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Research article

A certain subclass of bi-univalent functions associated with Bell numbers and $q-$Srivastava Attiya operator

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Abstract: In the present study, we introduced general a subclass of bi-univalent functions by using the Bell numbers and $q-$Srivastava Attiya operator. Also, we investigate coefficient estimates and famous Fekete-Szegö inequality for functions belonging to this interesting class.

Keywords: bi-univalent function; $q-$Srivastava Attiya operator; Bell numbers; coefficient estimates
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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The well-known Koebe one-quarter theorem [8] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $1/4$. Thus, every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbb{D})$ and

$$f^{-1}(f(w)) = w, \quad (|w| < r_0(f),\ r_0(f) \geq 1/4)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and g to $\mathbb{D}$ are univalent in $\mathbb{D}$, where g is the analytic continuation of f^{-1} to the unit disk $\mathbb{D}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{D}$ given by 1.1. For a brief history of functions in the class Σ, see [3, 4, 16, 19]. Later, Srivastava et al.'s [24, 26–28] gave very important contributions to this theory. Recently, for coefficient estimates of the functions in some particular subclasses of bi-univalent functions, one may see [6, 7, 10, 15, 20, 25, 29, 30].

For analytic functions f and g in $\mathbb{D}$, f is said to be subordinate to g if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. This subordination is denote by $f(z) \prec g(z)$. In particular, when g is univalent in $\mathbb{D}$,

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}) \quad (z \in \mathbb{D}).$$

The $q-$difference operator, which was introduced by Jackson [13], is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (z \neq 0) \tag{1.3}$$

for $q \in (0,1)$. It is clear that $\lim\limits_{q \to 1^-} \partial_q f(z) = f'(z)$ and $\partial_q f(0) = f'(0)$, where f' is the ordinary derivative of the function. For more properties of ∂_q see [9, 11, 12].

Thus, for function $f \in \mathcal{A}$ we have

$$\partial_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{1.4}$$

where $[k]_q$ is given by

$$[k]_q = \frac{1-q^k}{1-q}, \quad [0]_q = 0 \tag{1.5}$$

and the $q-$ factorial is defined by

$$[k]_q! = \begin{cases} \prod\limits_{n=1}^{k} [n]_q, & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}. \tag{1.6}$$

As $q \to 1^-$, then we get $[k]_q \to k$. Thus, if we choose the function $g(z) = z^k$, while $q \to 1$, then we have

$$\partial_q g(z) = \partial_q z^k = [k]_q z^{k-1} = g'(z), \tag{1.7}$$

where g' is the ordinary derivative.

In order to derive our main results, we have to recall here the following lemmas.

Lemma 1.1. (*[8]*) *If $p \in \mathcal{P}$ then $|p_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{D}$ for which* $\mathrm{Re}\, p(z) > 0$,

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{1.8}$$

for $z \in \mathbb{D}$.

Lemma 1.2. (*[17]*) *If the function $p \in \mathcal{P}$ is given by the series 1.8 , then*

$$\begin{aligned} 2p_2 &= p_1^2 + x\left(4 - p_1^2\right), \\ 4p_3 &= p_1^3 + 2\left(4 - p_1^2\right) p_1 x - p_1\left(4 - p_1^2\right) x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z \end{aligned}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

For a fixed non-negative integer n, the Bell numbers B_n count the possible disjoint partitions of a set with n elements into non-empty subsets or, equivalently, the number of equivalence relations on it. The numbers B_n are named the Bell numbers after Eric Temple Bell (1883 − 1960) (see [1, 2]) who called then the "exponential numbers". The Bell numbers B_n $(n \geqq 0)$ are generated by the function e^{e^z-1} as follows: $e^{e^z-1} = \sum_{n=0}^{\infty} B_n\,(z^n/n!)\ (z \in \mathbb{R})$. The Bell numbers B_n satisfy the following recurrence relation involving binomial coefficients: $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. Clearly, we have $B_0 = B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$ and $B_6 = 203$. We now consider the function $\varphi(z) := e^{e^z-1}$ with its domain of definition as the open unit disk $\mathbb{D}$. Recently Srivastava and co-auhors studied geometric properties and coefficients bounds for starlike functions related to the Bell numbers (see [5, 14]).

On the other hand, Shah and Noor [21] introduced the $q-$analogue of the Hurwitz Lerch zeta function by the following series:

$$\phi_q(s, b; z) = \sum_{k=0}^{\infty} \frac{z^k}{[k+b]_q^s}, \tag{1.9}$$

where $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, and $\operatorname{Re}(s) > 1$ when $|z| = 1$. The a normalized form of 1.9 as follows:

$$\begin{aligned} \psi_q(s, b; z) &= [1+b]_q^s \left\{ \phi_q(s, b; z) - [b]_q^{-s} \right\} \\ &= z + \sum_{k=2}^{\infty} \left(\frac{[1+b]_q}{[k+b]_q} \right)^s z^k. \end{aligned} \tag{1.10}$$

From 1.10 and 1.1, Shah and Noor [21] defined the $q-$Srivastava Attiya operator $J_{q,b}^s f(z) : \mathcal{A} \to \mathcal{A}$

by

$$\begin{aligned} J_{q,b}^s f(z) &= \psi_q(s, b; z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \left(\frac{[1+b]_q}{[k+b]_q} \right)^s a_k z^k \end{aligned} \tag{1.11}$$

where $*$ denotes convolution (or the Hadamard product).

We note that:

(i) If $q \to 1^-$, then the function $\phi_q(s, b; z)$ reduces to the Hurwitz-Lerch zeta function and the operator $J_{q,b}^s$ coincides with the Srivastava-Attiya operator (see [22, 23]).

(ii) $J_{q,0}^1 f(z) = \int_0^z \frac{f(t)}{t} d_q t$ ($q-$Alexander operator).

(iii) $J_{q,b}^1 f(z) = \frac{[1+b]_q}{z^b} \int_0^z t^{b-1} f(t)\, d_q t$ ($q-$Bernardi operator [18]).

(iv) $J_{q,1}^1 f(z) = \frac{[2]_q}{z} \int_0^z t^{b-1} f(t)\, d_q t$ ($q-$Libera operator [18]).

In present paper, we defined a general subclass $\Sigma H_{q,b}^s(\tau, \lambda, \mu)$ of bi-univalent functions related to the Bell numbers by using $q-$Srivastava Attiya operator. Using the principles of subordination, the estimates for the coefficients $|a_2|$, $|a_3|$ and $\left|a_3 - \delta a_2^2\right|$ of the functions of the form 1.1 in the class $\Sigma H_{q,b}^s(\tau, \lambda, \mu)$ have been obtained. For some particular choices of τ, λ, μ and s the bounds determined.

2. Coefficient estimates

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + ...$$

in the unit disk $\mathbb{D}$ satisfying the condition $|w(z)| < 1$. There is an important relation between the classes Ω and $\mathcal{P}$ as follows:

$$w \in \Omega \;\Leftrightarrow\; \frac{1 + w(z)}{1 - w(z)} \in \mathcal{P} \text{ or } p \in \mathcal{P} \;\Leftrightarrow\; \frac{p(z) - 1}{p(z) + 1} \in \Omega. \tag{2.1}$$

Define the functions p and s in $\mathcal{P}$ given by

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

and

$$s(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots .$$

It follows that

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{p_1}{2} z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots \tag{2.2}$$

and

$$v(z) = \frac{s(z) - 1}{s(z) + 1} = \frac{s_1}{2} z + \frac{1}{2}\left(s_2 - \frac{s_1^2}{2}\right) z^2 + \cdots . \tag{2.3}$$

Definition 2.1. *A function $f \in \Sigma$ is said to be in the class $\Sigma H_{q,b}^{s}(\tau, \lambda, \mu)$ if the following conditions hold true for all $z, w \in \mathbb{D}$:*

$$1 + \frac{1}{\tau}\left[(1 - \lambda)\left[\frac{J_{q,b}^{s} f(z)}{z}\right]^{\mu} + \lambda \partial_q \left(J_{q,b}^{s} f(z)\right)\left(\frac{J_{q,b}^{s} f(z)}{z}\right)^{\mu - 1} - 1\right] \prec \varphi(z)$$

and

$$1 + \frac{1}{\tau}\left[(1 - \lambda)\left[\frac{J_{q,b}^{s} g(w)}{w}\right]^{\mu} + \lambda \partial_q \left(J_{q,b}^{s} g(w)\right)\left(\frac{J_{q,b}^{s} g(w)}{w}\right)^{\mu - 1} - 1\right] \prec \varphi(w)$$

where $\varphi(z) = e^{e^{z-1}}$, $g(w) = f^{-1}(w)$, $\tau \in \mathbb{C}\backslash\{0\}$, $\mu > 0$, $0 < q < 1$ and $\lambda \geq 0$.

Remark 2.1. *We note that, for suitable choices parameters, the class $\Sigma H_{q,b}^{s}(\tau, \lambda, \mu)$ reduces to the following classes.*

1) Let $\lambda = 1$ in $\Sigma H_{q,b}^{s}(\tau, \lambda, \mu)$. Then a function $f \in \Sigma$ is said to be in the class $\Sigma H_{q,b}^{s}(\tau, \mu)$ if the following subordinations hold for all $z, w \in \mathbb{D}$:

$$1 + \frac{1}{\tau}\left[\partial_q \left(J_{q,b}^{s} f(z)\right)\left(\frac{J_{q,b}^{s} f(z)}{z}\right)^{\mu - 1} - 1\right] \prec \varphi(z)$$

and

$$1+\frac{1}{\tau}\left[\partial_q\left(J^s_{q,b}g(w)\right)\left(\frac{J^s_{q,b}g(w)}{w}\right)^{\mu-1}-1\right]\prec\varphi(w)$$

2) Let $\lambda=1$ and $\tau=1$ in $\Sigma H^s_{q,b}(\tau,\lambda,\mu)$. Then a function $f\in\Sigma$ is said to be in the class $\Sigma H^s_{q,b}(\mu)$ if the following subordinations hold for all $z,w\in\mathbb{D}$:

$$\partial_q\left(J^s_{q,b}f(z)\right)\left(\frac{J^s_{q,b}f(z)}{z}\right)^{\mu-1}\prec\varphi(z)$$

and

$$\partial_q\left(J^s_{q,b}f(w)\right)\left(\frac{J^s_{q,b}f(w)}{w}\right)^{\mu-1}\prec\varphi(w).$$

3) Let $\mu=1$ in $\Sigma H^s_{q,b}(\tau,\lambda,\mu)$. Then a function $f\in\Sigma$ is said to be in the class $\Sigma H^s_{q,b}(\tau,\lambda)$ if the following subordinations hold for all $z,w\in\mathbb{D}$:

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{J^s_{q,b}f(z)}{z}+\lambda\partial_q\left(J^s_{q,b}f(z)\right)-1\right]\prec\varphi(z)$$

and

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{J^s_{q,b}g(w)}{w}+\lambda\partial_q\left(J^s_{q,b}g(w)\right)-1\right]\prec\varphi(w).$$

4) Let $\mu=1$ and $\tau=1$ in $\Sigma H^s_{q,b}(\tau,\lambda,\mu)$. Then a function $f\in\Sigma$ is said to be in the class $\Sigma H^s_{q,b}(\lambda)$ if the following subordinations hold for all $z,w\in\mathbb{D}$:

$$(1-\lambda)\frac{J^s_{q,b}f(z)}{z}+\lambda\partial_q\left(J^s_{q,b}f(z)\right)\prec\varphi(z)$$

and

$$(1-\lambda)\frac{J^s_{q,b}g(w)}{w}+\lambda\partial_q\left(J^s_{q,b}g(w)\right)\prec\varphi(w)$$

5) Let $\mu=1$, $\tau=1$ and $\lambda=0$ in $\Sigma H^s_{q,b}(\tau,\lambda,\mu)$. Then a function $f\in\Sigma$ is said to be in the class $\Sigma H^s_{q,b}$ if the following subordinations hold for all $z,w\in\mathbb{D}$:

$$\frac{J^s_{q,b}f(z)}{z}\prec\varphi(z)$$

and

$$\frac{J^s_{q,b}g(w)}{w}\prec\varphi(w).$$

6) Let $s=0$ in $\Sigma H^s_{q,b}(\tau,\lambda,\mu)$. Then a function $f\in\Sigma$ is said to be in the class $\Sigma H(\tau,\lambda,\mu)$ if the following subordinations hold for all $z,w\in\mathbb{D}$:

$$1+\frac{1}{\tau}\left[(1-\lambda)\left[\frac{f(z)}{z}\right]^{\mu}+\lambda\partial_q(f(z))\left(\frac{f(z)}{z}\right)^{\mu-1}-1\right]\prec\varphi(z)$$

and

$$1+\frac{1}{\tau}\left[(1-\lambda)\left[\frac{g(w)}{w}\right]^{\mu}+\lambda\partial_q(g(w))\left(\frac{g(w)}{w}\right)^{\mu-1}-1\right]\prec\varphi(w).$$

The following theorem derives the estimates for the coefficients $|a_2|$ and $|a_3|$ for the functions given by 1.1 that belong to the class $\Sigma H^{s}_{q,b}(\tau,\lambda,\mu)$.

Theorem 2.1. *Let f given by (1.1) be in the class $\Sigma H^{s}_{q,b}(\tau,\lambda,\mu)$. Then*

$$|a_2| \le \left|\left(\frac{[2+b]_q}{[1+b]_q}\right)^s\right| \min\left\{\frac{|\tau|}{\mu+\lambda q}, \sqrt{\frac{2|\tau|}{\mu(1+\mu)+2\lambda q(\mu+q)}}\right\} \tag{2.4}$$

and

$$|a_3| \le \left|\left(\frac{[3+b]_q}{[1+b]_q}\right)^s\right| \frac{|\tau|}{\mu+\lambda q(1+q)} \min\left\{1, \frac{|\tau|(\mu+\lambda q(1+q))}{(\mu+\lambda q)^2}\right\}. \tag{2.5}$$

Proof. Let $f \in \Sigma H^{s}_{q,b}(\tau,\lambda,\mu)$ and $g = f^{-1}$. Then, there are analytic functions $u, v \in \Omega$ satisfying

$$1+\frac{1}{\tau}\left[(1-\lambda)\left[\frac{J^s_{q,b}f(z)}{z}\right]^{\mu} + \lambda\partial_q\left(J^s_{q,b}f(z)\right)\left(\frac{J^s_{q,b}f(z)}{z}\right)^{\mu-1} - 1\right] = \varphi(u(z)) \tag{2.6}$$

and

$$1+\frac{1}{\tau}\left[(1-\lambda)\left[\frac{J^s_{q,b}g(w)}{w}\right]^{\mu} + \lambda\partial_q\left(J^s_{q,b}g(w)\right)\left(\frac{J^s_{q,b}g(w)}{w}\right)^{\mu-1} - 1\right] = \varphi(v(z)). \tag{2.7}$$

In other words, by using 2.1 in 2.6 and 2.7 we write

$$1+\frac{1}{\tau}\left[(1-\lambda)\left[\frac{J^s_{q,b}f(z)}{z}\right]^{\mu} + \lambda\partial_q\left(J^s_{q,b}f(z)\right)\left(\frac{J^s_{q,b}f(z)}{z}\right)^{\mu-1} - 1\right] = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) = e^{e^{\frac{p(z)-1}{p(z)+1}}-1} \tag{2.8}$$

and

$$1+\frac{1}{\tau}\left[(1-\lambda)\left[\frac{J^s_{q,b}g(w)}{w}\right]^{\mu} + \lambda\partial_q\left(J^s_{q,b}g(w)\right)\left(\frac{J^s_{q,b}g(w)}{w}\right)^{\mu-1} - 1\right] = \varphi\left(\frac{s(z)-1}{s(z)+1}\right) = e^{e^{\frac{s(z)-1}{s(z)+1}}-1}. \tag{2.9}$$

From 2.8 and 2.9, we have

$$\begin{aligned} & 1+\frac{(\mu+\lambda q)}{\tau}\left(\frac{[1+b]_q}{[2+b]_q}\right)^s a_2 z \\ & +\frac{1}{\tau}\left(\left(\frac{(\mu-1)(\mu+2\lambda q)}{2}\right)\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s} a_2^2 + (\mu+\lambda q(1+q))\left(\frac{[1+b]_q}{[3+b]_q}\right)^s a_3\right) z^2 \cdots \\ = \ & 1+\frac{p_1}{2}z+\frac{p_2}{2}z^2+\cdots \end{aligned}$$

and

$$\begin{aligned} & 1-\frac{(\mu+\lambda q)}{\tau}\left(\frac{[1+b]_q}{[2+b]_q}\right)^s a_2 w \\ & +\frac{1}{\tau}\left(\left(\lambda q(2q+\mu+1)+\frac{\mu(\mu+3)}{2}\right)\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s} a_2^2 + (-\mu-\lambda q(1+q))\left(\frac{[1+b]_q}{[3+b]_q}\right)^s a_3\right) w^2 \cdots \end{aligned}$$

$$1+\frac{s_1}{2}z+\frac{s_2}{2}z^2+\cdots.$$

Comparing the coefficients on the both sides of above last equalities, we have the relations

$$\frac{1}{\tau}(\mu+\lambda q)\,a_2\left(\frac{[1+b]_q}{[2+b]_q}\right)^s=\frac{p_1}{2},\tag{2.10}$$

$$\frac{1}{\tau}\left(\left(\frac{(\mu-1)(\mu+2\lambda q)}{2}\right)\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s}a_2^2+(\mu+\lambda q(1+q))\left(\frac{[1+b]_q}{[3+b]_q}\right)^s a_3\right)=\frac{p_2}{2},\tag{2.11}$$

$$-\frac{1}{\tau}(\mu+\lambda q)\,a_2\left(\frac{[1+b]_q}{[2+b]_q}\right)^s=\frac{s_1}{2}\tag{2.12}$$

and

$$\frac{1}{\tau}\left(\left(\lambda q(2q+\mu+1)+\frac{\mu(\mu+3)}{2}\right)\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s}a_2^2-(\mu+\lambda q(1+q))\left(\frac{[1+b]_q}{[3+b]_q}\right)^s a_3\right)=\frac{s_2}{2}.\tag{2.13}$$

Therefore, from the Eqs 2.10 and 2.12 , we find that

$$p_1=-s_1\tag{2.14}$$

and

$$\left[\frac{1}{\tau}(\mu+\lambda q)\left(\frac{[1+b]_q}{[2+b]_q}\right)^s\right]^2a_2^2=\frac{1}{8}\left(p_1^2+s_1^2\right),\tag{2.15}$$

which upon applying Lemma 1.1, yields

$$|a_2|\le\left|\left(\frac{[2+b]_q}{[1+b]_q}\right)^s\right|\frac{|\tau|}{\mu+\lambda q}.$$

On the other hand, by using 2.11 and 2.13, we obtain

$$\frac{1}{\tau}\left(\mu^2+\mu+2\lambda q\mu+2\lambda q^2\right)\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s}a_2^2=\frac{p_2+s_2}{2},\tag{2.16}$$

which yields

$$|a_2|\le\left|\left(\frac{[2+b]_q}{[1+b]_q}\right)^s\right|\sqrt{\frac{2|\tau|}{\mu^2+\mu+2\lambda q\mu+2\lambda q^2}}.$$

We now, investigate the upper bound of $|a_3|$. For this, by using 2.11 and 2.13, we have

$$\frac{2}{\tau}(\mu+\lambda q(1+q))\left(\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s}a_2^2-\left(\frac{[1+b]_q}{[3+b]_q}\right)^s a_3\right)=\frac{s_2-p_2}{2}.\tag{2.17}$$

Therefore for substituting 2.15 in 2.17, we have

$$\left(\frac{[1+b]_q}{[3+b]_q}\right)^s a_3=\frac{\tau^2\left(p_1^2+s_1^2\right)}{8(\mu+\lambda q)^2}+\frac{\tau(p_2-s_2)}{4(\mu+\lambda q(1+q))}\tag{2.18}$$

or

$$a_3 = \left(\frac{[3+b]_q}{[1+b]_q}\right)^s \frac{\tau}{4(\mu+\lambda q(1+q))}\left[(p_2 - s_2) + \frac{\tau(\mu+\lambda q(1+q))}{(\mu+\lambda q)^2} p_1^2\right]. \tag{2.19}$$

On the other hand, according to the Lemma 1.2 and 2.14, we write

$$\left.\begin{aligned} 2p_2 &= p_1^2 + x\left(4-p_1^2\right) \\ 2s_2 &= s_1^2 + y\left(4-s_1^2\right) \end{aligned}\right\} \Longrightarrow p_2 - s_2 = \frac{4-p_1^2}{2}(x-y) \tag{2.20}$$

and so, from 2.19 and 2.20, we have

$$a_3 = \left(\frac{[3+b]_q}{[1+b]_q}\right)^s \frac{\tau}{4(\mu+\lambda q(1+q))}\left[\frac{4-p_1^2}{2}(x-y) + \frac{\tau(\mu+\lambda q(1+q))}{(\mu+\lambda q)^2} p_1^2\right]. \tag{2.21}$$

If we apply triangle inequality to equation 2.21, we obtain

$$|a_3| \le \left|\left(\frac{[3+b]_q}{[1+b]_q}\right)^s\right| \frac{|\tau|}{4(\mu+\lambda q(1+q))}\left[\frac{4-p_1^2}{2}(|x|+|y|) + \frac{|\tau|(\mu+\lambda q(1+q))}{(\mu+\lambda q)^2} p_1^2\right].$$

Since the function $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ is in the class $\mathcal{P}$ for any $p \in \mathcal{P}$, there is no loss of generality in assuming $p_1 > 0$. Write $p_1 = \mathfrak{p}$, $\mathfrak{p} \in [0,2]$. Thus, for $|x| \le 1$ and $|y| \le 1$ we obtain

$$|a_3| \le \left|\left(\frac{[3+b]_q}{[1+b]_q}\right)^s\right| \frac{|\tau|}{4(\mu+\lambda q(1+q))}\left[4 + \left(\frac{|\tau|(\mu+\lambda q(1+q))}{(\mu+\lambda q)^2} - 1\right)\mathfrak{p}^2\right],$$

which upon applying Lemma 1.1, yields upper bound of $|a_3|$. □

Theorem 2.2. *If $f(z)$ given by (1.1) be in the class $\Sigma H^s_{q,b}(\tau,\lambda,\mu)$ and $\delta \in \mathbb{C}$, then*

$$\left|a_3 - \delta a_2^2\right| \le |\tau|\left(|K+L| + |K-L|\right)$$

where

$$\begin{aligned} K &= \left(\left(\frac{[3+b]_q}{[1+b]_q}\right)^s - \delta\left(\frac{[2+b]_q}{[1+b]_q}\right)^{2s}\right)\frac{1}{\mu^2+\mu+2\lambda q\mu+2\lambda q^2}, \\ L &= \left(\frac{[3+b]_q}{[1+b]_q}\right)^s \frac{1}{2(\mu+\lambda q+\lambda q^2)}. \end{aligned} \tag{2.22}$$

Proof. From the Eqs 2.16 and 2.18 we obtain

$$a_2^2 = \left(\frac{[2+b]_q}{[1+b]_q}\right)^{2s} \frac{\tau(p_2+s_2)}{2(\mu^2+\mu+2\lambda q\mu+2\lambda q^2)} \tag{2.23}$$

and

$$a_3 = \frac{\tau}{2}\left(\frac{[3+b]_q}{[1+b]_q}\right)^s \left(\frac{p_2+s_2}{\mu^2+\mu+2\lambda q\mu+2\lambda q^2} - \frac{s_2-p_2}{2(\mu+\lambda q+\lambda q^2)}\right). \tag{2.24}$$

Therefore, by using the equalities 2.23 and 2.24 for $\delta \in \mathbb{C}$, we have

$$a_3 - \delta a_2^2 = \frac{\tau}{2}\left(\frac{[3+b]_q}{[1+b]_q}\right)^s \left(\frac{p_2+s_2}{\mu^2+\mu+2\lambda q\mu+2\lambda q^2} - \frac{s_2-p_2}{2(\mu+\lambda q+\lambda q^2)}\right)$$

$$-\delta\left(\frac{[2+b]_q}{[1+b]_q}\right)^{2s}\frac{\tau\left(p_2+s_2\right)}{2\left(\mu^2+\mu+2\lambda q\mu+2\lambda q^2\right)}.$$

After the necessary arrangements, we rewrite the above last equality as

$$a_3-\delta a_2^2=\frac{\tau}{2}\left(\left(K+L\right)p_2+\left(K-L\right)s_2\right) \tag{2.25}$$

where K and L are given by 2.22. Taking the absolute value of 2.25, from Lemma 1.1 we obtain the desired inequality. □

Theorem 2.3. *If $f(z)$ given by (1.1) be in the class $\Sigma H^s_{q,b}\left(\tau,\lambda,\mu\right)$ and $\delta\in\mathbb{C}$, then*

$$\left|\left(\frac{[1+b]_q}{[3+b]_q}\right)^{s}a_3-\delta\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s}a_2^2\right|\le 2\left|\tau\right|\begin{cases}\frac{1}{2\left(\mu+\lambda q+\lambda q^2\right)}, & 0\le\left|\Psi\left(\delta\right)\right|\le\frac{1}{2\left(\mu+\lambda q+\lambda q^2\right)}\\ \left|\Psi\left(\delta\right)\right|, & \left|\Psi\left(\delta\right)\right|\ge\frac{1}{2\left(\mu+\lambda q+\lambda q^2\right)}\end{cases}$$

where

$$\Psi\left(\delta\right)=\frac{1-\delta}{\mu^2+\mu+2\lambda q\mu+2\lambda q^2}.$$

Proof. From Eq 2.17, we write

$$\left(\frac{[1+b]_q}{[3+b]_q}\right)^{s}a_3-\delta\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s}a_2^2=\frac{\tau\left(p_2-s_2\right)}{4\left(\mu+\lambda q+\lambda q^2\right)}+\left(1-\delta\right)\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s}a_2^2. \tag{2.26}$$

By substituting 2.16 in 2.26, we have

$$\begin{aligned}&\left(\frac{[1+b]_q}{[3+b]_q}\right)^{s}a_3-\delta\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s}a_2^2\\ =\;&\frac{\tau\left(p_2-s_2\right)}{4\left(\mu+\lambda q+\lambda q^2\right)}+\left(1-\delta\right)\frac{\tau\left(s_2+p_2\right)}{2\left(\mu^2+\mu+2\lambda q\mu+2\lambda q^2\right)}\\ =\;&\frac{\tau}{2}\left(\left(\Psi\left(\delta\right)+\frac{1}{2\left(\mu+\lambda q+\lambda q^2\right)}\right)p_2+\left(\Psi\left(\delta\right)-\frac{1}{2\left(\mu+\lambda q+\lambda q^2\right)}\right)s_2\right)\end{aligned}$$

where

$$\Psi\left(\delta\right)=\frac{1-\delta}{\mu^2+\mu+2\lambda q\mu+2\lambda q^2}.$$

Therefore, we conclude that

$$\left|\left(\frac{[1+b]_q}{[3+b]_q}\right)^{s}a_3-\delta\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s}a_2^2\right|\le 2\left|\tau\right|\begin{cases}\frac{1}{2\left(\mu+\lambda q+\lambda q^2\right)}, & 0\le\left|\Psi\left(\delta\right)\right|\le\frac{1}{2\left(\mu+\lambda q+\lambda q^2\right)}\\ \left|\Psi\left(\delta\right)\right|, & \left|\Psi\left(\delta\right)\right|\ge\frac{1}{2\left(\mu+\lambda q+\lambda q^2\right)}\end{cases},$$

which evidently complete the proof of the theorem. □

Corollary 2.1. *Let f given by (1.1) be in the class* $\Sigma H^{s}_{q,b}(\tau,\mu)$*. Then*

$$\begin{aligned}
|a_2| &\le \left|\left(\frac{[2+b]_q}{[1+b]_q}\right)^s\right| \min\left\{\frac{|\tau|}{\mu+q}, \sqrt{\frac{2|\tau|}{\mu(1+\mu)+2q(\mu+q)}}\right\},\\
|a_3| &\le \left|\left(\frac{[3+b]_q}{[1+b]_q}\right)^s\right| \frac{|\tau|}{\mu+q(1+q)} \min\left\{1, \frac{|\tau|(\mu+q(1+q))}{(\mu+q)^2}\right\},\\
\left|a_3-\delta a_2^2\right| &\le |\tau|\left(|K_1+L_1|+|K_1-L_1|\right)
\end{aligned}$$

and

$$\left|\left(\frac{[1+b]_q}{[3+b]_q}\right)^s a_3 - \delta\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s} a_2^2\right| \le 2|\tau| \begin{cases} \frac{1}{2(\mu+q+q^2)}, & 0\le |\Psi_1(\delta)| \le \frac{1}{2(\mu+q+q^2)} \\ |\Psi_1(\delta)|, & |\Psi_1(\delta)| \ge \frac{1}{2(\mu+q+q^2)} \end{cases},$$

where

$$\begin{aligned}
K_1 &= \left(\left(\frac{[3+b]_q}{[1+b]_q}\right)^s - \delta\left(\frac{[2+b]_q}{[1+b]_q}\right)^{2s}\right)\frac{1}{\mu^2+\mu+2q\mu+2q^2},\\
L_1 &= \left(\frac{[3+b]_q}{[1+b]_q}\right)^s \frac{1}{2(\mu+q+q^2)},\\
\Psi_1(\delta) &= \frac{1-\delta}{\mu^2+\mu+2q\mu+2q^2}.
\end{aligned}$$

Corollary 2.2. *Let f given by (1.1) be in the class* $\Sigma H^{s}_{q,b}(\tau,\lambda)$*. Then*

$$\begin{aligned}
|a_2| &\le \left|\left(\frac{[2+b]_q}{[1+b]_q}\right)^s\right| \min\left\{\frac{|\tau|}{1+\lambda q}, \sqrt{\frac{|\tau|}{1+\lambda q(1+q)}}\right\},\\
|a_3| &\le \left|\left(\frac{[3+b]_q}{[1+b]_q}\right)^s\right| \frac{|\tau|}{1+\lambda q(1+q)} \min\left\{1, \frac{|\tau|(1+\lambda q(1+q))}{(1+\lambda q)^2}\right\},\\
\left|a_3-\delta a_2^2\right| &\le |\tau|\left(|K_2+L_2|+|K_2-L_2|\right)
\end{aligned}$$

and

$$\left|\left(\frac{[1+b]_q}{[3+b]_q}\right)^s a_3 - \delta\left(\frac{[1+b]_q}{[2+b]_q}\right)^{2s} a_2^2\right| \le 2|\tau| \begin{cases} \frac{1}{2(1+\lambda q+\lambda q^2)}, & 0\le |\Psi_2(\delta)| \le \frac{1}{2(1+\lambda q+\lambda q^2)} \\ |\Psi_2(\delta)|, & |\Psi_2(\delta)| \ge \frac{1}{2(1+\lambda q+\lambda q^2)} \end{cases},$$

where

$$\begin{aligned}
K_2 &= \left(\left(\frac{[3+b]_q}{[1+b]_q}\right)^s - \delta\left(\frac{[2+b]_q}{[1+b]_q}\right)^{2s}\right)\frac{1}{2(1+\lambda q+2\lambda q^2)},\\
L_2 &= \left(\frac{[3+b]_q}{[1+b]_q}\right)^s \frac{1}{2(1+\lambda q+\lambda q^2)},\\
\Psi_2(\delta) &= \frac{1-\delta}{2(1+\lambda q+\lambda q^2)}.
\end{aligned}$$

Corollary 2.3. *Let f given by (1.1) be in the class* $\Sigma H(\tau,\lambda,\mu)$. *Then*

$$\begin{aligned} |a_2| &\le \min\left\{\frac{|\tau|}{\mu+\lambda q}, \sqrt{\frac{2|\tau|}{\mu(1+\mu)+2\lambda q(\mu+q)}}\right\}, \\ |a_3| &\le \frac{|\tau|}{\mu+\lambda q(1+q)}\min\left\{1, \frac{|\tau|(\mu+\lambda q(1+q))}{(\mu+\lambda q)^2}\right\} \end{aligned}$$

and

$$\left|a_3-\delta a_2^2\right| \le 2|\tau| \begin{cases} \frac{1}{2\left(1+\lambda q+\lambda q^2\right)}, & 0 \le |\Psi_3(\delta)| \le \frac{1}{2\left(1+\lambda q+\lambda q^2\right)} \\ |\Psi_3(\delta)|, & |\Psi_3(\delta)| \ge \frac{1}{2\left(1+\lambda q+\lambda q^2\right)} \end{cases},$$

where

$$\Psi_3(\delta) = \frac{1-\delta}{2\left(1+\lambda q+\lambda q^2\right)}.$$

3. Conclusions

In this paper, we defined a general subclass of bi-univalent functions related with $q-$Srivastava Attiya operator by using the Bell numbers and subordination. For the functions belonging to this class, we obtained non-sharp bounds for the initial coefficients and the Fekete-Szegö functional. Some interesting corollaries and applications of the results are also discussed.

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