Research article

Properties of the power-mean and their applications

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Abstract: Suppose $w, v > 0$, $w \neq v$ and $A_u (w, v)$ is the $u$-order power mean (PM) of $w$ and $v$. In this paper, we completely describe the convexity of $u \mapsto A_u (w, v)$ on $\mathbb{R}$ and $s \mapsto A_{u(s)} (w, v)$ with $u(s) = (\ln 2) / \ln (1/s)$ on $(0, \infty)$. These yield some new inequalities for PMs, and give an answer to an open problem.

Keywords: power mean; power-type mean; convexity; inequality

Mathematics Subject Classification: 26E60, 26A51

1. Introduction

A function $M : \mathbb{R}_+^2 \mapsto \mathbb{R}$ is called a bivariate mean (BM) if for all $w, v > 0$

$$\min (w, v) \leq M (w, v) \leq \max (w, v)$$

is valid. A BM is symmetric if for all $w, v > 0$

$$M (w, v) = M (v, w)$$

is valid. It is said to be homogeneous (of degree one) if for all $\lambda, w, v > 0$

$$M (\lambda w, \lambda v) = \lambda M (w, v)$$

is valid. If a BM $M$ is differentiable on $\mathbb{R}_+^2$, then the function $M_u : \mathbb{R}_+^2 \mapsto \mathbb{R}$ defined by

$$M_u (w, v) = M^{1/u} (w^u, v^u) \text{ if } u \neq 0 \text{ and } M_0 (w, v) = w^{M_1 (1,1)} v^{M_1 (1,1)}, \quad (1.1)$$
is called “\( u \)-order \( M \) mean”, where \( M_x (x, y) \), \( M_y (x, y) \) are the first-order partial derivatives in regard to the first and second components of \( M(x, y) \), respectively (see [1]). For example, the arithmetic mean (AM), logarithmic mean (LM) and identric mean (IM) are given by

\[
A(w, v) = \frac{w + v}{2}, \quad L(w, v) = \frac{w - v}{\ln w - \ln v}, \quad I(w, v) = e^{-1 \left( \frac{v^u}{w^u} \right)^{1/(v-w)}},
\]

respectively, then

\[
A_u(w, v) = \left( \frac{w^u + v^u}{2} \right)^{1/u} \quad \text{if} \quad u \neq 0 \quad \text{and} \quad A_0(w, v) = \sqrt{wv},
\]

\[
L_u(w, v) = \left( \frac{w^u - v^u}{u(\ln w - \ln v)} \right)^{1/p} \quad \text{if} \quad u \neq 0 \quad \text{and} \quad L_0(w, v) = \sqrt{wv},
\]

\[
I_u(w, v) = e^{-1/u} \left( \frac{v^u}{w^u} \right)^{1/(v-w)} \quad \text{if} \quad u \neq 0 \quad \text{and} \quad I_0(w, v) = \sqrt{wv}
\]

are \( u \)-order AM, \( u \)-order LM and \( u \)-order IM, respectively. As usual, the \( u \)-order AM is still called \( u \)-order PM. Correspondingly, since the form of \( M_u \) is similar to PM \( A_u \), it is also known simply as “power-type mean”. More general means than power-type mean including Stolarsky means, Gini means, and two-parameters functions, etc., which can be seen in [2–7].

For those means with parameters, there are many nice properties including monotonicity, (log-) convexity, comparability, additivity, stability and inequalities, which can be found in [8–17].

In this paper, we are interested in the properties of the PM \( A_u \). As is well-known that \( u \mapsto A_u(w, v) \) is increasing on \( \mathbb{R} \) (see [5]). The log-convexity of \( u \mapsto A_u(w, v) \), \( L_u(w, v) \) and \( I_u(w, v) \) is a direct consequence of [9, Conclusion 1.1] when \( q = 0 \), that is,

**Theorem 1.** The functions \( u \mapsto A_u(w, v) \), \( L_u(w, v) \) and \( I_u(w, v) \) are log-convex on \((-\infty, 0)\) and log-concave on \((0, \infty)\).

The log-convexity of the function \( u \mapsto A_u(w, v) \) was reproved in [19] by Begea, Bukor and Tóth. The authors proposed an open problem on the convexity of the function \( u \mapsto A_u(w, v) \):

**Problem 1.** Prove that

\[
\inf_{w,v > 0} \left\{ u : A_u(w, v) \text{ is concave for variable } u \in \mathbb{R} \right\} = \frac{1}{2} \ln 2,
\]

\[
\sup_{w,v > 0} \left\{ u : A_u(w, v) \text{ is convex for variable } u \in \mathbb{R} \right\} = \frac{1}{2}.
\]

Problem 1 was proven by Matejíčka in [20]. In 2016, Raïsouli and Sándor [16, Problem 1] proposed the following problem.

**Problem 2.** Let \( p, q, r \in \mathbb{R} \) with \( q > r > p \). Are there \( 0 < \beta, \alpha < 1 \) with \( \beta > \alpha \), such that the double inequality

\[
(1 - \alpha) A_p + \alpha A_q < A_r < (1 - \beta) A_p + \beta A_q
\]

holds? If it is positive, what are the best \( \beta \) and \( \alpha \)?
Clearly, this problem is partly related to the convexity of $u \mapsto A_u(w, v)$. Motivated by Problem 2, the main purpose of this paper is to investigate completely the convexity of $u \mapsto A_u(w, v)$ on $\mathbb{R}$ and $s \mapsto A_{u,s}(w, v)$ with $u(s) = (\ln 2) / \ln (1/s)$ on $(0, \infty)$. As applications, some new inequalities for power means are established, and an answer to Problem 2 is given. Finally, three problems on the convexity of certain power-type means and inequalities are proposed.

It should be noted that a homogeneous BM can be represented by the exponential functions. If $M(x, y)$ is a HM of positive arguments $x$ and $y$, then $M(x, y)$ can be represented as

$$M(x, y) = \sqrt{xy}M(e^x, e^y),$$

where $t = (1/2) \ln (x/y)$. Further, if $M(x, y)$ is symmetric, then $M(x, y)$ can be expressed in terms of hyperbolic functions (see [18, Lemma 3]). For example, in view of symmetry, we suppose $v > w > 0$. Then we find $t = (1/2) \ln (v/w) > 0$. Thus the PM $A_u(w, v)$, $u$-order LM $L_u(w, v)$ and $u$-order IM $I_u(w, v)$ can be represented as

$$A_u(w, v) = \cosh^{1/u}(ut), \quad L_u(w, v) = \left[ \sinh(ut) \right]^{1/u}, \quad I_u(w, v) = \exp \left[ \frac{t \tanh(ut)}{u} - 1 \right]$$

if $u \neq 0$.

The first result of the paper is the following theorem.

**Theorem 2.** The function $u \mapsto A_u(w, v)$ is convex on $(-\infty, \ln \sqrt{2})$ and concave on $(1/2, \infty)$ for all $w, v > 0$ with $w \neq v$. While $u \in (\ln \sqrt{2}, 1/2)$, the function $u \mapsto A_u(w, v)$ is concave then convex. Equivalently, the function

$$F_t(u) = \cosh^{1/u}(ut)$$

is convex (concave) for all $t > 0$ if and only if $u \leq \ln \sqrt{2}$ ($u \geq 1/2$). While $\ln \sqrt{2} < u < 1/2$, there is a $u_1 \in (\ln \sqrt{2}, 1/2)$ such that $F_t(u)$ is concave on $[\ln \sqrt{2}, u_1]$ and convex on $(u_1, 1/2)$.

**Remark 1.** Theorem 2 not only gives an answer to Problem 1, but also describes completely the convexity of the function $u \mapsto A_u(w, v)$ on $\mathbb{R}$.

**Remark 2.** By Theorems 1 and 2, we see that the function $u \mapsto A_u(w, v)$ has the following (log-)

<table>
<thead>
<tr>
<th>$u$</th>
<th>$(-\infty, 0)$</th>
<th>$(0, \ln \sqrt{2})$</th>
<th>$(\ln \sqrt{2}, 1/2)$</th>
<th>$(1/2, \infty)$</th>
</tr>
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<tbody>
<tr>
<td>$A_u$</td>
<td>$\cup$</td>
<td>$\cup$</td>
<td>$\cap$</td>
<td>$\cap$</td>
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<tr>
<td>$\ln A_u$</td>
<td>$\cup$</td>
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</tr>
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</table>

where and in what follows the symbols “$\cup$” and “$\cap$” denote the given function are convex and concave, “$\cap \cup$” and “$\cup \cap$” denote the given function are “concave then convex” and “convex then concave”, respectively.

The second and third results of the paper are the following theorems.

**Theorem 3.** Suppose $w, v > 0$ and $w \neq v$. The function $s \mapsto A_{u,s}(w, v)$ with $u = u(s) = (\ln 2) / \ln (1/s)$ is convex on $(e^{-2}, 1)$ and concave on $(1, \infty)$. While $s \in (0, e^{-2})$, the function $s \mapsto A_{u,s}(w, v)$ is convex then concave. Equivalently, the function

$$G_t(s) = \cosh^{1/u}(ut), \quad \text{where} \quad u = \frac{\ln 2}{\ln (1/s)}$$
is convex (concave) for all $t > 0$ if and only if $s \in (e^{-2}, 1)$ ($s \in (1, \infty)$). While $s \in (0, e^{-2})$, there is a $s_2^* \in (0, e^{-2})$ such that $G_t(s)$ is convex on $(0, s_2^*)$ and concave on $(s_2^*, e^{-2})$.

**Theorem 4.** Suppose $w, v > 0$ and $a \neq b$. The function $s \mapsto A_{u(s)}(w, v)$ with $u(s) = (\ln 2) / \ln (1/s)$ is log-concave on $(0, e^{-2}) \cup (1, \infty)$. Equivalently, the function $G_t(s)$ is log-concave for all $t > 0$ if and only if $s \in (0, e^{-2}) \cup (1, \infty)$.

**Remark 3.** By Theorems 3 and 4, the function $s \mapsto A_{u(s)}(w, v)$ has the following (log-) convexity:

<table>
<thead>
<tr>
<th>$s$</th>
<th>$(0, e^{-2})$</th>
<th>$(e^{-2}, 1)$</th>
<th>$(1, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{u(s)}$</td>
<td>$\cap$</td>
<td>$\cup$</td>
<td>$\cap$</td>
</tr>
<tr>
<td>$\ln A_{u(s)}$</td>
<td>$\cap$</td>
<td>$\cap$</td>
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2. Tools

To prove the lemmas listed in Sections 3–5, we need two tools. The first is the so-called L’Hospital Monotone Rule (LMR), which appeared in [21] (see also [22]).

**Proposition 1.** Suppose $-\infty \leq a < b \leq \infty$, $\phi$ and $\psi$ are differentiable functions on $(a, b)$. Suppose also the derivative $\psi'$ is nonzero and does not change sign on $(a, b)$, and $\phi(a^+) = \psi(a^+) = 0$ or $\phi(b^-) = \psi(b^-) = 0$. If $\phi'/\psi'$ is increasing (decreasing) on $(a, b)$ then so is $\phi/\psi$.

Before stating the second tool, we present first an important function $H_{\phi, \psi}$. Assume that $\phi$ and $\psi$ are differentiable functions on $(a, b)$ with $\psi' \neq 0$, where $-\infty \leq a < b \leq \infty$. It was introduced by Yang in [23, Eq (2.1)] that

$$H_{\phi, \psi} := \frac{\phi'}{\psi'} \psi - \phi,$$  \hspace{1cm} (2.1)

which we call Yang’s H-function. This function has some good properties, see [23, Properties 1 and 2], and plays an important role in the proof of a monotonicity criterion for the quotient of two functions, see for example, [24–28].

To study the monotonicity of the ratio $\phi/\psi$ on $(a, b)$, Yang [23, Property 1] presented two identities in term of $H_{\phi, \psi}$, which state that, if $\phi$ and $\psi$ are twice differentiable with $\psi \psi' \neq 0$ on $(a, b)$, then

$$\left(\frac{\phi}{\psi}\right)' = \frac{\psi'}{\psi^2} \left(\frac{\phi'}{\psi'} \psi - \phi\right) = \frac{\psi'}{\psi^2} H_{\phi, \psi},$$  \hspace{1cm} (2.2)

$$H'_{\phi, \psi} = \left(\frac{\phi'}{\psi'}\right)'.$$  \hspace{1cm} (2.3)

3. Proof of Theorem 2

In order to prove Theorem 2, we need the following lemma.

**Lemma 1.** Let $h_1(x) = f_1(x) / g_1(x)$, where

$$f_1(x) = (x \tanh x - \ln (\cosh x))^2,$$  \hspace{1cm} (3.1)
Then $h_1 (x)$ is strictly decreasing from $(0, \infty)$ onto $\left( \ln \sqrt{2}, 1/2 \right)$.

**Proof.** Differentiation yields

\[
\begin{align*}
g_1' (x) &= 2x \tanh x - \frac{x^2}{\cosh^2 x} - 2 \ln (\cosh x), \\
g_2' (x) &= 2x^2 \sinh x - \frac{2x}{\cosh^2 x},
\end{align*}
\]

where

\[
\begin{align*}
f_2 (x) &= x \tanh x - \ln \cosh x, \\
g_2 (x) &= x \tanh x;
\end{align*}
\]

\[
\begin{align*}
f_2' (x) &= \frac{x}{\cosh^2 x}, \\
g_2' (x) &= \frac{x + \cosh x \sinh x}{\cosh^2 x}.
\end{align*}
\]

Then

\[
\begin{align*}
f_2' (x) &= \frac{x}{g_2' (x)}, \\
g_2' (x) &= \frac{x + \cosh x \sinh x}{1 + \sinh (2x) / (2x)}.
\end{align*}
\]

Clearly, for $x \in (0, \infty)$, $g_1' (x) > 0$, and hence, $g_1 (x) > g_1 (0) = 0$. Since $\sinh (2x) / (2x)$ is strictly increasing for $x \in (0, \infty)$, it is readily seen that for $x \in (0, \infty)$, the function $f_2' (x) / g_2' (x)$ is strictly decreasing. Due to $f_2 (0) = g_2 (0) = 0$, so is $f_2 (x) / g_2 (x)$ by Proposition 1. Similarly, in view of $f_1 (0) = g_1 (x) = 0$, so is $f_1 (x) / g_1 (x) = h_1 (x)$ using Proposition 1 again. An easy computation gives

\[
\lim_{x \to 0} \frac{f_2 (x)}{g_1 (x)} = \frac{1}{2} \quad \text{and} \quad \lim_{x \to \infty} \frac{f_1 (x)}{g_1 (x)} = \frac{1}{2} \ln 2,
\]

thereby completing the proof. \hfill \Box

Now we shall prove Theorem 2.

**Proof of Theorem 2.** Differentiation yields

\[
F_i' (u) = \frac{t}{u} \cosh^{1/u-1} (ut) \sinh (ut) - \frac{1}{u^2} \cosh^{1/u} (ut) \ln \cosh (ut),
\]

\[
F_i'' (u) = \frac{t}{u^3} \sinh (ut) [(1 - u) (ut) \sinh (ut) - \cosh (ut) \ln \cosh (ut)] \cosh^{1/u-2} (ut)
\]

\[
+ \frac{t}{u^2} [ut \cosh (ut) - \sinh (ut)] \cosh^{1/u-1} (ut)
\]

\[
- \frac{1}{u^4} [ut \sinh (ut) - \cosh (ut) \ln \cosh (ut)] \cosh^{1/u-1} (ut) \ln \cosh (ut)
\]

\[
- \frac{1}{u^4} [ut \tanh (ut) - 2 \ln \cosh (ut)] \cosh^{1/u} (ut).
\]
Letting \( ut = x \) and simplifying give

\[
\frac{u^4}{\cosh^{1/u-2}(ut)} F''_t(u) = x(\sinh x) [(1 - u)x \sinh x - \cosh x \ln \cosh x] \\
+ ux (x \cosh x - \sinh x) \cosh x \\
- (x \sinh x - \cosh x \ln \cosh x) \cosh x \ln \cosh x \\
- u(x \tanh x - 2 \ln \cosh x) \cosh^2 x
\]

\[
= u \left[ 2 \cosh^2 x \ln \cosh x + x^2 - 2 x \cosh x \sinh x \right] \\
+ (x \sinh x - \cosh x \ln \cosh x)^2 = - [u - h_1(x)] g_1(x) \cosh^2 x,
\]

where \( h_1(x) \) and \( g_1(x) \) are given in Lemma 1. Since \( h_1(x) \) and \( g_1(x) \) are even on \((-\infty, \infty)\) and \( g_1(x) \equiv g_1(|x|) > 0 \) shown in Lemma 1, \( F''_t(u) \geq (\leq) 0 \) for \( t > 0 \) if and only if

\[
Q_1(t) = u - h_1(|ut|) \leq (\geq) 0.
\]

From Lemma 1 we find

\[
Q'_1(t) = -|u|h'_1(|ut|) > 0
\]

for all \( t > 0 \) and

\[
\lim_{t \to 0} Q_1(t) = u - \lim_{t \to 0} h_1(|ut|) = u - \frac{1}{2},
\]

\[
\lim_{t \to \infty} Q_1(t) = u - \lim_{t \to \infty} h_1(|ut|) = u - \frac{1}{2} \ln 2.
\]

We conclude thus that \( F''_t(u) > (<) 0 \) for all \( t > 0 \) if and only if

\[
u \leq \min \left\{ \frac{1}{2}, \frac{1}{2} \ln 2 \right\} = \frac{1}{2} \ln 2 \text{ or } u \geq \max \left\{ \frac{1}{2}, \frac{1}{2} \ln 2 \right\} = \frac{1}{2}.
\]

When \( \ln \sqrt{2} < u < 1/2 \), since \( Q'_1(t) > 0 \) with \( Q_1(0^+) = u - 1/2 < 0 \) and \( Q_1(\infty) = u - \ln \sqrt{2} > 0 \), there is a \( t_1 = t_1(u) \) such that \( Q_1(t) < 0 \) on \((0, t_1)\) and \( Q_1(t) > 0 \) on \((t_1, \infty)\), where \( t_1 \) is a solution of the equation

\[
Q_1(t) = u - h_1(|ut|) = 0.
\]

Since for \( x \in (0, \infty) \), the function \( h_1(x) \) is strictly decreasing, the inverse of \( h_1 \) exists and so is \( h_1^{-1} \).

Solving the equation (3.3) for \( t \) yields

\[
t = \frac{h_1^{-1}(u)}{u} = T_1(u).
\]

Noting that \( 1/u \) and \( h_1^{-1}(u) \) are both positive and decreasing, so is \( t = T_1(u) \). This implies \( u = T_1^{-1}(t) \) exists and strictly decreasing on \((0, \infty)\). It then follows that

\[
t \in (0, t_1) \iff u \in \left( T_1^{-1}(t_1), 1/2 \right) = (u_1, 1/2),
\]

\[
t \in (t_1, \infty) \iff u \in \left( \ln \sqrt{2}, T_1^{-1}(t_1) \right) = \left( \ln \sqrt{2}, u_1 \right),
\]

where \( u_1 = T_1^{-1}(t_1) \).

We thus arrive at that

\[
F''_t(u) \begin{cases} > 0 & \text{if } u \in (u_1, 1/2), \\
< 0 & \text{if } u \in (\ln \sqrt{2}, u_1),
\end{cases}
\]

which completes the proof. \( \square \)
4. Proof of Theorem 3

Lemma 2. The function

\[ h_2(x) = \frac{(\ln 2) (x \sinh x - (\cosh x) \ln \cosh x) \cosh x - (x \sinh x - (\cosh x) \ln \cosh x)^2}{x^2} \]

is strictly decreasing from \((0, \infty)\) onto \((0, \ln \sqrt{2})\).

Proof. We write

\[ h_2(x) = \frac{(x \tanh x - \ln \cosh x) \ln 2 - (x \tanh x - \ln \cosh x)^2}{x^2 / \cosh^2 x} := \frac{f_3(x)}{g_3(x)}, \]

where

\[ f_3(x) = (x \tanh x - \ln \cosh x) \ln 2 - (x \tanh x - \ln \cosh x)^2, \]
\[ g_3(x) = \frac{x^2}{\cosh^2 x}. \]

It is easy to check that

\[ f_3(0) = g_3(0) = f_3(\infty) = g_3(\infty) = 0. \]

Differentiation yields

\[ f'_3(x) = \frac{x \ln 2}{\cosh^2 x} - 2 (x \tanh x - \ln \cosh x) \frac{x}{\cosh^2 x} := \frac{x}{\cosh^2 x} f'_4(x), \]
\[ g'_3(x) = 2x \frac{\cosh x - x \sinh x}{\cosh^3 x} = \frac{x}{\cosh^2 x} g'_4(x), \]

where

\[ f_4(x) = \ln 2 - 2 (x \tanh x - \ln \cosh x), \]
\[ g_4(x) = 2 - 2x \tanh x; \]

\[ f'_4(x) = -\frac{2x}{\cosh^2 x}, \]
\[ g'_4(x) = -2 \frac{x + \cosh x \sinh x}{\cosh^2 x}. \]

Then

\[ \frac{f'_3(x)}{g'_3(x)} = \frac{\ln 2 - 2 (x \tanh x - \ln \cosh x)}{2 - 2x \tanh x} = \frac{f_4(x)}{g_4(x)}, \]
\[ \frac{f'_4(x)}{g'_4(x)} = \frac{x}{x + \cosh x \sinh x} = \frac{1}{1 + \sinh (2x) / (2x)}, \]

where \(g_4(x) \neq 0\). As shown in the proof of Lemma 1, \(f'_4(x) / g'_4(x)\) is strictly decreasing on \((0, \infty)\).
Since \( f'_3(x) < 0 \) with \( f_3(0) = \ln 2 \) and \( f_4(\infty) = -\ln 2 \), there is an \( x_1 > 0 \) such that \( f_4(x) > 0 \) on \((0,x_1)\) and \( f_4(x) < 0 \) on \((x_1,\infty)\). Likewise, the facts that \( g'_4(x) < 0 \) with \( g_4(0) = 2 \) and \( g_4(\infty) = -\infty \) implies that there is an \( x_2 > 0 \) such that \( g_4(x) > 0 \) on \((0,x_2)\) and \( g_4(x) < 0 \) on \((x_2,\infty)\). We claim that \( x_1 < \ln 3 < x_2 \). In fact, since

\[
\begin{align*}
f_4(\ln 3) &= \ln 2 - \frac{8}{5} \ln 3 + 2 \ln \frac{5}{3} < 0, \\
g_4(\ln 3) &= 2 - \frac{8}{5} \ln 3 > 0,
\end{align*}
\]

it is deduced that \( x_1 \in (0, \ln 3) \) and \( x_2 \in (\ln 3, \infty) \), and therefore, \( x_1 < \ln 3 < x_2 \).

We next prove that \( h_2 = f_3/g_3 \) is strictly decreasing on \((0,\infty)\) by distinguishing two cases.

**Case 1:** \( x \in (0,x_2) \). Due to \( x_1 < \ln 3 < x_2 \), we have \( f_4(x_2) < 0 \), \( g_4(x_2) = 0 \). Since \( \left(f'_4/g'_4\right)^{\prime} < 0 \) for \( x \in (0,\infty) \), \( g_4 > 0 \) for \( x \in (0,x_2) \), by the second identity (2.3) it is seen that \( H'_{f_4,g_4} = \left(f'_4/g'_4\right)^{\prime} g_4 < 0 \) for \( x \in (0, x_2) \). On the other hand, we see that

\[
H_{f_4,g_4}(x_2) = \lim_{x \to x_2^{-}} \left[ \frac{f'_4(x)}{g'_4(x)} g_4(x) - f_4(x) \right] = -f_4(x_2) > 0. \tag{4.1}
\]

Then \( H_{f_4,g_4}(x) > H_{f_4,g_4}(x_2) > 0 \) for \( x \in (0, x_2) \). Due to \( g'_4(x) < 0 \), it follows from the first identity (2.2) that

\[
\left(\frac{f_4}{g_4}\right)^{\prime} = \frac{g'_4}{g_4} H_{f_4,g_4} < 0 \text{ for } x \in (0,x_2).
\]

In view of \( f_3(0) = g_3(0) = 0 \), by Proposition 1 we find that \( h_2 = f_3/g_3 \) is strictly decreasing on \((0,x_2)\).

**Case 2:** \( x \in (x_2,\infty) \). We have \( f_4(x_2) < 0 \), \( g_4(x_2) = 0 \). Since \( \left(f'_4/g'_4\right)^{\prime} < 0 \) for \( x \in (0,\infty) \), \( g_4 < 0 \) for \( x \in (x_2,\infty) \), by the second identity (2.3) it is deduced that \( H'_{f_4,g_4} = \left(f'_4/g'_4\right)^{\prime} g_4 > 0 \) for \( x \in (x_2,\infty) \). This together with (4.1) gives that \( H_{f_4,g_4}(x) > H_{f_4,g_4}(x_2) > 0 \) for \( x \in (x_2,\infty) \). Due to \( g'_4(x) < 0 \), it follows that

\[
\left(\frac{f_4}{g_4}\right)^{\prime} = \frac{g'_4}{g_4} H_{f_4,g_4} < 0 \text{ for } x \in (x_2,\infty).
\]

In view of \( f_3(\infty) = g_3(\infty) = 0 \), by Proposition 1 we deduce that \( h_2 = f_3/g_3 \) is strictly decreasing on \((x_2,\infty)\).

Taking into account Cases 1 and 2 as well the continuity of the function \( g_3(x) \) at \( x = x_2 \), we conclude that \( h_2 = f_3/g_3 \) is strictly decreasing on \((0,\infty)\). An easy calculation yields \( h_2(0) = \ln \sqrt{2} \) and \( h_2(\infty) = 0 \), and the proof is completed. \( \square \)

Now we shall prove Theorem 3.

**Proof of Theorem 3.** Differentiation give

\[
\begin{align*}
G'_i(s) &= \left[ ut \sinh(ut) - \cosh(ut) \ln \cosh(ut) \right] \frac{\cosh^{1/u-1}(ut)}{u^2} \frac{\ln 2}{s \ln^2 s} \\
&= \left[ ut \sinh(ut) - \cosh(ut) \ln \cosh(ut) \right] \frac{\cosh^{1/u-1}(ut)}{s \ln 2},
\end{align*}
\]

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\[ G''_t (s) = \left[ ut^2 \cosh (ut) - t \sinh (ut) \ln \cosh (ut) \right] \frac{\ln 2}{s \ln^2 s} \cosh^{1/u-1} (ut) \]

\[ + \left[ ut \sinh (ut) - \cosh (ut) \ln \cosh (ut) \right] \frac{1}{s \ln 2} \cosh^{1/u-2} (ut) \ln 2 \]

\[ \times \frac{(1 - u) ut \sinh (ut) - \cosh (ut) \ln \cosh (ut) \cosh^{1/u-2} (ut) \ln 2}{u^2} \]

\[ - \left[ ut \sinh (ut) - \cosh (ut) \ln \cosh (ut) \right] \frac{\cosh^{1/u-1} (ut)}{s^2 \ln 2}. \]

Letting \( ut = x \) and simplifying give

\[
\frac{s^2 \ln^2 2}{\cosh^{1/u-2} (ut)} G''_t (s) = u \left( x^2 \cosh x - x \sinh x \ln \cosh x \right) \cosh x
\]

\[ + (x \sinh x - \cosh x \ln \cosh x) \]

\[ \times [(1 - u) x \sinh x - \cosh x \ln \cosh x]
\]

\[ - (\ln 2) (x \sinh x - \cosh x \ln \cosh x) \cosh x \]

\[ = ux^2 - [(\ln 2) (x \sinh x - \cosh x \ln \cosh x) \cosh x
\]

\[ - (x \sinh x - \cosh x \ln \cosh x)^2] = x^2 \left[ u - h_2 (x) \right], \]

where \( h_2 (x) \) is as in Lemma 2. Since \( h_2 (x) \) is even on \((-\infty, \infty)\), \( G''_t (s) \geq (\leq) 0 \) for all \( t > 0 \) if and only if

\[ Q_2 (t) = u - h_2 (|ut|) \geq (\leq) 0 \]

for \( t > 0 \). From Lemma 2 we find

\[ Q'_2 (t) = -|u| h_2 (|ut|) > 0 \]

for all \( t > 0 \) and

\[ \lim_{t \to 0} Q_2 (t) = u - \lim_{t \to 0} h_2 (|ut|) = u - \frac{1}{2} \ln 2, \]

\[ \lim_{t \to \infty} Q_2 (t) = u - \lim_{t \to \infty} h_2 (|ut|) = u. \]

We conclude thus that \( G''_t (s) \geq (\leq) 0 \) for all \( t > 0 \) if and only if

\[ u \geq \max \left\{ 0, \frac{1}{2} \ln 2 \right\} = \frac{1}{2} \ln 2 \quad \text{or} \quad u \leq \min \left\{ 0, \frac{1}{2} \ln 2 \right\} = 0, \]

which, by the relation \( u = (\ln 2) / (\ln (1/s)) \), implies that \( e^{-2} \leq s < 1 \) or \( s > 1 \).

When \( 0 < u < (\ln \sqrt{2}) \), that is, \( s \in (0, \ln \sqrt{2}) \), since \( Q'_2 (t) > 0 \), \( Q_2 (0^+) = u - \ln \sqrt{2} < 0 \) and \( Q_2 (\infty) = u > 0 \), there is a \( t_2 > 0 \) such that \( Q_2 (t) < 0 \), \( t \in (0, t_2) \) and \( Q_2 (t) > 0 \), \( t \in (t_2, \infty) \), where \( t_2 \) is a solution of the equation

\[ Q_2 (t) = u - h_2 (|ut|) = 0. \quad (4.2) \]

Since the function \( h_2 (x) \), \( x > 0 \) is strictly decreasing, the inverse of \( h_2 \) exists and so is \( h_2^{-1} \). Solving the Eq (4.2) for \( t \) yields

\[ t = \frac{h_2^{-1} (u)}{u} = T_2 (u). \]
Because that $1/u$ and $h_2^{-1}(u)$ are both positive and strictly decreasing, so is $t = T_2(u)$. This implies $u = T_2^{-1}(t)$ exists and strictly decreasing on $(0, \infty)$. It then follows that

$$t \in (0, t_2) \iff u \in (T_2^{-1}(t_2), \ln \sqrt{2}) = (u_2, \ln \sqrt{2}),$$

$$t \in (t_2, \infty) \iff u \in (0, T_2^{-1}(t_2)) = (0, u_2),$$

where $u_2 = T_2^{-1}(t_2) \in (0, \ln \sqrt{2})$. We thus deduce that $G''_i(s) < 0$ for $u \in (u_2, \ln \sqrt{2})$ and $G''_i(s) > 0$ for $u \in (0, u_2)$. Due to $u = (\ln 2) / \ln (1/s)$, it follows that $G''_i(s) < 0$ on $u \in \left(s_2^*, e^{-2}\right)$ and $G''_i(s) > 0$ on $(0, s_2^*)$, where $s_2^* = 2^{-1/u_2}$. This completes the proof.

\[\Box\]

5. Proof of Theorem 4

**Lemma 3.** The function

$$h_3(x) = \frac{x \tanh x - \ln (\cosh x)}{x^2 / \cosh^2 x} \ln 2$$

is strictly increasing from $(0, \infty)$ onto $(\ln \sqrt{2}, \infty)$.

**Proof.** As shown in Lemmas 1 and 2, $x \tanh x - \ln \cosh x = f_2(x)$ and $x^2 / \cosh^2 x = g_3(x)$ with $f_2(0) = g_3(0) = 0$. Since $f_2'(x) = x / \cosh^2 x > 0$, we have $f_2(x) > f_2(0) = 0$ for $x > 0$. Note that

$$g_3'(x) \quad f_2'(x) = 2 - 2x \tanh x,$$

$$\left[ \frac{g_3'(x)}{f_2'(x)} \right] \quad -2x + \cosh x \sinh x \quad < 0.$$

By Proposition 1 we deduce that $g_3(x)/f_2(x)$ is strictly decreasing on $(0, \infty)$, which, due to $g_3(x)/f_2(x) > 0$, implies that $h_3(x) = [f_2(x)/g_3(x)] \ln 2$ is strictly increasing on $(0, \infty)$. A simple computation yields

$$\lim_{x \to 0} h_3(x) = \frac{1}{2} \ln 2$$

and

$$\lim_{x \to \infty} h_3(x) = \infty,$$

which completes the proof.

\[\Box\]

Based on Lemma 3, we now check Theorem 4.

**Proof of Theorem 4.** Differentiation yields

$$[\ln G_i(s)]' = \left[ ut \tanh (ut) - \ln \cosh (ut) \right] \frac{1}{u^2} \ln 2 \quad \frac{u^2}{s \ln^2 s}$$

$$= \frac{ut \tanh (ut) - \ln \cosh (ut)}{s \ln 2},$$

$$[\ln G_i(s)]'' = \frac{ut^2 \cosh^2 (ut)}{s \ln^2 s} \frac{1}{s \ln 2} - \frac{ut \tanh (ut) - \ln \cosh (ut)}{s^2 \ln 2}.$$
Letting \( ut = x \) and simplifying lead to
\[
\frac{\ln 2}{x^2} \left( \ln \cosh x \right) = u - \frac{x \tanh x - \ln \cosh x}{x^2/\cosh^2 x} \ln 2 = u - h_3(x),
\]
where \( h_3(x) \) is given in Lemma 3. Since \( h_3(x) \) is even on \((-\infty, \infty)\), \( [\ln G_t(s)]'' \geq (\leq) 0 \) for \( t > 0 \) if and only if
\[
Q_3(t) = u - h_3(|ut|) \geq (\leq) 0
\]
for \( t > 0 \). From Lemma 3 we get
\[
Q_3'(t) = -|u|h_3(|ut|) < 0
\]
for \( t > 0 \) and
\[
\lim_{t \to 0} Q_3(t) = u - \lim_{t \to 0} h_3(|ut|) = u - \frac{1}{2} \ln 2,
\]
\[
\lim_{t \to \infty} Q_3(t) = u - \lim_{t \to \infty} h_3(|ut|) = -\infty.
\]
We conclude thus that \( [\ln G_t(s)]'' \leq 0 \) for all \( t > 0 \) if and only if \( u \leq \ln \sqrt{2} \), which, by the relation \( u = (\ln 2)/\ln (1/s) \), implies that \( 0 < s \leq e^{-2} \) or \( s > 1 \). This completes the proof. \( \square \)

6. Several new inequalities

Using Theorems 2 and 4, we get the following corollary.

**Corollary 1.** Suppose \( w, v > 0, w \neq v \). If \( p < r < q \leq \ln \sqrt{2} \), then the double inequality
\[
A_p(w, v)^{1-\beta_0}A_q(w, v)^{\beta_0} < A_r(w, v) < (1 - \alpha_0)A_p(w, v) + \alpha_0A_q(w, v)
\]
holds, where
\[
\alpha_0 = \frac{r - p}{q - p} \quad \text{and} \quad \beta_0 = \frac{2^{-1/r} - 2^{-1/p}}{2^{-1/q} - 2^{-1/p}}.
\]
The second inequality of (6.1) is reversed if \( 1/2 \leq p < r < q \).

**Proof.** By Theorem 4, the function \( s \mapsto \ln A_{\alpha(s)}(w, v) \) is concave on \((0, e^{-2}] \cup (1, \infty)\). Then for \( s_i \in (0, e^{-2}] \) or \( s_i \in (1, \infty) \), \( i = 1, 2, 3 \), using the property of convex functions we have
\[
\frac{\ln A_{\alpha(s_2)}(w, v) - \ln A_{\alpha(s_1)}(w, v)}{s_2 - s_1} > \frac{\ln A_{\alpha(s_3)}(w, v) - \ln A_{\alpha(s_1)}(w, v)}{s_3 - s_1},
\]
which is equivalent to
\[
\ln A_{\alpha(s_2)}(w, v) > \frac{s_3 - s_2}{s_3 - s_1} \ln A_{\alpha(s_1)}(w, v) + \frac{s_2 - s_1}{s_3 - s_1} \ln A_{\alpha(s_3)}(w, v).
\]
Let \( (u(s_1), u(s_2), u(s_3)) = (p, r, q) \). Then by the relation \( u(s) = (\ln 2)/\ln (1/s) \) we get \( s_1, s_2, s_3 = (2^{-1/p}, 2^{-1/r}, 2^{-1/q}) \) with \( \ln \sqrt{2} \leq p < r < q \). The inequality (6.4) thus becomes to the left hand side inequality of (6.1).
From Theorem 2, the function \( u \mapsto A_u (w, v) \) is convex on \((-\infty, \ln \sqrt{2})\) and concave on \((1/2, \infty)\), where \( w, v > 0, w \neq v \). Then for \( p < r < q \leq \ln \sqrt{2} \) the right hand side inequality of (6.1) holds, which is reversed if \( 1/2 \leq p < r < q \). This completes the proof. \( \square \)

Using Theorems 1 and 3, we obtain the following corollary.

**Corollary 2.** Suppose \( w, v > 0, w \neq v \). If \( \ln \sqrt{2} \leq p < r < q \), then the double inequality

\[
A_p (w, v) \left( x \right)^{1-\alpha_0} A_q (w, v) c < A_r (w, v) \left( x \right) < \left( 1 - \beta_0 \right) A_p (w, v) + \beta_0 A_q (w, v) \quad (6.5)
\]

holds, where \( \alpha_0 \) and \( \beta_0 \) are given in (6.2) are the best constants. The double inequality (6.5) is reversed if \( p < r < q < 0 \) with the best constants \( \alpha_0 \) and \( \beta_0 \).

**Proof.** By Theorem 1 the function \( u \mapsto \ln A_u (w, v) \) is convex on \((-\infty, 0)\) and concave on \((0, \infty)\). This implies that, for \( 0 < p < r < q \) \( (p < r < q < 0) \), the inequality

\[
\frac{q - r}{q - p} \ln A_p (w, v) + \frac{r - p}{q - p} \ln A_q (w, v) < (>) \ln A_r (w, v)
\]

holds, that is,

\[
A_p (w, v) \left( x \right)^{1-\alpha_0} A_q (w, v) c < A_r (w, v)
\]

By Theorem 3, the function \( s \mapsto A_{s(\alpha)} (w, v) \) is convex on \([e^{-2}, 1)\) and concave on \((1, \infty)\). Then for \( s_i \in [e^{-2}, 1), i = 1, 2, 3 \), with \( s_1 < s_2 < s_3 \), by the property of concave functions we have

\[
\frac{A_{s(\alpha)} (w, v) - A_{s(\alpha)} (w, v)}{s_2 - s_1} < \frac{A_{s(\alpha)} (w, v) - A_{s(\alpha)} (w, v)}{s_3 - s_1}
\]

which is equivalent to

\[
A_{s(\alpha)} (w, v) < \frac{s_3 - s_2}{s_3 - s_1} A_{s(\alpha)} (w, v) + \frac{s_2 - s_1}{s_3 - s_1} A_{s(\alpha)} (w, v)
\]

Let \( (u (s_1), u (s_2), u (s_3)) = (p, r, q) \). Then by the relation \( u (s) = (\ln 2) / (\ln 1/s) \) we get \( s_1, s_2, s_3 = (2^{-1/p}, 2^{-1/r}, 2^{-1/q}) \) with \( \ln \sqrt{2} \leq p < r < q \). The inequality (6.7) thus becomes to the right hand side inequality of (6.5).

If \( s_i \in (1, \infty), i = 1, 2, 3 \), with \( s_1 < s_2 < s_3 \), by the property of concave functions, the inequality (6.6) is reversed, and so is the right hand side inequality of (6.5) if \( p < r < q < 0 \).

Without loss of generality, we suppose that \( 0 < w < v \). Then \( \zeta = \ln \sqrt{v/w} > 0 \). Due to

\[
\lim_{v \to \infty} \frac{\ln A_r (w, v) - \ln A_p (w, v)}{\ln A_q (w, v) - \ln A_p (w, v)} = \lim_{s \to 0} \frac{\ln \cosh^{1/r} (r \zeta) - \ln \cosh^{1/p} (p \zeta)}{\ln \cosh^{1/q} (q \zeta) - \ln \cosh^{1/p} (p \zeta)} = \frac{r - p}{q - p} = \alpha_0,
\]

\[
\lim_{v \to \infty} \frac{A_r (w, v) - A_p (w, v)}{A_q (w, v) - A_p (w, v)} = \lim_{s \to 0} \frac{\cosh^{1/r} (r \zeta) - \cosh^{1/p} (p \zeta)}{\cosh^{1/q} (q \zeta) - \cosh^{1/p} (p \zeta)} = \frac{2^{-1/r} - 2^{-1/p}}{2^{-1/q} - 2^{-1/p}} = \beta_0
\]

for \( \max \{p, q, r\} < 0 \) or \( \min \{p, q, r\} > 0 \). \( \alpha_0 \) and \( \beta_0 \) are the best. This completes the proof. \( \square \)
Similarly, by means of Theorems 1 and 4 we can prove the following corollary, all the details of proof are omitted here.

**Corollary 3.** Suppose \( w, v > 0, w \neq v \). If \( p < r < q < 0 \), then

\[
A_p(w, v)^{1-\beta_0} A_q(w, v)^{\beta_0} < A_r(w, v) < A_p(w, v)^{1-\alpha_0} A_q(w, v)^{\alpha_0},
\]

where \( \alpha_0 \) and \( \beta_0 \) are given in (6.2).

By means of Corollaries 1 and 2, we have

**Corollary 4.** Suppose \( p, q, r \in \mathbb{R}, p < r < q \). (i) If \( p \geq 1/2 \), then for \( w, v > 0 \), \( w \neq v \) the double mean-inequality

\[
(1 - \beta) A_p(w, v) + \beta A_q(w, v) > A_r(w, v) > (1 - \alpha) A_p(w, v) + \alpha A_q(w, v)
\]

is valid if and only if

\[
\alpha \leq \alpha_0 = \frac{r - p}{q - p} \quad \text{and} \quad \beta \geq \beta_0 = \frac{2^{-1/r} - 2^{-1/p}}{2^{-1/q} - 2^{-1/p}}.
\]

(ii) If \( q < 0 \), then for \( w, v > 0 \), \( w \neq v \) the double inequality (6.8) is reversed if and only if \( \alpha \geq \alpha_0 \) and \( \beta \leq \beta_0 \).

**Proof.** (i) **Necessity.** Since \( w, v > 0 \) with \( w \neq v \), we suppose \( v > w > 0 \). Then \( \varsigma = \ln \sqrt{v/w} > 0 \). If the first inequality of (6.8) holds for all \( v > w > 0 \), then

\[
\alpha \leq \lim_{\varsigma \to 0} \frac{\cosh^{1/r}(r \varsigma) - \cosh^{1/p}(p \varsigma)}{\cosh^{1/q}(q \varsigma) - \cosh^{1/p}(p \varsigma)} = \frac{r - p}{q - p} = \alpha_0.
\]

If the second inequality of (6.8) is valid for \( v > w > 0 \), then

\[
\beta \geq \lim_{\varsigma \to 0} \frac{\cosh^{1/r}(r \varsigma) - \cosh^{1/p}(p \varsigma)}{\cosh^{1/q}(q \varsigma) - \cosh^{1/p}(p \varsigma)} = \frac{2^{-1/r} - 2^{-1/p}}{2^{-1/q} - 2^{-1/p}} = \beta_0.
\]

**Sufficiency.** By Corollaries 1 and 2, the reverse of the right hand side inequality in (6.1) for \( \alpha = \alpha_0 \) and the inequality (6.5) for \( \beta = \beta_0 \) both hold if \( 1/2 \leq p < r < q \), that is, for \( w, v > 0 \), \( w \neq v \) and \((\alpha, \beta) = (\alpha_0, \beta_0)\), the double inequality (6.8) is valid. It is easy to find that, for \( \alpha \leq \alpha_0 \),

\[
A_r(w, v) > (1 - \alpha_0) A_p(w, v) + \alpha_0 A_q(w, v) \geq (1 - \alpha) A_p(w, v) + \alpha A_q(w, v),
\]

and for \( \beta \geq \beta_0 \),

\[
(1 - \beta) A_p(w, v) + \beta A_q(w, v) > (1 - \beta_0) A_p(w, v) + \beta_0 A_q(w, v) > A_r(w, v).
\]

This proves the sufficiency.

(ii) The second assertion of this theorem can be proven in a similar way. This completes the proof. \( \square \)

**Remark 4.** Clearly, Corollary 4 gives an answer to Problem 2.
7. Conclusions

In this paper, we completely described the convexity of $u \mapsto A_u(w, v)$ on $\mathbb{R}$ and $s \mapsto A_{u(s)}(w, v)$, $\ln A_{u(s)}(w, v)$ with $u(s) = (\ln 2) / \ln (1/s)$ on $(0, \infty)$ by using two tools. From which we obtained several new sharp inequalities involving the power means (Corollaries 1–4), where Corollary 4 gives an answer to Problem 2. Moreover, we gave another new proof of Problem 1.

Final inspired by Theorems 1–4, we propose the following problem.

**Problem 3.** For all $w, v > 0$, $w \neq v$, determine the best $p \in \mathbb{R}$ such that the functions $p \mapsto L_p(w, v)$, $I_p(w, v)$ are convex or concave.

The second problem is inspired by Corollary 3 and Problem 2.

**Problem 4.** Suppose $p, q, r \in \mathbb{R}$ with $p < r < q$ and $v, w > 0$ with $v \neq w$. Determine the best $\alpha, \beta \in (0, 1)$ with $\alpha < \beta$ such that the double inequality

$$A_p(w, v)^{1-\beta} A_q(w, v)^\beta < A_r(w, v) < A_p(w, v)^{1-\alpha} A_q(w, v)^\alpha$$

is valid.

It was shown in [29, Lemma 6] (see also [30, 31]) that the function $p \mapsto 2^{1/p} A_p(w, v)$ is strictly decreasing and log-convex on $(0, \infty)$. Motivated by this, it is natural to propose the following problem.

**Problem 5.** Describe the convexity of the function $p \mapsto 2^{1/p} A_p(w, v)$ on $(-\infty, 0)$ and $(0, \infty)$.

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Conflict of interest

The authors declare no conflict of interest.

References


