Mathematics

## Research article

# Positive periodic solution for third-order singular neutral differential equation with time-dependent delay 

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#### Abstract

In this paper, we investigate a class of third-order singular neutral differential equations with time-dependent delay. Applying Krasnoselskii's fixed point theorem, we prove the existence results of a positive periodic solution for this neutral equation.


Keywords: positive periodic solution; neutral operator; singularity; Krasnoselskii's fixed point theorem; two available operators
Mathematics Subject Classification: 34B16, 34B18, 34C25

## 1. Introduction

The main purpose of this paper is to consider the existence of a positive periodic solution for thirdorder neutral differential equation with a singularity

$$
\begin{equation*}
(u(t)-c u(t-\tau(t)))^{\prime \prime \prime}+a(t) u(t)=f(t, u(t))+e(t), \tag{1.1}
\end{equation*}
$$

where $\tau \in C(\mathbb{R}, \mathbb{R})$ is a $\omega$-periodic function, $a \in C\left(\mathbb{R}, \mathbb{R}^{+}\right)$is $\omega$-periodic functions, $c$ is a constant and $c \in\left(-\frac{m}{M+m}, \frac{m}{M+m}\right)$, here $M:=\max _{t \in[0, \omega]}|a(t)|$ and $m:=\min _{t \in[0, \omega]}|a(t)|, e \in L^{1}(\mathbb{R})$ is an $\omega$-periodic function, the nonlinear term $f \in \operatorname{Car}\left(\mathbb{R} \times \mathbb{R}^{+}, \mathbb{R}\right)$ is a $L^{2}$-Carathéodory function and is an $\omega$-periodic function on $t, f$ has a singularity of repulsive type at the origin, i.e.,

$$
\lim _{u \rightarrow 0^{+}} f(t, u)=+\infty, \quad \text { uniformly in } t .
$$

Note that when $c=0$ the neutral operator $u \mapsto\left(u(t)-c u(t-\tau(t))^{\prime \prime \prime}\right.$ reduces to the linear operator $u \mapsto u^{\prime \prime \prime}$ and then equation (1.1) is of the differential equation form

$$
\begin{equation*}
u^{\prime \prime \prime}+a(t) u=f(t, u)+e(t) . \tag{1.2}
\end{equation*}
$$

During the last two decades, there are some good amount of works on periodic solutions for neutral differential equations (see $[2,3,6-8,11,14,16,17,19,22,23]$ and the references cited therein). Some classical tools have been used to study neutral differential equation in the literature, including the fixed point index theorem [2, 16], Krasnoselskii's fixed point theorem [3,6-8], fixed point theorem of Leray-Schauder type [14], Mawhin's continuous theorem [11, 23], Continuation theorem of coincidence degree theory [19], the fixed point theorem in cones [17, 22]. For example, Wu and Wang [22] in 2007 discussed a kind of second-order neutral differential equation

$$
\begin{equation*}
(u(t)-c u(t-\tau))^{\prime \prime}+a(t) u(t)=\lambda b(t) f(u(t-\delta(t))), \tag{1.3}
\end{equation*}
$$

where $c \in\left(-\frac{m}{M+m}, 0\right), b, \delta \in C(\mathbb{R}, \mathbb{R})$ are $\omega$-periodic functions, $\lambda$ is a constant and $0<\lambda<1$. By a fixed point theorem in cones and the property of neutral operator $\left(A_{1} u\right)(t):=u(t)-c u(t-\tau)$, they obtained sufficient conditions for the existence of positive periodic solutions to (1.3). Afterwards, Ren et al. [17] in 2011 considered second-order neutral differential equation with variable delay as follows

$$
\begin{equation*}
(u(t)-c u(t-\tau(t)))^{\prime \prime \prime}+a(t) u(t)=f(t, u(t-\tau(t)), \tag{1.4}
\end{equation*}
$$

where $|c|<1$. The authors presented the existence result for a positive periodic solution for (1.4) by applications of Krasnoselskii's fixed point theorem.

Besides, recently there have been published some results on second-order or third-order singular equations (see $[4,5,9,10,12,13,15,18,20,21,24,25]$ ). Torres [18] in 2007 investigated the existence of periodic solutions for the following second-order equation with a singularity of repulsive type

$$
\begin{equation*}
u^{\prime \prime}+a(t) u=f(t, u)+e(t) . \tag{1.5}
\end{equation*}
$$

His proof was based on Schauder's fixed point theorem. After that, Ma et al. [15] in 2014 improved Torres's result and given an assumption which is relatively weaker than condition in [18].

We are mainly motivated by the recent work $[15,16,18,22]$ and focus on Eq (1.1). By employing two available operators and applying Krasnoselskii's fixed point theorem, we obtain the existence of a positive periodic solution for (1.1). We would like to emphasize that the inclusion of the neutral operator in the singularity implies a new technical difficulty concerning the right choice of the operator.

## 2. Preparation

Firstly, we recall Krasnoselskii's fixed point theorem, which can be found in [1].
Lemma 2.1. Let $Y$ be a Banach space. Assume $\mathcal{K}$ is a bounded closed convex subset of $Y$. If $Q$, $\mathcal{S}: \mathcal{K} \rightarrow Y$ satisfy
(i) $Q u+\mathcal{S} y \in \mathcal{K}, \forall u, y \in \mathcal{K}$;
(ii) $\mathcal{S}$ is a contractive operator;
(iii) $Q$ is a completely continuous operator in $\mathcal{K}$.

Then $Q+\mathcal{S}$ has a fixed point in $\mathcal{K}$.
Consider the following third-order linear nonhomogeneous differential equation

$$
\left\{\begin{array}{l}
u^{\prime \prime \prime}+a(t) u=h(t)  \tag{2.1}\\
u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega), u^{\prime \prime}(0)=u^{\prime \prime}(\omega)
\end{array}\right.
$$

where $h \in C_{\omega}^{+}:=\{h \in C(\mathbb{R},(0+\infty)): h(t+\omega) \equiv h(t), \forall t \in \mathbb{R}\}$. Obviously, the calculation of the Green's function of (2.1) is very complicated. In order to get around the calculation of the Green's function of (2.1), we discuss the Green's function of differential equation as follows

$$
\left\{\begin{array}{l}
u^{\prime \prime \prime}+M u=h(t)  \tag{2.2}\\
u(0)=u(\omega), u^{\prime}(0)=u^{\prime}(\omega), u^{\prime \prime}(0)=u^{\prime \prime}(\omega)
\end{array}\right.
$$

where $M=\max _{t \in[0, \omega]}|a(t)|$ is defined in Section 1. The Eq (2.2) has a unique $\omega$-periodic solution

$$
\begin{equation*}
u(t)=\int_{0}^{\omega} G(t, s) h(s) \mathrm{d} s \tag{2.3}
\end{equation*}
$$

We introduce the positiveness of the Green's function $G(t, s)$, which can be found in [8, Lemma 2.2].
Lemma 2.2. (see [8, Lemma 2.2]) Assume that $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ holds, then the Green's function $G(t, s)$ satisfies

$$
0<l:=\frac{1}{3 M^{\frac{2}{3}}\left(\exp \left(M^{\frac{1}{3}} \omega\right)-1\right)} \leq G(t, s) \leq \frac{3+2 \exp \left(-\frac{M^{\frac{1}{3}} \omega}{2}\right)}{3 M^{\frac{2}{3}}\left(1-\exp \left(-\frac{M^{\frac{1}{3}} \omega}{2}\right)\right)^{2}}:=L
$$

for all $(t, s) \in[0, \omega] \times[0, \omega]$. Furthermore, $\int_{0}^{\omega} G(t, s) \mathrm{d} s=\frac{1}{M}$.
On the other hand, we give the property of neutral operator $(A u)(t):=u(t)-c u(t-\tau(t))$.
Lemma 2.3. (see [16, Lemma 2.1]) If $|c|<1$, then the operator $A u$ has a continuous inverse $A^{-1} u$ on

$$
X:=\{u \in C(\mathbb{R}, \mathbb{R}): u(t+\omega) \equiv u(t), \forall t \in \mathbb{R}\},
$$

satisfying
(1) $\left(A^{-1} f\right)(t)=f(t)+\sum_{j=1}^{\infty} c^{j} f\left(s-\sum_{i=1}^{j-1} \tau\left(D_{i}\right)\right), \quad \forall f \in X ;$
(2) $\left|\left(A^{-1} f\right)(t)\right| \leq \frac{\|f\|}{1-|c|}, \quad \forall f \in X$,
where $t-\tau(t)=s$ and $D_{j}=s-\sum_{i=1}^{j-1} \tau\left(D_{i}\right),\|f\|:=\max _{t \in[0, \omega]}|f(t)|$.
Let $v(t)=(A u)(t)$, then from Lemma 2.3, we obtain that $u(t)=\left(A^{-1} v\right)(t)$. Hence (1.1) can be transformed into

$$
\begin{equation*}
v^{\prime \prime \prime}(t)+a(t)\left(A^{-1} v\right)(t)=f\left(t, A^{-1} v(t)\right)+e(t) \tag{2.4}
\end{equation*}
$$

which can be further rewritten as

$$
\begin{equation*}
v^{\prime \prime \prime}(t)+a(t) v(t)-a(t) \mathcal{H}(v(t))=f\left(t, A^{-1} v(t)\right)+e(t), \tag{2.5}
\end{equation*}
$$

where $\mathcal{H}(v(t))=v(t)-\left(A^{-1} v\right)(t)=-c\left(A^{-1} v\right)(t-\tau(t))$.
Furthermore, we consider

$$
\begin{equation*}
v^{\prime \prime \prime}(t)+a(t) v(t)-a(t) \mathcal{H}(v(t))=h(t), \quad \text { for } h \in C_{\omega}^{+} . \tag{2.6}
\end{equation*}
$$

Formula (2.6) is rewritten in the following form

$$
\begin{equation*}
v^{\prime \prime \prime}+M v=(M-a(t)) v+a(t) \mathcal{H}(v(t))+h(t) . \tag{2.7}
\end{equation*}
$$

Define operators $\mathcal{T}, \mathcal{B}: X \rightarrow X$ by

$$
(\mathcal{T} h)(t)=\int_{0}^{\omega} G(t, s) h(s) d s, \quad(\mathcal{B} v)(t)=(M-a(t)) v+a(t) \mathcal{H}(v(t)) .
$$

Obviously, $(\mathcal{T} h)(t)>0$ if $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$, for all $t \in[0, \omega]$ and $h \in C_{\omega}^{+}$. By Lemma 2.3, $\|\mathcal{B}\| \leq M-m+\frac{M|c|}{1-|c|}$ if $|c|<1$. By (2.3), the solution of (2.7) can be written as the following form

$$
v(t)=(\mathcal{T} h)(t)+(\mathcal{T} \mathcal{B} v)(t) .
$$

In view of $c \in\left(-\frac{m}{M+m}, \frac{m}{M+m}\right)$, we arrive at

$$
\begin{equation*}
\|\mathcal{T} \mathcal{B}\| \leq\|\mathcal{T}\|\|\mathcal{B}\| \leq \frac{M-m+m|c|}{M(1-|c|)}<1, \tag{2.8}
\end{equation*}
$$

where we used the fact $\int_{0}^{\omega} G(t, s) d s=\frac{1}{M}$. Hence

$$
\begin{equation*}
v(t)=(I-\mathcal{T} \mathcal{B})^{-1}(\mathcal{T} h)(t) . \tag{2.9}
\end{equation*}
$$

Remark 2.1. If $|c|>1$, by Lemma [16, Lemma 2.1] and (2.8)-(2.9), we get $\|\mathcal{T} \mathcal{B}\| \leq 1-\frac{m}{M}+\frac{|c|}{|c|-1}$. Since $1-\frac{m}{M}+\frac{|c|}{|c|-1}>1$, we can not get $(I-\mathcal{T} \mathcal{B})^{-1}$. Therefore, the above method does not apply to the case of $|c|>1$.

Define an operator $\mathcal{P}: X \rightarrow X$ by

$$
(\mathcal{P h})(t)=(I-\mathcal{T} \mathcal{B})^{-1}(T h)(t) .
$$

If $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$, then (2.6) has the unique periodic solution $v(t)=(\mathscr{P} h)(t)$. Moveover, we get the following conclusion.

Lemma 2.4. Assume that $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}, c<0$ and $|c|<\min \left\{\frac{m}{M+m}, \sigma\right\}$ hold. Then $\mathcal{P}$ satisfies

$$
\begin{equation*}
(\mathcal{T} h)(t) \leq(\mathcal{P} h)(t) \leq \frac{M(1-|c|)}{m-(M+m)|c|}\|\mathcal{T} h\|, \text { for } h \in C_{\omega}^{+}, \tag{2.10}
\end{equation*}
$$

where $\sigma:=\frac{l}{L}$ and $0<\sigma \leq 1$.
Proof. By the Neumann expansion of $\mathcal{P}$, we have

$$
\begin{align*}
\mathcal{P} & =(I-\mathcal{T B})^{-1} \mathcal{T} \\
& =\left(I+\mathcal{T B}+(\mathcal{T B})^{2}+\cdots+(\mathcal{T B})^{n}+\cdots\right) \mathcal{T}  \tag{2.11}\\
& =\mathcal{T}+\mathcal{T} \mathcal{B} \mathcal{T}+(\mathcal{T B})^{2} T+\cdots+(\mathcal{T B})^{n} \mathcal{T}+\cdots .
\end{align*}
$$

From (2.11) and recalling that $\|\mathcal{T} \mathcal{H}\| \leq \frac{M-m+m\|c\|}{M(1-\|c\|)}<1$, we get

$$
(\mathcal{P} h)(t)=(I-\mathcal{T} \mathcal{B})^{-1}(\mathcal{T} h)(t) \leq \frac{\|\mathcal{T} h\|}{I-\|\mathcal{T} B\|} \leq \frac{M(1-|c|)}{m-(M+m)|c|}\|\mathcal{T} h\| .
$$

On the other hand, applying Lemma 2.2 and $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$, we obtain

$$
\begin{aligned}
(\mathcal{T} h)(t) & =\int_{0}^{\omega} G(t, s) h(s) d s \\
& \geq l \int_{0}^{\omega} h(s) d s \\
& =\frac{l}{L} L \int_{0}^{\omega} h(s) d s \\
& \geq \sigma \max _{t \in[0, \omega]} \int_{0}^{\omega} G(t, s) h(s) d s \\
& =\sigma\|\mathcal{T} h\|>0 .
\end{aligned}
$$

Since $c \in\left(-\frac{m}{M+m}, 0\right)$ and $|c| \leq \sigma$, using Lemma 2.3, we arrive at

$$
\begin{aligned}
\left(A^{-1} \mathcal{T} h\right)(t) & =(\mathcal{T} h)(t)+\sum_{j=1}^{\infty} c^{j}(\mathcal{T} h)\left(s-\sum_{i=1}^{j-1} \tau\left(D_{i}\right)\right) \\
& =(\mathcal{T} h)(t)+\sum_{j \geq 1} c^{j}(\mathcal{T} h)\left(s-\sum_{i=1}^{j-1} \tau\left(D_{i}\right)\right)-\sum_{j \geq 1}|c|^{j}(\mathcal{T} h)\left(s-\sum_{i=1}^{j-1} \tau\left(D_{i}\right)\right) \\
& \geq \frac{\sigma\|\mathcal{T} h\|}{1-c^{2}}-\frac{|c|\|\mathcal{T} h\|}{1-c^{2}} \\
& =\frac{(\sigma-|c|) \mid \boldsymbol{T} h \|}{1-c^{2}} \geq 0,
\end{aligned}
$$

and from (2.11), we see that

$$
(\mathcal{B T} h)(t)=(M-a(t))(\mathcal{T} h)(t)+a(t)\left(-c\left(A^{-1} \mathcal{T} h\right)(t-\tau(t))\right) \geq 0, \text { for } h \in C_{\omega}^{+} .
$$

Clearly, $(\mathcal{T} \mathcal{B T} h)(t) \geq 0$ if $h \in C_{\omega}^{+}$. Then we have from the above analysis that

$$
(\mathcal{P} h)(t)=(\mathcal{T} h)(t)+(\mathcal{T} \mathcal{B T} h)(t)+\left((\mathcal{T} \mathcal{B})^{2} \mathcal{T} h\right)(t)+\left((\mathcal{T} \mathcal{B})^{3} \mathcal{T} h\right)(t)+\cdots \geq(\mathcal{T} h)(t), \text { for } h \in C_{\omega}^{+} .
$$

Lemma 2.5. Assume that $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ and $c \in\left(0, \frac{m}{M+m}\right)$ hold. Then $P$ satisfies

$$
\begin{equation*}
\frac{m-(M+m) c}{M(1-c)}(\mathcal{T} h)(t) \leq(\mathcal{P} h)(t) \leq \frac{M(1-c)}{m-(M+m) c}\|\mathcal{T} h\|, \quad \text { for } h \in C_{\omega}^{+} . \tag{2.12}
\end{equation*}
$$

Proof. In view of $\|\mathcal{T} \mathcal{B}\|<1$, similarly as the proof of Lemma 2.4, we get that $(\mathcal{P} h)(t) \leq \frac{M(1-c)}{m-(M+m) c}\|\mathcal{T} h\|$.
Since $c \in\left(0, \frac{m}{M+m}\right)$, we can not get $(\mathcal{T B} h)(t) \geq 0$ for all $h \in C_{\omega}^{+}$. From (2.11), it is clear

$$
\begin{aligned}
\mathcal{P} & =\left(I+\mathcal{T B}+(\mathcal{T B})^{2}+(\mathcal{T B})^{3}+\cdots\right) \mathcal{T} \\
& =\left(I+(\mathcal{T B})^{2}+(\mathcal{T B})^{4}+\cdots\right) \mathcal{T}+\left(\mathcal{T B}+(\mathcal{T B})^{3}+(\mathcal{T B})^{5}+\cdots\right) \mathcal{T} \\
& =\left(I+(\mathcal{T B})^{2}+(\mathcal{T B})^{4}+\cdots\right) \mathcal{T}+\left(I+(\mathcal{T B})^{2}+(\mathcal{T B})^{4}+\cdots\right) \mathcal{T} \mathcal{T} \\
& =\left(I+(\mathcal{T B})^{2}+(\mathcal{T B})^{4}+\cdots\right)(I+\mathcal{T B}) \mathcal{T} .
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
(\mathcal{P} h)(t) & \geq(I+\mathcal{T} \mathcal{B})(\mathcal{T} h)(t) \geq(I-\|\mathcal{T} \mathcal{B}\|)(\mathcal{T} h)(t) \\
& \geq \frac{m-(m+M) c}{M(1-c)}(\mathcal{T} h)(t)>0, \quad \text { for } h \in C_{\omega}^{+} .
\end{aligned}
$$

Define operators $Q, \mathcal{S}: X \rightarrow X$ by

$$
\begin{equation*}
(Q u)(t)=\mathcal{P}(f(t, u)+e(t)), \quad(\mathcal{S} u)(t)=c u(t-\tau(t)) . \tag{2.13}
\end{equation*}
$$

From (2.6) and (2.13), the existence of periodic solutions to (1.1) is equivalent to the existence of solutions to operator equation as follows

$$
\begin{equation*}
Q u+\mathcal{S} u=u, \quad \text { in } X . \tag{2.14}
\end{equation*}
$$

## 3. Periodic solution for (1.1) in the case that $c \in\left(0, \frac{m}{M+m}\right]$

In this section, we establish the existence of a positive periodic solution for (1.1) in the case that $c \in\left(0, \frac{m}{M+m}\right]$ by using Krasnoselskii's fixed point theorem. Define the function $\gamma: \mathbb{R} \rightarrow \mathbb{R}$

$$
\gamma(t):=\int_{0}^{\omega} G(t, s) e(s) d s,
$$

and

$$
\gamma^{*}:=\max _{t \in \mathbb{R}} \gamma(t), \quad \gamma_{*}:=\min _{t \in \mathbb{R}} \gamma(t) .
$$

By analysis of $\gamma(t)$, we consider the following three cases.
Case (I) $\gamma_{*}>0$.
Theorem 3.1. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ and $c \in\left(0, \frac{m}{m+M}\right]$ hold. Furthermore, assume that the following conditions hold:
$\left(H_{1}\right)$ There exist continuous, non-negative functions $g(u), h(u)$ and $\zeta(t)$ such that

$$
0 \leq f(t, u) \leq \zeta(t)(g(u)+h(u)) \quad \text { for all }(t, u) \in[0, \omega] \times(0, \infty),
$$

and $g(u)>0$ is non-increasing and $h(u)$ is non-decreasing in $u \in(0, \infty)$.
$\left(H_{2}\right)$ There exists a positive constant $R>0$ such that

$$
\frac{M}{m-(M+m) c}\left(g\left(\frac{m-(M+m) c}{M(1-c)} \gamma_{*}\right)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+\gamma^{*}\right) \leq R,
$$

where $\Lambda(t)=\int_{0}^{\omega} G(t, s) \zeta(s) d s$ and $\Lambda^{*}:=\max _{t \in \mathbb{R}} \Lambda(t)$.
If $\gamma_{*}>0$, then (1.1) has at least one positive periodic solution.
Proof. An $\omega$-periodic solution of (1.1) is just a fixed point of the following operator equation

$$
\begin{equation*}
(Q u)(t)+(\mathcal{S} u)(t)=u(t) . \tag{3.1}
\end{equation*}
$$

Let $R$ be the positive constant satisfying $\left(H_{2}\right)$ and

$$
r:=\frac{m-(M+m) c}{M(1-c)} \gamma_{*} .
$$

Then we have $R>r>0$ since $R>\gamma^{*}>\frac{m-(M+m) c}{M(1-c)} \gamma_{*}$. Now we define the set

$$
\begin{equation*}
\mathcal{K}=\{u \in X: r \leq u(t) \leq R \text { for all } t\} . \tag{3.2}
\end{equation*}
$$

Obviously, $\mathcal{K}$ is a bounded closed convex set in $X$. Moreover, for any $u \in \mathcal{K}$, it is easy to verify that $Q, \mathcal{S}$ are continuous and $(Q u)(t+\omega)=(Q u)(t),(\mathcal{S} u)(t+\omega)=(\mathcal{S u})(t)$, that is, $Q(\mathcal{K}) \subset X, \mathcal{S}(\mathcal{K}) \subset X$.

Next we claim that $Q u+\mathcal{S} y \in \mathcal{K}$, for any $u, y \in \mathcal{K}$. By Lemma 2.5 and non-negative sign of $G(t, s)$ and $f(t, u)$, we have

$$
\begin{align*}
& (Q u)(t)+(\mathcal{S y})(t) \\
= & \mathcal{P}(f(t, u(t))+e(t)))+c y(t-\tau(t)) \\
\geq & \left.\frac{m-(M+m) c}{M(1-c)} \mathcal{T}(f(t, u(t))+e(t))\right)+c y(t-\tau(t))  \tag{3.3}\\
= & \frac{m-(M+m) c}{M(1-c)}\left(\int_{0}^{\omega} G(t, s)(f(s, u(s)) d s+\gamma(t))+c y(t-\tau(t))\right. \\
\geq & \frac{m-(M+m) c}{M(1-c)} \gamma_{*}:=r>0,
\end{align*}
$$

where we used the fact $\int_{0}^{\omega} G(t, s) d s=\frac{1}{M}$.
On the other hand, using Lemma 2.5, we have

$$
\begin{aligned}
& (Q u)(t)+(\mathcal{S y})(t) \\
= & \mathcal{P}(f(t, u(t))+e(t)))+c y(t-\tau(t)) \\
\leq & \left.\left.\frac{M(1-c)}{m-(M+m) c} \max _{t \in[0, \omega]} \right\rvert\, \mathcal{T}(f(t, u(t))+e(t))\right) \mid+c y(t-\tau(t)) \\
= & \frac{M(1-c)}{m-(M+m) c} \max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)\right|+c y(t-\tau(t)) \\
\leq & \frac{M(1-c)}{m-(M+m) c}\left\{\max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s\right|+\|\gamma\|\right\}+c y(t-\tau(t)),
\end{aligned}
$$

since $\gamma_{*}>0$, then $\gamma(t)>0$ and $\|\gamma\|:=\max _{t \in[0, \omega]}|\gamma(t)|=\gamma^{*}$. By conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$, we have

$$
\begin{align*}
& (Q u)(t)+(\mathcal{S} y)(t) \\
\leq & \frac{M(1-c)}{m-(M+m) c}\left\{\max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) \zeta(s)(g(u(s))+h(u(s))) d s\right|+\gamma^{*}\right\}+c y(t-\tau(t))  \tag{3.4}\\
\leq & \frac{M(1-c)}{m-(M+m) c}\left\{g(r)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+\gamma^{*}\right\}+c R \\
\leq & R .
\end{align*}
$$

Combining (3.3) and (3.4), we get $Q u+\mathcal{S} y \in \mathcal{K}$ for all $u, y \in \mathcal{K}$.

Furthermore, for any $u_{1}, u_{2} \in \mathcal{K}$, we have

$$
\begin{aligned}
\left\|\left(\mathcal{S} u_{1}\right)(t)-\left(\mathcal{S} u_{2}\right)(t)\right\| & =\left|c u_{1}(t-\tau(t))-c u_{2}(t-\tau(t))\right| \\
& \leq\left|c\| \| u_{1}-u_{2}\right| \mid,
\end{aligned}
$$

which implies that

$$
\begin{equation*}
\left\|\left(\mathcal{S} u_{1}\right)(t)-\left(\mathcal{S} u_{2}\right)(t)\right\| \leq|c|\left\|u_{1}-u_{2}\right\| . \tag{3.5}
\end{equation*}
$$

In view of $|c|<1, S$ is a contractive operator.
By [16, Theorem 3.1], we get that $Q$ is a completely continuous. Therefore, by Krasnoselskii's fixed point theorem, $Q+\mathcal{S}$ has a fixed point $u \in \mathcal{K}$, that is to say, (1.1) has a positive $\omega$-periodic solution $u(t)$ with $u \in[r, R]$.
Remark 3.1. If $|c|>1$, from (3.11), we can not get $\mathcal{S}$ is a contractive operator. Therefore, the above method of Theorem 3.1 does not apply to the case of $|c|>1$.
Corollary 3.1. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ and $c \in\left(0, \frac{m}{M+m}\right]$ hold. Assume the following condition holds:
$\left(F_{1}\right)$ There exist a continuous function $d(t)>0$ and a constant $\rho>0$ such that satisfy

$$
0 \leq f(t, u) \leq \frac{d(t)}{u^{\rho}}, \quad \text { for all }(t, u) \in[0, \omega] \times(0, \infty)
$$

If $\gamma_{*}>0$, then (1.1) has at least one positive periodic solution.
Proof. We apply Theorem 3.1. We take

$$
\zeta(t)=d(t), \quad g(u)=\frac{1}{u^{\rho}}, \quad h(u)=0 .
$$

Then condition $\left(H_{1}\right)$ is satisfied. Next, we consider the condition $\left(H_{2}\right)$ is also satisfied. In fact, we take $R>0$ with

$$
\frac{M}{m-(M+m) c}\left(\frac{M^{\rho}(1-c)^{\rho} \Psi^{*}}{(m-(M+m) c)^{\rho} \gamma_{*}^{\rho}}+\gamma^{*}\right) \leq R
$$

since $c \in\left(0, \frac{m}{M+m}\right)$ and $\Psi(t):=\int_{0}^{\omega} G(t, s) d(s) d t$.
Corollary 3.2. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ and $c \in\left(0, \frac{m}{M+m}\right]$ hold. Assume the following condition holds:
$\left(F_{2}\right)$ There exist a continuous function $d(t)>0$ and constants $\rho>0,0 \leq \eta<1$ such that satisfy

$$
0 \leq f(t, u) \leq \frac{d(t)}{u^{\rho}}+d(t) u^{\eta}, \quad \text { for all }(t, u) \in[0, \omega] \times(0, \infty)
$$

If $\gamma_{*}>0$, then (1.1) has at least one positive periodic solution.
Proof. We apply Theorem 3.1. We take

$$
\zeta(t)=d(t), \quad g(u)=\frac{1}{u^{\rho}}, \quad h(u)=u^{\eta} .
$$

Then condition $\left(H_{1}\right)$ is satisfied and the existence condition $\left(H_{2}\right)$ is also satisfied. The existence condition $\left(H_{2}\right)$ becomes

$$
\begin{equation*}
\frac{M}{m-(M+m) c}\left(\Psi^{*}\left(\frac{M^{\rho}(1-c)^{\rho}}{(m-(M+m) c)^{\rho} \gamma_{*}^{\rho}}+(R)^{\eta}\right)+\gamma^{*}\right) \leq R, \tag{3.6}
\end{equation*}
$$

Since $\rho>0,0 \leq \eta<1$ and $c \in\left(0, \frac{m}{M+m}\right)$, we can choose $R>0$ large enough such that (3.6) is satisfied.

In the following, we investigate (1.1) in the case that attractive-repulsive singularities.
Corollary 3.3. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ and $c \in\left(0, \frac{m}{M+m}\right]$ hold. Assume that the following condition holds:
( $F_{3}$ ) There exist positive constants $\alpha>\beta>0$ and $\mu>0$ such that

$$
f(t, u)=\frac{1}{u^{\alpha}}-\frac{\mu}{u^{\beta}} .
$$

If $\gamma_{*}>0$, then there exists a positive constant $\mu_{1}$ such that (1.1) has at least one positive periodic solution for each $0 \leq \mu \leq \mu_{1}$.
Proof. We apply Theorem 3.1. Take

$$
g(x)=\frac{1}{u^{\alpha}}, \quad h(u) \equiv 0, \quad \zeta(t) \equiv 1 .
$$

Firstly, we consider that condition $\left(H_{2}\right)$ is satisfied. Take $R>0$ with

$$
R=\frac{M}{m-(M+m) c}\left(\frac{1}{M\left(\gamma_{*}\right)^{\alpha}}+\gamma^{*}\right)
$$

where $\int_{0}^{\omega} G(t, s) d s=\frac{1}{M}$. Next, we consider that the condition $\left(H_{1}\right)$ is also satisfied. In fact, $f(t, u) \geq 0$ if and only if $\mu \leq u^{\beta-\alpha}$.

In view of $\beta<\alpha$, then we have $\mu<R^{\beta-\alpha}$. As a consequence, the result holds for

$$
\mu_{1}:=\left(\frac{M}{m-(M+m) c}\left(\frac{1}{M\left(\gamma_{*}\right)^{\alpha}}+\gamma^{*}\right)\right)^{\beta-\alpha} .
$$

By Theorem 3.1, we consider a special case of $c$, i.e., $c=0$.
Theorem 3.2. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ and $c=0$ hold, and $f(t, u)$ satisfies condition $\left(H_{1}\right)$. Furthermore, assume that the following conditions holds:
$\left(H_{2}^{*}\right)$ There exists a positive constant $R>0$ such that

$$
\frac{M}{m}\left(g\left(\gamma_{*}\right)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+\gamma^{*}\right) \leq R .
$$

If $\gamma_{*}>0$, then (1.2) has at least one positive periodic solution.
Remark 3.2. Theorem 3.2 extends and improves [26, Theorem 3.3].
Case (II) $\gamma_{*}=0$.
Theorem 3.3. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ and $c \in\left(0, \frac{m}{M+m}\right]$ hold. And $f(t, u)$ satisfies $\left(H_{1}\right)$. Furthermore, assume that the following conditions hold:
$\left(H_{3}\right)$ For each $L>0$, there exists a continuous function $\phi_{L}>0$ such that $f(t, u) \geq \phi_{L}(t)$ for all $(t, u) \in[0, \omega] \times(0, L]$.
$\left(H_{4}\right)$ There exists $R>0$ such that $R>\left(\Phi_{R}\right)_{*}^{\prime}:=\frac{m-(M+m) c}{M(1-c)}\left(\Phi_{R}\right)_{*}$ and

$$
\frac{M}{m-c(m+M)}\left(g\left(\left(\Phi_{R}\right)_{*}^{\prime}\right)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+\gamma^{*}\right) \leq R,
$$

where $\Phi_{R}(t)=\int_{0}^{\omega} G(t, s)\left(\phi_{R}\right)(s) d s$.
If $\gamma_{*}=0$, then (1.1) has at least one positive periodic solution.

Proof. We follow the same strategy and notation as in the proof of Theorem 3.1. Let $R$ be the positive constant satisfying $\left(H_{4}\right)$ and let $r:=\frac{m-(M+m) c}{M(1-c)}\left(\Phi_{R}\right)_{*}$; then $R>r>0$ since $R>\left(\Phi_{R}\right)_{*}^{\prime}$. Next we prove that $Q u+\mathcal{S} y \in \mathcal{K}$ for all $u, y \in \mathcal{K}$.

For each $u, y \in \mathcal{K}$ and for all $t \in[0, \omega]$, by Lemma 2.5 and $\left(H_{3}\right)$, we get

$$
\begin{align*}
& (Q u)(t)+(\mathcal{S} y)(t) \\
= & \mathcal{P}(f(t, u(t))+e(t)))+c y(t-\tau(t)) \\
\geq & \left.\frac{m-(M+m) c}{M(1-c)} \mathcal{T}(f(t, u(t))+e(t))\right)+c y(t-\tau(t)) \\
= & \frac{m-(M+m) c}{M(1-c)}\left(\int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)\right)+c y(t-\tau(t))  \tag{3.7}\\
\geq & \frac{m-(M+m) c}{M(1-c)}\left(\int_{0}^{\omega} G(t, s) \phi_{R}(s) d s+\gamma(t)\right)+c y(t-\tau(t)) \\
\geq & \frac{m-(M+m) c}{M(1-c)}\left(\Phi_{R}\right)_{*}:=r>0 .
\end{align*}
$$

On the other hand, using Lemma 2.5, we see that

$$
\begin{align*}
& (Q u)(t)+(\mathcal{S} y)(t) \\
= & \mathcal{P}(f(t, u(t))+e(t)))+c y(t-\tau(t)) \\
\leq & \left.\left.\frac{M(1-c)}{m-(M+m) c} \max _{t \in[0, \omega]} \right\rvert\, \mathcal{T}(f(t, u(t))+e(t))\right) \mid+c y(t-\tau(t)) \\
= & \frac{M(1-c)}{m-(M+m) c} \max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)\right|+c y(t-\tau(t))  \tag{3.8}\\
\leq & \frac{M(1-c)}{m-(M+m) c} \max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)\right|+c y(t-\tau(t)) \\
\leq & \frac{M(1-c)}{m-(M+m) c}\left\{\max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s\right|+\|\gamma\|\right\}+c y(t-\tau(t)) .
\end{align*}
$$

By conditions $\left(H_{1}\right)$ and $\left(H_{4}\right)$, we have

$$
\begin{align*}
& (Q u)(t)+(\mathcal{S y})(t) \\
\leq & \frac{M(1-c)}{m-(M+m) c}\left\{\max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) \zeta(s)(g(u(s))+h(u(s))) d s\right|+\|\gamma\|\right\}+c y(t-\tau(t))  \tag{3.9}\\
\leq & \frac{M(1-c)}{m-(M+m) c}\left\{g(r)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+\|\gamma\|\right\}+c R \\
\leq & R .
\end{align*}
$$

Combining (3.7) and (3.9), we get $Q u+\mathcal{S} y \in \mathcal{K}$ for all $u, y \in \mathcal{K}$.
Similarly, we get that $Q$ is a completely continuous and $\mathcal{S}$ is a contractive operator in $X$. Therefore, by Krasnoselskii's fixed point theorem, our result is proven.

Corollary 3.4. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ and $c \in\left(0, \frac{m}{M+m}\right]$ hold. Assume that the following condition holds:
$\left(F_{4}\right)$ There exist continuous functions $d(t), \hat{d}(t)>0$ and $0<\rho<1$ such that satisfy

$$
0 \leq \frac{\hat{d}(t)}{u^{\rho}} \leq f(t, u) \leq \frac{d(t)}{u^{\rho}}, \quad \text { for all }(t, u) \in[0, \omega] \times(0, \infty)
$$

If $\gamma_{*}=0$, then (1.1) has at least one positive periodic solution.
Proof. We apply Theorem 3.3. We take

$$
\phi_{L}(t)=\frac{\hat{d}(t)}{L^{\rho}}, \quad \zeta(t)=d(t), \quad g(u)=\frac{1}{u^{\rho}}, \quad h(u)=0 .
$$

Then conditions $\left(H_{1}\right)$ and $\left(H_{3}\right)$ are satisfied, and the existence condition $\left(H_{4}\right)$ becomes

$$
\begin{equation*}
R>\frac{(m-(M+m) c) \hat{\Psi}_{*}}{M(1-c) R^{\rho}}=r, \quad \frac{M}{m-(M+m) c}\left(\left(\frac{M(1-c) R^{\rho}}{(m-(M+m) c) \hat{\Psi}_{*}}\right)^{\rho} \Psi^{*}+\|\gamma\|\right) \leq R \tag{3.10}
\end{equation*}
$$

where $\hat{\Psi}=\int_{0}^{\omega} G(t, s) \hat{d}(t) d t$. Note that $\Psi_{*}>0$, since $0<\rho<1$, we can choose appropriate $R>0$ so that (3.10) is satisfied and the proof is complete.

Case (III) $\gamma^{*}<0$.
Theorem 3.4. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}$ and $c \in\left(0, \frac{m}{M+m}\right]$ hold. And $f(t, u)$ satisfies $\left(H_{1}\right)$ and $\left(H_{3}\right)$. Furthermore, assume that the following condition holds:
$\left(H_{5}\right)$ There exists $R>0$ such that $R>\frac{m-(M+m) c}{M(1-c)}\left(\left(\Phi_{R}\right)_{*}+\gamma_{*}\right)>0$ and

$$
\frac{M}{m-(M+m) c} g\left(\frac{m-(M+m) c}{M(1-c)}\left(\left(\Phi_{R}\right)_{*}+\gamma_{*}\right)\right)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*} \leq R .
$$

If $\gamma^{*}<0$, then (1.1) has at least one positive periodic solution.
Proof. We follow the same strategy and notation as in the proof of Theorem 3.1. Let $R$ be the positive constant satisfying $\left(H_{5}\right)$ and let $r:=\frac{m-(M+m) c}{M(1-c)}\left(\left(\Phi_{R}\right)_{*}+\gamma_{*}\right)$; then $R>r>0$ since $R>\frac{m-(M+m) c}{M(1-c)}\left(\left(\Phi_{R}\right)_{*}+\gamma_{*}\right)$. Next we prove that $Q u+\mathcal{S} y \in \mathcal{K}$ for all $u, y \in \mathcal{K}$.

For each $u, y \in \mathcal{K}$ and for all $t \in[0, \omega]$, by Lemma 2.5 and $\left(H_{3}\right)$, we deduce

$$
\begin{align*}
& (Q u)(t)+(\mathcal{S} y)(t) \\
= & \mathcal{P}(f(t, u(t))+e(t)))+c y(t-\tau(t)) \\
\geq & \left.\frac{m-(M+m) c}{M(1-c)} \mathcal{T}(f(t, u(t))+e(t))\right)+c y(t-\tau(t)) \\
= & \frac{m-(M+m) c}{M(1-c)}\left(\int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)\right)+c y(t-\tau(t))  \tag{3.11}\\
\geq & \frac{m-(M+m) c}{M(1-c)}\left(\int_{0}^{\omega} G(t, s) \phi_{R}(s) d s+\gamma(t)\right)+c y(t-\tau(t)) \\
\geq & \frac{m-(M+m) c}{M(1-c)}\left(\left(\Phi_{R}\right)_{*}+\gamma_{*}\right):=r>0 .
\end{align*}
$$

On the other hand, applying Lemma 2.5 and (3.8), we arrive at

$$
(Q u)(t)+(\mathcal{S} y)(t)
$$

$$
\begin{aligned}
& \leq \frac{M(1-c)}{m-(M+m) c} \max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)\right|+c y(t-\tau(t)) \\
& \leq \frac{M(1-c)}{m-(M+m) c} \max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s\right|+c y(t-\tau(t)),
\end{aligned}
$$

since $\gamma^{*} \leq 0, G(t, s)$ and $f\left(t, u(t)\right.$ are non-negative, $\left(\Phi_{R}\right)_{*}+\gamma_{*}>0$, then we know $\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)\right| \leq\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s\right|$. By conditions $\left(H_{1}\right)$ and $\left(H_{5}\right)$, we get

$$
\begin{align*}
& (Q u)(t)+(S y)(t) \\
\leq & \frac{M(1-c)}{m-(M+m) c} \max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) \zeta(s)(g(u(s))+h(u(s))) d s\right|+c y(t-\tau(t)) \\
\leq & \frac{M(1-c)}{m-(M+m) c} g(r)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+c R  \tag{3.12}\\
\leq & R .
\end{align*}
$$

Combining (3.11) and (3.12), we get $Q u+\mathcal{S} y \in \mathcal{K}$ for all $u, y \in \mathcal{K}$.
Similarly, we get that $Q$ is a completely continuous and $\mathcal{S}$ is a contractive operator in $X$. Therefore, by Krasnoselskii's fixed point theorem, our result is proven.
Corollary 3.5. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}, c \in\left(0, \frac{m}{M+m}\right]$ and $\left(F_{4}\right)$ hold. If $\gamma^{*}<0$ and $\gamma_{*} \geq\left(\hat{\Psi}_{*} \frac{(m-(M+m))^{\rho}}{\left(M \Psi^{*}\right)^{\rho}} \rho^{2}\right)^{\frac{1}{1-\rho^{2}}}\left(1-\frac{1}{\rho^{2}}\right)$, then (1.1) has at least one positive periodic solution.
Proof. We apply Theorem 3.4. We take

$$
\phi_{L}(t)=\frac{\hat{d}(t)}{L^{\rho}}, \quad \zeta(t)=d(t), \quad g(u)=\frac{1}{u^{\rho}}, \quad h(u)=0 .
$$

Then conditions $\left(H_{1}\right)$ and $\left(H_{3}\right)$ hold. Next, we consider that the condition $\left(H_{5}\right)$ is also satisfied. In fact, take $R>0$ with

$$
R:=\frac{M}{m-(M+m) c}\left(\frac{\Psi^{*}}{r^{\rho}}\right),
$$

then $\frac{m-(M+m) c}{M(1-c)}\left(\left(\Phi_{R}\right)_{*}+\gamma_{*}\right) \geq r$ holds if $r$ verifies

$$
\frac{m-(M+m) c}{M(1-c)}\left(\hat{\Psi}_{*} \frac{(m-(M+m) c)^{\rho}}{\left(M \Psi^{*}\right)^{\rho}}(r)^{\rho^{2}}+\gamma_{*}\right) \geq r,
$$

or equivalently,

$$
\gamma_{*} \geq f(r):=\frac{m-(M+m) c}{M(1-c)} r-\hat{\Psi}_{*} \frac{(m-(M+m) c)^{\rho}}{\left(M \Psi^{*}\right)^{\rho}}(r)^{\rho^{2}} .
$$

The function $f(r)$ possesses a minimum at $r_{0}:=\left(\hat{\Psi}_{*} \frac{(m-(M+m) c)^{\rho}}{\left(M \Psi^{*}\right) \rho^{2}} \rho^{2}\right)^{\frac{1}{1-\rho^{2}}}$. Let $r=r_{0}$. Then the $\left(\Phi_{R}\right)_{*}+\gamma_{*}>$ 0 holds if $\gamma_{*} \geq f\left(r_{0}\right)$, which is just the condition $\gamma_{*} \geq\left(\hat{\Psi}_{*} \frac{(m-(M+m) c)^{\rho+1}}{\left(M \Psi^{*}\right) \rho^{M} M(1-c)} \rho^{2}\right)^{\frac{1}{1-\rho^{2}}}\left(1-\frac{1}{\rho^{2}}\right)$. The the condition $\left(H_{5}\right)$ holds directly by the choice of $R$, and it would remain to prove that $R=\frac{M}{m-(M+m)^{c}}\left(\frac{\Psi^{*}}{\left(r_{0}\right)^{\rho}}\right)>r_{0}$. This is easily verified through elementary computations.

In the end of this section, we illustrate our results with one example.
Example 3.1. Consider the following a singular equation

$$
\begin{equation*}
\left(u-\frac{1}{8} u\left(t-\cos ^{2} t\right)\right)^{\prime \prime \prime}+\frac{1}{8}(\sin (2 t)+2) u=\frac{\cos (2 t)+4}{u^{\rho}}+(\cos (2 t)+4) u^{\frac{1}{2}}+e^{\sin ^{2} t}, \tag{3.13}
\end{equation*}
$$

where $\rho$ is a real constant and $\rho>0$.
Comparing Eq (3.13) with $\operatorname{Eq}$ (1.1), it is easy to see that

$$
c=\frac{1}{8}, \tau=\cos ^{2} t, a(t)=\frac{1}{8}(\sin (2 t)+2), \omega=\pi, f(t, u)=\frac{\cos (2 t)+4}{u^{\rho}}+(\cos (2 t)+4) u^{\frac{1}{2}}, e(t)=e^{\sin ^{2} t} .
$$

Furthermore, we have

$$
m=\frac{1}{8}, M=\frac{3}{8}<0.456<\frac{64}{81 \sqrt{3}}, c \in\left(0, \frac{1}{4}\right), d(t)=\cos (2 t)+4, \eta=\frac{1}{2},
$$

and

$$
\gamma(t)=\int_{0}^{\omega} G(t, s) e(t) d s>0
$$

Obviously, condition $\left(F_{2}\right)$ holds, and $\gamma_{*}>0$. Therefore, applying Corollary 3.2, we get that Eq (3.13) has at least one positive $\pi$-periodic solution.

## 4. Periodic solution for (1.1) in the case that $c \in\left(-\frac{m}{M+m}, 0\right)$

In this section, we investigate the existence of positive periodic solutions for (1.1) in the case that $c \in\left(-\frac{m}{M+m}, 0\right)$ by using Krasnoselskii's fixed point theorem.

Case (I) $\gamma_{*}>0$.
Theorem 4.1. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}, c<0$ and $|c|<\min \left\{\frac{m}{M+m}, \sigma\right\}$ hold, and $f(t, u)$ satisfies conditions $\left(H_{1}\right)$. Furthermore, assume that the following condition holds:
$\left(H_{2}^{* *}\right)$ There exists a positive constant $R$ such that $\frac{\gamma_{*}}{1+|c|}<R<\frac{\gamma_{*}}{|c|}$ and

$$
\frac{M(1-|c|)}{m-(M+m)|c|}\left(g\left(\gamma_{*}-|c| R\right)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+\gamma^{*}\right) \leq R
$$

If $\gamma_{*}>0$, then (1.1) has at least one positive periodic solution.
Proof. We follow the same strategy and notation as in the proof of Theorem 3.1. Let $R$ be the positive constant satisfying $\left(H_{2}^{* *}\right)$ and let $r:=\gamma_{*}-|c| R$; then $R>r>0$ since $R>\frac{\gamma_{*}}{1+|c|}$. Next we prove that $Q u+\mathcal{S} y \in \mathcal{K}$ for all $u, y \in \mathcal{K}$.

For any $u, y \in \mathcal{K}$, by Lemma 2.4 and non-negative sign of $G(t, s)$ and $f(t, u)$, we see that

$$
\begin{align*}
&(Q u)(t)+(\mathcal{S y})(t) \\
&= \mathcal{P}(f(t, u(t))+e(t))+c y(t-\tau(t)) \\
& \geq \mathcal{T}(f(t, u(t))+e(t))+c y(t-\tau(t))  \tag{4.1}\\
&= \int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)-|c| y(t-\tau(t)) \\
& \geq \gamma_{*}-|c| R:=r>0 .
\end{align*}
$$

On the other hand, applying Lemma 2.4, we obtain

$$
\begin{aligned}
& (Q u)(t)+(\mathcal{S y})(t) \\
= & \mathcal{P}(f(t, u(t))+e(t))+c y(t-\tau(t)) \\
\leq & \frac{M(1-|c|)}{m-(M+m)|c|} \max _{t \in[0, \omega]}|\mathcal{T}(f(t, u(t))+e(t))|+c y(t-\tau(t)) \\
= & \frac{M(1-|c|)}{m-(M+m)|c|} \max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)\right|-|c| y(t-\tau(t)) \\
\leq & \frac{M(1-|c|)}{m-(M+m)|c|}\left(\max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s\right|+\gamma(t)\right),
\end{aligned}
$$

since $c<0$ and $y \in \mathcal{K}$. By conditions $\left(H_{1}\right)$ and $\left(H_{2}^{* *}\right)$, we get

$$
\begin{align*}
& Q u)(t)+(S y)(t) \\
\leq & \frac{M(1-|c|)}{m-(M+m)|c|}\left\{\max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) \zeta(s)(g(u(s))+h(u(s))) d s\right|+\gamma^{*}\right\} \\
\leq & \frac{M(1-|c|)}{m-(M+m)|c|}\left\{g(r)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+\gamma^{*}\right\}  \tag{4.2}\\
\leq & R .
\end{align*}
$$

Combining (4.1) and (4.2), we get $Q u+\mathcal{S} y \in \mathcal{K}$ for all $u, y \in \mathcal{K}$.
Similarly, we get that $Q$ is a completely continuous and $\mathcal{S}$ is a contractive operator in $\mathcal{X}$. Therefore, by Krasnoselskii's fixed point theorem, our results is proven.
Corollary 4.1. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}, c<0,|c|<\min \left\{\frac{m}{M+m}, \sigma\right\}$ and $\left(F_{1}\right)$ hold. If

$$
\begin{equation*}
\gamma_{*}>\left(\frac{(m-(M+m)|c|}{\rho|c| M(1-|c|) \Psi^{*}}\right)^{\frac{1}{\rho+1}}+\frac{1}{\rho}\left(\frac{(m-(M+m)|c|}{\rho|c| M(1-|c|) \Psi^{*}}\right)^{-\frac{2 \rho+1}{\rho+1}}+\frac{|c| M(1-|c|)}{m-(M+m)|c|} \gamma^{*}>0, \tag{4.3}
\end{equation*}
$$

then (1.1) has at least one positive periodic solution.
Proof. We apply Theorem 4.1. We take

$$
\zeta(t)=d(t), \quad g(u)=\frac{1}{u^{\rho}}, \quad h(u)=0 .
$$

Then condition $\left(H_{1}\right)$ is satisfied. Next, we consider the condition $\left(H_{2}^{* *}\right)$ is also satisfied. In fact, taking $R=\frac{M(1-|c|)}{m-(M+m)|c|}\left(\frac{\Psi^{*}}{r^{p}}+\gamma^{*}\right)$, the $\gamma_{*}-|c| R \geq r$ holds if and only if $r$ verifies

$$
\gamma_{*}-\frac{|c| M(1-|c|)}{m-(M+m)|c|}\left(\frac{\Psi^{*}}{r^{\rho}}+\gamma^{*}\right) \geq r
$$

or equivalently,

$$
\gamma_{*} \geq f(r):=r+\frac{|c| M(1-|c|)}{m-(M+m)|c|}\left(\frac{\Psi^{*}}{r^{\rho}}+\gamma^{*}\right) .
$$

The function $f(r)$ possesses a minimum at $r_{0}:=\left(\frac{(m-(M+m)) c \mid}{\rho|c| M\left(1-c|c| \psi^{*}\right.}\right)^{\frac{1}{\rho+1}}$. Let $r=r_{0}$. Then $\gamma_{*}-c R \geq r$ holds if $\gamma_{*} \geq f\left(r_{0}\right)$, which is just the condition (4.3). The ( $H_{2}^{* *}$ ) holds directly by the choice of $R$, and it would remain to prove that $R=\frac{M(1-c \mid)}{m-(M+m) c|c|}\left(\frac{\Psi^{*}}{r_{0}^{\rho}}+\gamma^{*}\right)>r_{0}$. This is easily verified through elementary computations.

Case (II) $\gamma_{*}=0$.
Theorem 4.2. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}, c<0$ and $|c|<\min \left\{\frac{m}{M+m}, \sigma\right\}$ hold. And $f(t, u)$ satisfies $\left(H_{1}\right)$ and $\left(H_{3}\right)$. Furthermore, assume that the following condition holds:
$\left(H_{4}^{* *}\right)$ There exists $R>0$ such that $\frac{\left(\Phi_{R}\right)_{*}}{1+|c|}<R<\frac{\left(\Phi_{R}\right)_{*}}{|c|}$ and

$$
\frac{M(1-|c|)}{m-(M+m)|c|}\left(g\left(\left(\Phi_{R}\right)_{*}-|c| R\right)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+\|\gamma\|\right) \leq R .
$$

If $\gamma_{*}>0$, then (1.1) has at least one positive periodic solution.
Proof. We follow the same strategy and notation as in the proof of Theorem 4.1. Let $R$ be the positive constant satisfying $\left(H_{4}^{* *}\right)$ and let $r:=\left(\Phi_{R}\right)_{*}-|c| R$; then $R>r>0$ since $R>\frac{\left(\Phi_{R}\right)_{s}}{1+|c|}$. Next we prove that $Q u+\mathcal{S} y \in \mathcal{K}$ for all $u, y \in \mathcal{K}$.

For any $u, y \in \mathcal{K}$, by Lemma 2.5 and $\left(H_{3}\right)$, we have

$$
\begin{align*}
&(Q u)(t)+(\mathcal{S} y)(t) \\
&= \mathcal{P}(f(t, u(t))+e(t))+c y(t-\tau(t)) \\
& \geq \mathcal{T}(f(t, u(t))+e(t))+c y(t-\tau(t)) \\
&= \int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)-|c| y(t-\tau(t))  \tag{4.4}\\
& \geq \int_{0}^{\omega} G(t, s) \phi_{R}(s) d s+\gamma(t)-|c| y(t-\tau(t)) \\
& \geq\left(\Phi_{R}\right)_{*}-|c| R:=r>0 .
\end{align*}
$$

On the other hand, applying Lemma 2.4, we get

$$
\begin{aligned}
& (Q u)(t)+(\mathcal{S y})(t) \\
= & \mathcal{P}(f(t, u(t))+e(t))+c y(t-\tau(t)) \\
\leq & \frac{M(1-|c|)}{m-(M+m)|c|} \max _{t \in[0, \omega]}|\mathcal{T}(f(t, u(t))+e(t))|+c y(t-\tau(t)) \\
= & \frac{M(1-|c|)}{m-(M+m)|c|} \max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s+\gamma(t)\right|-|c| y(t-\tau(t)) \\
\leq & \frac{M(1-|c|)}{m-(M+m)|c|}\left(\max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) f(s, u(s)) d s\right|+\|\gamma\|\right),
\end{aligned}
$$

since $c<0$ and $y \in K$. By conditions $\left(H_{1}\right)$ and $\left(H_{4}^{* *}\right)$, it is clear

$$
\begin{align*}
& (Q u)(t)+(S y)(t) \\
\leq & \frac{M(1-|c|)}{m-(M+m)|c|}\left(\max _{t \in[0, \omega]}\left|\int_{0}^{\omega} G(t, s) \zeta(s)(g(u(s))+h(u(s))) d s\right|+\|\gamma\|\right) \\
\leq & \frac{M(1-|c|)}{m-(M+m)|c|}\left(g(r)\left(1+\frac{h(R)}{g(R)}\right) \Lambda^{*}+\|\gamma\|\right)  \tag{4.5}\\
\leq & R .
\end{align*}
$$

Combining (4.4) and (4.5), we get $Q u+\mathcal{S} y \in \mathcal{K}$ for all $u, y \in \mathcal{K}$.
Similarly, we get that $Q$ is a completely continuous and $\mathcal{S}$ is a contractive operator in $X$. Therefore, by Krasnoselskii's fixed point theorem, our results is proven.
Corollary 4.2. Suppose $M<\frac{64 \pi^{3}}{81 \sqrt{3} \omega^{3}}, c<0,|c|<\min \left\{\frac{m}{M+m}, \sigma\right\}$ and $\left(F_{5}\right)$ hold. If $\gamma_{*}=0$, then (1.1) has at least one positive periodic solution.

Proof. We apply Theorem 4.2. We take

$$
\phi_{L}(t)=\frac{\hat{d}(t)}{L^{\rho}}, \quad \zeta(t)=d(t), \quad g(u)=\frac{1}{u^{\rho}}, \quad h(u)=0 .
$$

Then conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$ hold, and the existence condition $\left(H_{4}^{* *}\right)$ becomes

$$
\begin{equation*}
\frac{\hat{\Psi}_{*}}{|c| R^{\rho}}>R>\frac{\hat{\Psi}_{*}}{(1+|c|) R^{\rho}}=r \tag{4.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{M(1-|c|)}{m-(M+m)|c|}\left(\left(\frac{(1+|c|) R^{\rho}}{\hat{\Psi}_{*}}\right)^{\rho} \Psi^{*}+\|\gamma\|\right) \leq R . \tag{4.7}
\end{equation*}
$$

Note that $\Psi_{*}>0$, since $0<\rho<1$, we can choose appropriate $R>0$ so that (4.6) and (4.7) are satisfied and the proof is complete.

## 5. Conclusion

The paper is devoted to the existence of positive periodic solutions for (1.1), where the nonlinear function $f$ has a singularity at $u=0$ and sub-linearity condition at $u=\infty$ for an appropriately chosen parameter. By employing Green's function and the Krasnoselskii fixed point theorem in cones, we prove the existence of positive periodic solutions to (1.1) with time-dependent delay for the first time. We would like to emphasize that the inclusion of the neutral operator in the singularity implies a new technical difficulty concerning the right choice of the operator.

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## Conflict of interest

The authors declare that they have no competing interests.

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