y)$ can be represented as

$$M(x,y) = \sqrt{xy}M(e^t, e^{-t}),$$

where $t = (1/2)\ln(x/y)$. Further, if $M(x,y)$ is symmetric, then $M(x,y)$ can be expressed in terms of hyperbolic functions (see [18, Lemma 3]). For example, in view of symmetry, we suppose $v > w > 0$. Then we find $t = (1/2)\ln(v/w) > 0$. Thus the PM $A_u(w,v)$, u-order LM $L_u(w,v)$ and u-order IM $I_u(w,v)$ can be represented as

$$\frac{A_u(w,v)}{\sqrt{wv}} = \cosh^{1/u}(ut), \quad \frac{L_u(w,v)}{\sqrt{wv}} = \left[\frac{\sinh(ut)}{ut}\right]^{1/u}, \quad \frac{I_u(w,v)}{\sqrt{wv}} = \exp\left[\frac{t}{\tanh(ut)} - \frac{1}{u}\right]$$

if $u \neq 0$.

The first result of the paper is the following theorem.

Theorem 2. *The function $u \mapsto A_u(w,v)$ is convex on $\left(-\infty, \ln\sqrt{2}\right)$ and concave on $(1/2,\infty)$ for all $w,v > 0$ with $w \neq v$. While $u \in \left(\ln\sqrt{2}, 1/2\right)$, the function $u \mapsto A_u(w,v)$ is concave then convex. Equivalently, the function*

$$F_t(u) = \cosh^{1/u}(ut)$$

is convex (concave) for all $t > 0$ if and only if $u \leq \ln\sqrt{2}$ ($u \geq 1/2$). While $\ln\sqrt{2} < u < 1/2$, there is a $u_1 \in \left(\ln\sqrt{2}, 1/2\right)$ such that $F_t(u)$ is concave on $\left(\ln\sqrt{2}, u_1\right)$ and convex on $(u_1, 1/2)$.

Remark 1. *Theorem 2 not only gives an answer to Problem 1, but also describes completely the convexity of the function $u \mapsto A_u(w,v)$ on $\mathbb{R}$.*

Remark 2. *By Theorems 1 and 2, we see that the function $u \mapsto A_u(w,v)$ has the following (log-) convexity:*

u	$(-\infty,0)$	$\left(0,\ln\sqrt{2}\right)$	$\left(\ln\sqrt{2},1/2\right)$	$(1/2,\infty)$
A_u	∪	∪	∩∪	∩
$\ln A_u$	∪	∩	∩	∩

where and in what follows the symbols "∪" and "∩" denote the given function are convex and concave, "∩∪" and "∪∩" denote the given function are "concave then convex" and "convex then concave", respectively.

The second and third results of the paper are the following theorems.

Theorem 3. *Suppose $w,v > 0$ and $w \neq v$. The function $s \mapsto A_{u(s)}(w,v)$ with $u = u(s) = (\ln 2)/\ln(1/s)$ is convex on $\left(e^{-2}, 1\right)$ and concave on $(1,\infty)$. While $s \in \left(0, e^{-2}\right)$, the function $s \mapsto A_{u(s)}(w,v)$ is convex then concave. Equivalently, the function*

$$G_t(s) = \cosh^{1/u}(ut), \ \text{ where } u = \frac{\ln 2}{\ln(1/s)}$$

is convex (concave) for all $t > 0$ *if and only if* $s \in \left(e^{-2}, 1\right)$ $(s \in (1, \infty))$. *While* $s \in \left(0, e^{-2}\right)$, *there is a* $s_2^* \in \left(0, e^{-2}\right)$ *such that* $G_t(s)$ *is convex on* $\left(0, s_2^*\right)$ *and concave on* $\left(s_2^*, e^{-2}\right)$.

Theorem 4. *Suppose* $w, v > 0$ *and* $a \neq b$. *The function* $s \mapsto A_{u(s)}(w, v)$ *with* $u(s) = (\ln 2) / \ln(1/s)$ *is log-concave on* $\left(0, e^{-2}\right) \cup (1, \infty)$. *Equivalently, the function* $G_t(s)$ *is log-concave for all* $t > 0$ *if and only if* $s \in \left(0, e^{-2}\right) \cup (1, \infty)$.

Remark 3. *By Theorems 3 and 4, the function* $s \mapsto A_{u(s)}(w, v)$ *has the following (log-) convexity:*

s	$\left(0, e^{-2}\right)$	$\left(e^{-2}, 1\right)$	$(1, \infty)$
$A_{u(s)}$	$\cup\cap$	$\cup$	$\cap$
$\ln A_{u(s)}$	$\cap$		$\cap$

2. Tools

To prove the lemmas listed in Sections 3–5, we need two tools. The first is the so-called L'Hospital Monotone Rule (LMR), which appeared in [21] (see also [22]).

Proposition 1. *Suppose* $-\infty \leq a < b \leq \infty$, ϕ *and* ψ *are differentiable functions on* (a, b). *Suppose also the derivative* ψ' *is nonzero and does not change sign on* (a, b), *and* $\phi(a^+) = \psi(a^+) = 0$ *or* $\phi(b^-) = \psi(b^-) = 0$. *If* ϕ'/ψ' *is increasing (decreasing) on* (a, b) *then so is* ϕ/ψ.

Before stating the second tool, we present first an important function $H_{\phi,\psi}$. Assume that ϕ and ψ are differentiable functions on (a, b) with $\psi' \neq 0$, where $-\infty \leq a < b \leq \infty$. It was introduced by Yang in [23, Eq (2.1)] that

$$H_{\phi,\psi} := \frac{\phi'}{\psi'}\psi - \phi, \tag{2.1}$$

which we call Yang's H-function. This function has some good properties, see [23, Properties 1 and 2], and plays an important role in the proof of a monotonicity criterion for the quotient of two functions, see for example, [24–28].

To study the monotonicity of the ratio ϕ/ψ on (a, b), Yang [23, Property 1] presented two identities in term of $H_{\phi,\psi}$, which state that, if ϕ and ψ are twice differentiable with $\psi\psi' \neq 0$ on (a, b), then

$$\left(\frac{\phi}{\psi}\right)' = \frac{\psi'}{\psi^2}\left(\frac{\phi'}{\psi'}\psi - \phi\right) = \frac{\psi'}{\psi^2}H_{\phi,\psi}, \tag{2.2}$$

$$H'_{\phi,\psi} = \left(\frac{\phi'}{\psi'}\right)'\psi. \tag{2.3}$$

3. Proof of Theorem 2

In order to prove Theorem 2, we need the following lemma.

Lemma 1. *Let* $h_1(x) = f_1(x) / g_1(x)$, *where*

$$f_1(x) = (x \tanh x - \ln(\cosh x))^2, \tag{3.1}$$

$$g_1(x) = 2x\tanh x - \frac{x^2}{\cosh^2 x} - 2\ln(\cosh x). \tag{3.2}$$

Then $h_1(x)$ is strictly decreasing from $(0,\infty)$ onto $\left(\ln\sqrt{2}, 1/2\right)$.

Proof. Differentiation yields

$$\begin{aligned} f_1'(x) &= \frac{2x}{\cosh^2 x}(x\tanh x - \ln\cosh x) := \frac{2x}{\cosh^2 x} f_2(x), \\ g_1'(x) &= 2\frac{x^2\sinh x}{\cosh^3 x} := \frac{2x}{\cosh^2 x} g_2(x), \end{aligned}$$

where

$$f_2(x) = x\tanh x - \ln\cosh x, \quad g_2(x) = x\tanh x;$$
$$f_2'(x) = \frac{x}{\cosh^2 x}, \quad g_2'(x) = \frac{x + \cosh x\sinh x}{\cosh^2 x}.$$

Then

$$\begin{aligned} \frac{f_1'(x)}{g_1'(x)} &= \frac{f_2(x)}{g_2(x)}, \\ \frac{f_2'(x)}{g_2'(x)} &= \frac{x}{x + \cosh x\sinh x} = \frac{1}{1 + \sinh(2x)/(2x)}. \end{aligned}$$

Clearly, for $x \in (0,\infty)$, $g_1'(x) > 0$, and hence, $g_1(x) > g_1(0) = 0$. Since $\sinh(2x)/(2x)$ is strictly increasing for $x \in (0,\infty)$, it is readily seen that for $x \in (0,\infty)$, the function $f_2'(x)/g_2'(x)$ is strictly decreasing. Due to $f_2(0) = g_2(0) = 0$, so is $f_2(x)/g_2(x)$ by Proposition 1. Similarly, in view of $f_1(0) = g_1(x) = 0$, so is $f_1(x)/g_1(x) = h_1(x)$ using Proposition 1 again. An easy computation gives

$$\lim_{x\to 0}\frac{f_1(x)}{g_1(x)} = \frac{1}{2} \text{ and } \lim_{x\to\infty}\frac{f_1(x)}{g_1(x)} = \frac{1}{2}\ln 2,$$

thereby completing the proof. □

Now we shall prove Theorem 2.

Proof of Theorem 2. Differentiation yields

$$F_t'(u) = \frac{t}{u}\cosh^{1/u-1}(ut)\sinh(ut) - \frac{1}{u^2}\cosh^{1/u}(ut)\ln\cosh(ut),$$

$$\begin{aligned} F_t''(u) = {} & \frac{t}{u^3}\sinh(ut)\left[(1-u)(ut)\sinh(ut) - \cosh(ut)\ln\cosh(ut)\right]\cosh^{1/u-2}(ut) \\ & + \frac{t}{u^2}\left[ut\cosh(ut) - \sinh(ut)\right]\cosh^{1/u-1}(ut) \\ & - \frac{1}{u^4}\left[ut\sinh(ut) - \cosh(ut)\ln\cosh(ut)\right]\cosh^{1/u-1}(ut)\ln\cosh(ut) \\ & - \frac{1}{u^3}\left[ut\tanh(ut) - 2\ln\cosh(ut)\right]\cosh^{1/u}(ut). \end{aligned}$$

Letting $ut = x$ and simplifying give

$$\begin{aligned}\frac{u^4}{\cosh^{1/u-2}(ut)}F_t''(u) &= x(\sinh x)[(1-u)x\sinh x - \cosh x \ln\cosh x] \\ &\quad +ux(x\cosh x - \sinh x)\cosh x \\ &\quad -(x\sinh x - \cosh x\ln\cosh x)\cosh x\ln\cosh x \\ &\quad -u(x\tanh x - 2\ln\cosh x)\cosh^2 x\end{aligned}$$

$$\begin{aligned}&= u\left[2\cosh^2 x\ln\cosh x + x^2 - 2x\cosh x\sinh x\right] \\ &\quad +(x\sinh x - \cosh x\ln\cosh x)^2 = -[u-h_1(x)]g_1(x)\cosh^2 x,\end{aligned}$$

where $h_1(x)$ and $g_1(x)$ are given in Lemma 1. Since $h_1(x)$ and $g_1(x)$ are even on $(-\infty,\infty)$ and $g_1(x) = g_1(|x|) > 0$ shown in Lemma 1, $F_t''(u) \geq (\leq)\, 0$ for $t > 0$ if and only if

$$Q_1(t) = u - h_1(|ut|) \leq (\geq 0).$$

From Lemma 1 we find

$$Q_1'(t) = -|u|\,h_1'(|ut|) > 0$$

for all $t > 0$ and

$$\begin{aligned}\lim_{t\to 0} Q_1(t) &= u - \lim_{t\to 0} h_1(|ut|) = u - \frac{1}{2}, \\ \lim_{t\to\infty} Q_1(t) &= u - \lim_{t\to\infty} h_1(|ut|) = u - \frac{1}{2}\ln 2.\end{aligned}$$

We conclude thus that $F_t''(u) > (<)\, 0$ for all $t > 0$ if and only if

$$u \leq \min\left\{\frac{1}{2}, \frac{1}{2}\ln 2\right\} = \frac{1}{2}\ln 2 \text{ or } u \geq \max\left\{\frac{1}{2}, \frac{1}{2}\ln 2\right\} = \frac{1}{2}.$$

When $\ln\sqrt{2} < u < 1/2$, since $Q_1'(t) > 0$ with $Q_1(0^+) = u - 1/2 < 0$ and $Q_1(\infty) = u - \ln\sqrt{2} > 0$, there is a $t_1 = t_1(u)$ such that $Q_1(t) < 0$ on $(0, t_1)$ and $Q_1(t) > 0$ on (t_1, ∞), where t_1 is a solution of the equation

$$Q_1(t) = u - h_1(|ut|) = 0. \tag{3.3}$$

Since for $x \in (0,\infty)$, the function $h_1(x)$ is strictly decreasing, the inverse of h_1 exists and so is h_1^{-1}. Solving the equation (3.3) for t yields

$$t = \frac{h_1^{-1}(u)}{u} = T_1(u).$$

Noting that $1/u$ and $h_1^{-1}(u)$ are both positive and decreasing, so is $t = T_1(u)$. This implies $u = T_1^{-1}(t)$ exists and strictly decreasing on $(0,\infty)$. It then follows that

$$\begin{aligned}t &\in (0, t_1) \iff u \in \left(T_1^{-1}(t_1), 1/2\right) = (u_1, 1/2), \\ t &\in (t_1, \infty) \iff u \in \left(\ln\sqrt{2}, T_1^{-1}(t_1)\right) = \left(\ln\sqrt{2}, u_1\right),\end{aligned}$$

where $u_1 = T_1^{-1}(t_1)$.

We thus arrive at that

$$F_t''(u)\begin{cases} > 0 & \text{if } u \in (u_1, 1/2), \\ < 0 & \text{if } u \in (\ln\sqrt{2}, u_1),\end{cases}$$

which completes the proof. □

4. Proof of Theorem 3

Lemma 2. *The function*

$$h_2(x) = \frac{(\ln 2)(x\sinh x - (\cosh x)\ln\cosh x)\cosh x - (x\sinh x - (\cosh x)\ln\cosh x)^2}{x^2}$$

is strictly decreasing from $(0,\infty)$ *onto* $\left(0, \ln\sqrt{2}\right)$

Proof. We write

$$h_2(x) = \frac{(x\tanh x - \ln\cosh x)\ln 2 - (x\tanh x - \ln\cosh x)^2}{x^2/\cosh^2 x} := \frac{f_3(x)}{g_3(x)},$$

where

$$\begin{aligned} f_3(x) &= (x\tanh x - \ln\cosh x)\ln 2 - (x\tanh x - \ln\cosh x)^2, \\ g_3(x) &= \frac{x^2}{\cosh^2 x}. \end{aligned}$$

It is easy to check that

$$f_3(0) = g_3(0) = f_3(\infty) = g_3(\infty) = 0.$$

Differentiation yields

$$\begin{aligned} f_3'(x) &= \frac{x\ln 2}{\cosh^2 x} - 2(x\tanh x - \ln\cosh x)\frac{x}{\cosh^2 x} := \frac{x}{\cosh^2 x} f_4(x), \\ g_3'(x) &= 2x\frac{\cosh x - x\sinh x}{\cosh^3 x} = \frac{x}{\cosh^2 x} g_4(x), \end{aligned}$$

where

$$\begin{aligned} f_4(x) &= \ln 2 - 2(x\tanh x - \ln\cosh x), \\ g_4(x) &= 2 - 2x\tanh x; \end{aligned}$$

$$\begin{aligned} f_4'(x) &= -\frac{2x}{\cosh^2 x}, \\ g_4'(x) &= -2\frac{x + \cosh x\sinh x}{\cosh^2 x}. \end{aligned}$$

Then

$$\begin{aligned} \frac{f_3'(x)}{g_3'(x)} &= \frac{\ln 2 - 2(x\tanh x - \ln\cosh x)}{2 - 2x\tanh x} = \frac{f_4(x)}{g_4(x)}, \\ \frac{f_4'(x)}{g_4'(x)} &= \frac{x}{x + \cosh x\sinh x} = \frac{1}{1 + \sinh(2x)/(2x)}, \end{aligned}$$

where $g_4(x) \neq 0$. As shown in the proof of Lemma 1, $f_4'(x)/g_4'(x)$ is strictly decreasing on $(0,\infty)$.

Since $f_4'(x) < 0$ with $f_4(0) = \ln 2$ and $f_4(\infty) = -\ln 2$, there is an $x_1 > 0$ such that $f_4(x) > 0$ on $(0, x_1)$ and $f_4(x) < 0$ on (x_1, ∞). Likewise, the facts that $g_4'(x) < 0$ with $g_4(0) = 2$ and $g_4(\infty) = -\infty$ implies that there is an $x_2 > 0$ such that $g_4(x) > 0$ on $(0, x_2)$ and $g_4(x) < 0$ on (x_2, ∞). We claim that $x_1 < \ln 3 < x_2$. In fact, since

$$\begin{aligned} f_4(\ln 3) &= \ln 2 - \frac{8}{5}\ln 3 + 2\ln\frac{5}{3} < 0, \\ g_4(\ln 3) &= 2 - \frac{8}{5}\ln 3 > 0, \end{aligned}$$

it is deduced that $x_1 \in (0, \ln 3)$ and $x_2 \in (\ln 3, \infty)$, and therefore, $x_1 < \ln 3 < x_2$.

We next prove that $h_2 = f_3/g_3$ is strictly decreasing on $(0, \infty)$ by distinguishing two cases.

Case 1: $x \in (0, x_2)$. Due to $x_1 < \ln 3 < x_2$, we have $f_4(x_2) < 0$, $g_4(x_2) = 0$. Since $\left(f_4'/g_4'\right)' < 0$ for $x \in (0, \infty)$, $g_4 > 0$ for $x \in (0, x_2)$, by the second identity (2.3) it is seen that $H_{f_4,g_4}' = \left(f_4'/g_4'\right)' g_4 < 0$ for $x \in (0, x_2)$. On the other hand, we see that

$$H_{f_4,g_4}(x_2) = \lim_{x \to x_2^+}\left[\frac{f_4'(x)}{g_4'(x)} g_4(x) - f_4(x)\right] = -f_4(x_2) > 0. \tag{4.1}$$

Then $H_{f_4,g_4}(x) > H_{f_4,g_4}(x_2) > 0$ for $x \in (0, x_2)$. Due to $g_4'(x) < 0$, it follows from the first identity (2.2) that

$$\left(\frac{f_4}{g_4}\right)' = \frac{g_4'}{g_4^2} H_{f_4,g_4} < 0 \text{ for } x \in (0, x_2).$$

In view of $f_3(0) = g_3(0) = 0$, by Proposition 1 we find that $h_2 = f_3/g_3$ is strictly decreasing on $(0, x_2)$.

Case 2: $x \in (x_2, \infty)$. We have $f_4(x_2) < 0$, $g_4(x_2) = 0$. Since $\left(f_4'/g_4'\right)' < 0$ for $x \in (0, \infty)$, $g_4 < 0$ for $x \in (x_2, \infty)$, by the second identity (2.3) it is deduced that $H_{f_4,g_4}' = \left(f_4'/g_4'\right)' g_4 > 0$ for $x \in (x_2, \infty)$. This together with (4.1) gives that $H_{f_4,g_4}(x) > H_{f_4,g_4}(x_2) > 0$ for $x \in (x_2, \infty)$. Due to $g_4'(x) < 0$, it follows that

$$\left(\frac{f_4}{g_4}\right)' = \frac{g_4'}{g_4^2} H_{f_4,g_4} < 0 \text{ for } x \in (x_2, \infty).$$

In view of $f_3(\infty) = g_3(\infty) = 0$, by Proposition 1 we deduce that $h_2 = f_3/g_3$ is strictly decreasing on (x_2, ∞).

Taking into account Cases 1 and 2 as well the continuity of the function $g_3(x)$ at $x = x_2$, we conclude that $h_2 = f_3/g_3$ is strictly decreasing on $(0, \infty)$. An easy calculation yields $h_2(0) = \ln\sqrt{2}$ and $h_2(\infty) = 0$, and the proof is completed. □

Now we shall prove Theorem 3.

Proof of Theorem 3. Differentiation give

$$\begin{aligned} G_t'(s) &= [ut\sinh(ut) - \cosh(ut)\ln\cosh(ut)]\frac{\cosh^{1/u-1}(ut)}{u^2}\frac{\ln 2}{s\ln^2 s} \\ &= [ut\sinh(ut) - \cosh(ut)\ln\cosh(ut)]\frac{\cosh^{1/u-1}(ut)}{s\ln 2}, \end{aligned}$$

$$\begin{aligned} G_t''(s) &= \left[ut^2\cosh(ut) - t\sinh(ut)\ln\cosh(ut)\right]\frac{\ln 2}{s\ln^2 s}\frac{\cosh^{1/u-1}(ut)}{s\ln 2} \\ &\quad + [ut\sinh(ut) - \cosh(ut)\ln\cosh(ut)] \\ &\quad \times\frac{(1-u)\,ut\sinh(ut) - \cosh(ut)\ln\cosh(ut)}{s\ln 2}\frac{\cosh^{1/u-2}(ut)}{u^2}\frac{\ln 2}{s\ln^2 s} \\ &\quad - [ut\sinh(ut) - \cosh(ut)\ln\cosh(ut)]\frac{\cosh^{1/u-1}(ut)}{s^2\ln 2}. \end{aligned}$$

Letting $ut = x$ and simplifying give

$$\begin{aligned} \frac{s^2\ln^2 2}{\cosh^{1/u-2}(ut)}G_t''(s) &= u\left(x^2\cosh x - x\sinh x\ln\cosh x\right)\cosh x \\ &\quad + (x\sinh x - \cosh x\ln\cosh x) \\ &\quad \times[(1-u)\,x\sinh x - \cosh x\ln\cosh x] \\ &\quad - (\ln 2)(x\sinh x - \cosh x\ln\cosh x)\cosh x \\ \\ &= ux^2 - [(\ln 2)(x\sinh x - \cosh x\ln\cosh x)\cosh x \\ &\quad - (x\sinh x - \cosh x\ln\cosh x)^2\Big] = x^2[u - h_2(x)], \end{aligned}$$

where $h_2(x)$ is as in Lemma 2. Since $h_2(x)$ is even on $(-\infty,\infty)$, $G_t''(s) \geq (\leq)\, 0$ for all $t > 0$ if and only if

$$Q_2(t) = u - h_2(|ut|) \geq (\leq 0)$$

for $t > 0$. From Lemma 2 we find

$$Q_2'(t) = -|u|\, h_2(|ut|) > 0$$

for all $t > 0$ and

$$\begin{aligned} \lim_{t\to 0} Q_2(t) &= u - \lim_{t\to 0} h_2(|ut|) = u - \frac{1}{2}\ln 2, \\ \lim_{t\to\infty} Q_2(t) &= u - \lim_{t\to\infty} h_2(|ut|) = u. \end{aligned}$$

We conclude thus that $G_t''(s) \geq (\leq)\, 0$ for all $t > 0$ if and only if

$$u \geq \max\left\{0, \frac{1}{2}\ln 2\right\} = \frac{1}{2}\ln 2 \ \text{ or } \ u \leq \min\left\{0, \frac{1}{2}\ln 2\right\} = 0,$$

which, by the relation $u = (\ln 2)/\ln(1/s)$, implies that $e^{-2} \leq s < 1$ or $s > 1$.

When $0 < u(s) < \ln\sqrt{2}$, that is, $s \in \left(0, \ln\sqrt{2}\right)$, since $Q_2'(t) > 0$, $Q_2(0^+) = u - \ln\sqrt{2} < 0$ and $Q_2(\infty) = u > 0$, there is a $t_2 > 0$ such that $Q_2(t) < 0$, $t \in (0, t_2)$ and $Q_2(t) > 0$, $t \in (t_2, \infty)$, where t_2 is a solution of the equation

$$Q_2(t) = u - h_2(|ut|) = 0. \tag{4.2}$$

Since the function $h_2(x), (x > 0)$ is strictly decreasing, the inverse of h_2 exists and so is h_2^{-1}. Solving the Eq (4.2) for t yields

$$t = \frac{h_2^{-1}(u)}{u} = T_2(u).$$

Because that $1/u$ and $h_2^{-1}(u)$ are both positive and strictly decreasing, so is $t = T_2(u)$. This implies $u = T_2^{-1}(t)$ exists and strictly decreasing on $(0, \infty)$. It then follows that

$$\begin{aligned} t &\in (0, t_2) \iff u \in \left(T_2^{-1}(t_2), \ln\sqrt{2}\right) = \left(u_2, \ln\sqrt{2}\right), \\ t &\in (t_2, \infty) \iff u \in \left(0, T_2^{-1}(t_2)\right) = (0, u_2), \end{aligned}$$

where $u_2 = T_2^{-1}(t_2) \in \left(0, \ln\sqrt{2}\right)$. We thus deduce that $G_t''(s) < 0$ for $u \in \left(u_2, \ln\sqrt{2}\right)$ and $G_t''(s) > 0$ for $u \in (0, u_2)$. Due to $u = (\ln 2)/\ln(1/s)$, it follows that $G_t''(s) < 0$ on $u \in \left(s_2^*, e^{-2}\right)$ and $G_t''(s) > 0$ on $\left(0, s_2^*\right)$, where $s_2^* = 2^{-1/u_2}$. This completes the proof. □

5. Proof of Theorem 4

Lemma 3. *The function*

$$h_3(x) = \frac{x\tanh x - \ln(\cosh x)}{x^2/\cosh^2 x}\ln 2$$

is strictly increasing from $(0, \infty)$ *onto* $\left(\ln\sqrt{2}, \infty\right)$.

Proof. As shown in Lemmas 1 and 2, $x\tanh x - \ln\cosh x = f_2(x)$ and $x^2/\cosh^2 x = g_3(x)$ with $f_2(0) = g_3(0) = 0$. Since $f_2'(x) = x/\cosh^2 x > 0$, we have $f_2(x) > f_2(0) = 0$ for $x > 0$. Note that

$$\begin{aligned} \frac{g_3'(x)}{f_2'(x)} &= 2 - 2x\tanh x, \\ \left[\frac{g_3'(x)}{f_2'(x)}\right]' &= -2\frac{x + \cosh x\sinh x}{\cosh^2 x} < 0. \end{aligned}$$

By Proposition 1 we deduce that $g_3(x)/f_2(x)$ is strictly decreasing on $(0, \infty)$, which, due to $g_3(x)/f_2(x) > 0$, implies that $h_3(x) = [f_2(x)/g_3(x)]\ln 2$ is strictly increasing on $(0, \infty)$. A simple computation yields

$$\lim_{x\to 0} h_3(x) = \frac{1}{2}\ln 2 \text{ and } \lim_{x\to\infty} h_3(x) = \infty,$$

which completes the proof. □

Based on Lemma 3, we now check Theorem 4.

Proof of Theorem 4. Differentiation yields

$$\begin{aligned} [\ln G_t(s)]' &= [ut\tanh(ut) - \ln\cosh(ut)]\frac{1}{u^2}\frac{\ln 2}{s\ln^2 s} \\ &= \frac{ut\tanh(ut) - \ln\cosh(ut)}{s\ln 2}, \end{aligned}$$

$$[\ln G_t(s)]'' = \frac{ut^2}{\cosh^2(ut)}\frac{\ln 2}{s\ln^2 s}\frac{1}{s\ln 2} - \frac{ut\tanh(ut) - \ln\cosh(ut)}{s^2\ln 2}$$

$$= \frac{(ut)^2}{\cosh^2(ut)}\frac{u}{s^2\ln^2 2} - \frac{ut\tanh(ut) - \ln\cosh(ut)}{s^2\ln 2},$$

Letting $ut = x$ and simplifying lead to

$$\frac{s^2\ln^2 2}{x^2}\left(\cosh^2 x\right)[\ln G_t(s)]'' = u - \frac{x\tanh x - \ln\cosh x}{x^2/\cosh^2 x}\ln 2 = u - h_3(x),$$

where $h_3(x)$ is given in Lemma 3. Since $h_3(x)$ is even on $(-\infty, \infty)$, $[\ln G_t(s)]'' \geq (\leq) 0$ for $t > 0$ if and only if

$$Q_3(t) = u - h_3(|ut|) \geq (\leq 0)$$

for $t > 0$. From Lemma 3 we get

$$Q_3'(t) = -|u|\, h_3(|ut|) < 0$$

for $t > 0$ and

$$\begin{aligned}\lim_{t\to 0} Q_3(t) &= u - \lim_{t\to 0} h_3(|ut|) = u - \frac{1}{2}\ln 2,\\ \lim_{t\to\infty} Q_3(t) &= u - \lim_{t\to\infty} h_3(|ut|) = -\infty.\end{aligned}$$

We conclude thus that $[\ln G_t(s)]'' \leq 0$ for all $t > 0$ if and only if $u \leq \ln\sqrt{2}$, which, by the relation $u = (\ln 2)/\ln(1/s)$, implies that $0 < s \leq e^{-2}$ or $s > 1$. This completes the proof. □

6. Several new inequalities

Using Theorems 2 and 4, we get the following corollary.

Corollary 1. *Suppose $w, v > 0$, $w \neq v$. If $p < r < q \leq \ln\sqrt{2}$, then the double inequality*

$$A_p(w,v)^{1-\beta_0} A_q(w,v)^{\beta_0} < A_r(w,v) < (1-\alpha_0)A_p(w,v) + \alpha_0 A_q(w,v) \tag{6.1}$$

holds, where

$$\alpha_0 = \frac{r-p}{q-p} \quad \text{and} \quad \beta_0 = \frac{2^{-1/r} - 2^{-1/p}}{2^{-1/q} - 2^{-1/p}}. \tag{6.2}$$

The second inequality of (6.1) is reversed if $1/2 \leq p < r < q$.

Proof. By Theorem 4, the function $s \mapsto \ln A_{u(s)}(w,v)$ is concave on $(0, e^{-2}] \cup (1, \infty)$. Then for $s_i \in (0, e^{-2}]$ or $s_i \in (1, \infty)$, $i = 1, 2, 3$, using the property of convex functions we have

$$\frac{\ln A_{u(s_2)}(w,v) - \ln A_{u(s_1)}(w,v)}{s_2 - s_1} > \frac{\ln A_{u(s_3)}(w,v) - \ln A_{u(s_1)}(w,v)}{s_3 - s_1}, \tag{6.3}$$

which is equivalent to

$$\ln A_{u(s_2)}(w,v) > \frac{s_3 - s_2}{s_3 - s_1}\ln A_{u(s_1)}(w,v) + \frac{s_2 - s_1}{s_3 - s_1}\ln A_{u(s_3)}(w,v). \tag{6.4}$$

Let $(u(s_1), u(s_2), u(s_3)) = (p, r, q)$. Then by the relation $u(s) = (\ln 2)/\ln(1/s)$ we get $(s_1, s_2, s_3) = \left(2^{-1/p}, 2^{-1/r}, 2^{-1/q}\right)$ with $\ln\sqrt{2} \leq p < r < q$. The inequality (6.4) thus becomes to the left hand side inequality of (6.1).

From Theorem 2, the function $u \mapsto A_u(w,v)$ is convex on $\left(-\infty, \ln\sqrt{2}\right)$ and concave on $(1/2,\infty)$, where $w,v>0$, $w\neq v$. Then for $p<r<q\leq \ln\sqrt{2}$ the right hand side inequality of (6.1) holds, which is reversed if $1/2\leq p<r<q$. This completes the proof. □

Using Theorems 1 and 3, we obtain the following corollary.

Corollary 2. *Suppose $w,v>0$, $w\neq v$. If $\ln\sqrt{2}\leq p<r<q$, then the double inequality*

$$A_p(w,v)^{1-\alpha_0}A_q(w,v)^{\alpha_0}<A_r(w,v)<(1-\beta_0)A_p(w,v)+\beta_0A_q(w,v) \tag{6.5}$$

holds, where α_0 and β_0 are given in (6.2) are the best constants. The double inequality (6.5) is reversed if $p<r<q<0$ with the best constants α_0 and β_0.

Proof. By Theorem 1 the function $u\mapsto \ln A_u(w,v)$ is convex on $(-\infty,0)$ and concave on $(0,\infty)$. This implies that, for $0<p<r<q$ $(p<r<q<0)$, the inequality

$$\frac{q-r}{q-p}\ln A_p(w,v)+\frac{r-p}{q-p}\ln A_q(w,v)<(>)\ln A_r(w,v)$$

holds, that is,

$$A_p(w,v)^{1-\alpha_0}A_q(w,v)^{\alpha_0}<(>)A_r(w,v).$$

By Theorem 3, the function $s\mapsto A_{u(s)}(w,v)$ is convex on $[e^{-2},1)$ and concave on $(1,\infty)$. Then for $s_i\in[e^{-2},1)$, $i=1,2,3$, with $s_1<s_2<s_3$, by the property of concave functions we have

$$\frac{A_{u(s_2)}(w,v)-A_{u(s_1)}(w,v)}{s_2-s_1}<\frac{A_{u(s_3)}(w,v)-A_{u(s_1)}(w,v)}{s_3-s_1}, \tag{6.6}$$

which is equivalent to

$$A_{u(s_2)}(w,v)<\frac{s_3-s_2}{s_3-s_1}A_{u(s_1)}(w,v)+\frac{s_2-s_1}{s_3-s_1}A_{u(s_3)}(w,v). \tag{6.7}$$

Let $(u(s_1),u(s_2),u(s_3))=(p,r,q)$. Then by the relation $u(s)=(\ln 2)/\ln(1/s)$ we get $(s_1,s_2,s_3)=\left(2^{-1/p},2^{-1/r},2^{-1/q}\right)$ with $\ln\sqrt{2}\leq p<r<q$. The inequality (6.7) thus becomes to the right hand side inequality of (6.5).

If $s_i\in(1,\infty)$, $i=1,2,3$, with $s_1<s_2<s_3$, by the property of concave functions, the inequality (6.6) is reversed, and so is the right hand side inequality of (6.5) if $p<r<q<0$.

Without loss of generality, we suppose that $0<w<v$. Then $\varsigma=\ln\sqrt{v/w}>0$. Due to

$$\begin{aligned}&\lim_{v\to w}\frac{\ln A_r(w,v)-\ln A_p(w,v)}{\ln A_q(w,v)-\ln A_p(w,v)}\\ =\ &\lim_{\varsigma\to 0}\frac{\ln\cosh^{1/r}(r\varsigma)-\ln\cosh^{1/p}(p\varsigma)}{\ln\cosh^{1/q}(q\varsigma)-\ln\cosh^{1/p}(p\varsigma)}=\frac{r-p}{q-p}=\alpha_0,\end{aligned}$$

$$\begin{aligned}&\lim_{v\to\infty}\frac{A_r(w,v)-A_p(w,v)}{A_q(w,v)-A_p(w,v)}\\ =\ &\lim_{\varsigma\to\infty}\frac{\cosh^{1/r}(r\varsigma)-\cosh^{1/p}(p\varsigma)}{\cosh^{1/q}(q\varsigma)-\cosh^{1/p}(p\varsigma)}=\frac{2^{-1/r}-2^{-1/p}}{2^{-1/q}-2^{-1/p}}=\beta_0\end{aligned}$$

for $\max\{p,q,r\}<0$ or $\min\{p,q,r\}>0$, α_0 and β_0 are the best. This completes the proof. □

Similarly, by means of Theorems 1 and 4 we can prove the following corollary, all the details of proof are omitted here.

Corollary 3. *Suppose $w, v > 0$, $w \neq v$. If $p < r < q < 0$, then*

$$A_p(w,v)^{1-\beta_0} A_q(w,v)^{\beta_0} < A_r(w,v) < A_p(w,v)^{1-\alpha_0} A_q(w,v)^{\alpha_0},$$

where α_0 and β_0 are given in (6.2).

By means of Corollaries 1 and 2, we have

Corollary 4. *Suppose $p, q, r \in \mathbb{R}$, $p < r < q$. (i) If $p \geq 1/2$, then for $w, v > 0$, $w \neq v$ the double mean-inequality*

$$(1-\beta) A_p(w,v) + \beta A_q(w,v) > A_r(w,v) > (1-\alpha) A_p(w,v) + \alpha A_q(w,v) \tag{6.8}$$

is valid if and only if

$$\alpha \leq \alpha_0 = \frac{r-p}{q-p} \text{ and } \beta \geq \beta_0 = \frac{2^{-1/r} - 2^{-1/p}}{2^{-1/q} - 2^{-1/p}}.$$

(ii) If $q < 0$, then for $w, v > 0$, $w \neq v$ the double inequality (6.8) is reversed if and only if $\alpha \geq \alpha_0$ and $\beta \leq \beta_0$.

Proof. (i) **Necessity**. Since $w, v > 0$ with $w \neq v$, we suppose $v > w > 0$. Then $\varsigma = \ln \sqrt{v/w} > 0$. If the first inequality of (6.8) holds for all $v > w > 0$, then

$$\alpha \leq \lim_{\varsigma \to 0} \frac{\cosh^{1/r}(r\varsigma) - \cosh^{1/p}(p\varsigma)}{\cosh^{1/q}(q\varsigma) - \cosh^{1/p}(p\varsigma)} = \frac{r-p}{q-p} = \alpha_0.$$

If the second inequality of (6.8) is valid for $v > w > 0$, then

$$\beta \geq \lim_{\varsigma \to \infty} \frac{\cosh^{1/r}(r\varsigma) - \cosh^{1/p}(p\varsigma)}{\cosh^{1/q}(q\varsigma) - \cosh^{1/p}(p\varsigma)} = \frac{2^{-1/r} - 2^{-1/p}}{2^{-1/q} - 2^{-1/p}} = \beta_0.$$

Sufficiency. By Corollaries 1 and 2, the reverse of the right hand side inequality in (6.1) for $\alpha = \alpha_0$ and the inequality (6.5) for $\beta = \beta_0$ both hold if $1/2 \leq p < r < q$, that is, for $w, v > 0$, $w \neq v$ and $(\alpha, \beta) = (\alpha_0, \beta_0)$, the double inequality (6.8) is valid. It is easy to find that, for $\alpha \leq \alpha_0$,

$$A_r(w,v) > (1-\alpha_0) A_p(w,v) + \alpha_0 A_q(w,v) \geq (1-\alpha) A_p(w,v) + \alpha A_q(w,v),$$

and for $\beta \geq \beta_0$,

$$(1-\beta) A_p(w,v) + \beta A_q(w,v) > (1-\beta_0) A_p(w,v) + \beta_0 A_q(w,v) > A_r(w,v).$$

This proves the sufficiency.

(ii) The second assertion of this theorem can be proven in a similar way. This completes the proof. □

Remark 4. *Clearly, Corollary 4 gives an answer to Problem 2.*

7. Conclusions

In this paper, we completely described the convexity of $u \mapsto A_u(w, v)$ on $\mathbb{R}$ and $s \mapsto A_{u(s)}(w, v)$, $\ln A_{u(s)}(w, v)$ with $u(s) = (\ln 2) / \ln(1/s)$ on $(0, \infty)$ by using two tools. From which we obtained several new sharp inequalities involving the power means (Corollaries 1–4), where Corollary 4 gives an answer to Problem 2. Moreover, we gave another new proof of Problem 1.

Final inspired by Theorems 1–4, we propose the following problem.

Problem 3. *For all $w, v > 0$, $w \neq v$, determine the best $p \in \mathbb{R}$ such that the functions $p \mapsto L_p(w, v)$, $I_p(w, v)$ are convex or concave .*

The second problem is inspired by Corollary 3 and Problem 2.

Problem 4. *Suppose $p, q, r \in \mathbb{R}$ with $p < r < q$ and $v, w > 0$ with $v \neq w$. Determine the best $\alpha, \beta \in (0, 1)$ with $\alpha < \beta$ such that the double inequality*

$$A_p(w, v)^{1-\beta} A_q(w, v)^{\beta} < A_r(w, v) < A_p(w, v)^{1-\alpha} A_q(w, v)^{\alpha}$$

is valid.

It was shown in [29, Lemma 6] (see also [30, 31]) that the function $p \mapsto 2^{1/p} A_p(w, v)$ is strictly decreasing and log-convex on $(0, \infty)$. Motivated by this, it is natural to propose the following problem.

Problem 5. *Describe the convexity of the function $p \mapsto 2^{1/p} A_p(w, v)$ on $(-\infty, 0)$ and $(0, \infty)$.*

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Conflict of interest

The authors declare no conflict of interest.

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