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Research article

Monotonicity properties and bounds for the complete p-elliptic integrals

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Abstract: In the article, we establish some monotonicity properties for certain functions involving the complete p-elliptic integrals of the first and second kinds, and find several sharp bounds for the p-elliptic integrals. Our results are the generalizations and improvements of some previously known results for the classical complete elliptic integrals.

Keywords: complete elliptic integral; complete p-elliptic integral; generalized trigonometric function; monotonicity; bound
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1. Introduction

Let $r \in [0, 1)$. Then the Legendre's complete elliptic integrals [1–6] of the first and second kinds are defined by

$$\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - r^2 \sin^2\theta}}, \quad \mathcal{K}(0) = \frac{\pi}{2}, \quad \mathcal{K}(1^-) = +\infty, \tag{1.1}$$

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2\theta}\, d\theta, \quad \mathcal{E}(0) = \frac{\pi}{2}, \quad \mathcal{E}(1^-) = 1, \tag{1.2}$$

respectively. It is well known that the complete elliptic integrals and integral inequalities [7–20] have wide applications in mathematics and physics, including the formula of the arc length of an ellipse, the evaluation of the circumferences ratio π, the computations of electromagnetic fields and related quantities, the study of simple pendulum period, and so on. Convexity and monotonicity are the indispensable tools in the study of inequality theory [21–28], the generalizations and variants for the convexity have attracted the attention of many researchers [29–38] in recent decades, and many inequalities have been established via the convexity and monotonicity theory [39–44].

The recent interest of the complete elliptic integrals is motivated by their applications in geometric function theory due to many conformal invariants and distortion functions in the theory of quasi-conformal mappings can be expressed by the complete elliptic integrals.

Alzer and Qiu [45] proved that the double inequality

$$\begin{aligned}&\frac{\pi}{2}-\log 2+\alpha(1-\sqrt{1-r^2})+\log\left(1+\frac{1}{\sqrt{1-r^2}}\right)<\mathcal{K}(r)\\&<\frac{\pi}{2}-\log 2+\beta(1-\sqrt{1-r^2})+\log\left(1+\frac{1}{\sqrt{1-r^2}}\right)\end{aligned}\tag{1.3}$$

holds for all $r\in(0,1)$ with the best constant $\alpha=\pi/4-1/2$ and $\beta=3\log 2-\pi/2$.

Wang et al. [46] proved that the function $r\mapsto r[\pi/2-\mathcal{E}(r)]/[r-(1-r^2)\mathrm{arctanh}(r)]$ is strictly decreasing from $(0,1)$ onto $(\pi/2-1,3\pi/16)$, and the double inequality

$$\frac{\pi}{2}-\frac{\pi}{16}\frac{r-(1-r^2)\mathrm{arctanh}(r)}{r}<\mathcal{E}(r)<\frac{\pi}{2}-\left(\frac{\pi}{2}-1\right)\frac{r-(1-r^2)\mathrm{arctanh}(r)}{r}\tag{1.4}$$

holds for all $r\in(0,1)$. Here and in what follows we denote $\mathrm{arctanh}(\cdot)$ the inverse hyperbolic tangent function.

In 2018, Yang et al. [47] proved that the function $r\mapsto e^{\mathcal{K}(r)}-c/\sqrt{1-r^2}$ is strictly decreasing on $(0,1)$ if and only if $c\geq 4$, strictly increasing on $(0,1)$ if and only if $c\leq \pi e^{\pi/2}/4=3.77\cdots$, the double inequality

$$\log\frac{4}{\sqrt{1-r^2}}<\mathcal{K}(r)<\log\left(e^{\pi/2}-4+\frac{4}{\sqrt{1-r^2}}\right)\tag{1.5}$$

holds for all $r\in(0,1)$, and the two-sided inequality

$$\log\left(e^{\pi/2}-s+\frac{s}{\sqrt{1-r^2}}\right)<\mathcal{K}(r)<\log\left(e^{\pi/2}-t+\frac{t}{\sqrt{1-r^2}}\right)\tag{1.6}$$

takes place for all $r\in(0,1)$ if and only if $s\leq \pi e^{\pi/2}/4$ and $t\geq 4$.

Takeuchi [48] introduced a new form of the generalized elliptic integrals with one real parameter p, called the complete p-elliptic integrals. For $p\in(1,+\infty)$ and $r\in[0,1)$, the complete p-elliptic integrals of the first and second kinds are respectively defined by

$$\mathcal{K}_p=\mathcal{K}_p(r)=\int_0^{\pi_p/2}\frac{d\theta}{(1-r^p\sin_p^p\theta)^{1-1/p}},\quad \mathcal{K}_p(0)=\frac{\pi_p}{2},\quad \mathcal{K}_p(1^-)=+\infty\tag{1.7}$$

and

$$\mathcal{E}_p=\mathcal{E}_p(r)=\int_0^{\pi_p/2}(1-r^p\sin_p^p\theta)^{1/p}d\theta,\quad \mathcal{E}_p(0)=\frac{\pi_p}{2},\quad \mathcal{E}_p(1^-)=1,\tag{1.8}$$

where $\sin_p\theta$ is the generalized sine function, defined by the inverse function of

$$\sin_p{}^{-1}\theta=\int_0^{\theta}\frac{dt}{(1-t^p)^{1/p}},\quad 0\leq\theta\leq 1,$$

and π_p is the generalized circumference ratio defined by

$$\pi_p = 2\int_0^1 \frac{dt}{(1-t^p)^{1/p}} = \frac{2\pi}{p\sin(\pi/p)}.$$

Note that $\sin_2\theta = \sin\theta$ and $\pi_2 = \pi$. From (1.1), (1.2), (1.7) and (1.8) we know that $\mathcal{K}_2(r) = \mathcal{K}(r)$ and $\mathcal{E}_2(r) = \mathcal{E}(r)$.

Takeuchi [48, 49] provided the derivative formulas and identity for $\mathcal{K}_p$ and $\mathcal{E}_p$ as follows

$$\frac{d\mathcal{K}_p(r)}{dr} = \frac{\mathcal{E}_p - r'^p\mathcal{K}_p}{rr'^p}, \quad \frac{d\mathcal{E}_p(r)}{dr} = \frac{\mathcal{E}_p - \mathcal{K}_p}{r},$$

$$\frac{d(\mathcal{K}_p - \mathcal{E}_p)}{dr} = \frac{r^{p-1}\mathcal{E}_p(r)}{r'^p}, \quad \frac{d(\mathcal{E}_p - r'^p\mathcal{K}_p)}{dr} = (p-1)r^{p-1}\mathcal{K}_p(r)$$

and

$$\mathcal{K}_p(r')\mathcal{E}_p(r) + \mathcal{K}_p(r)\mathcal{E}_p(r') - \mathcal{K}_p(r)\mathcal{K}_p(r') = \frac{\pi_p}{2}.$$

where and in what follows, we denote $r' = \sqrt[p]{1-r^p}$ for $r \in [0,1]$. Using (1.5), Takeuchi [48, 50] found the formulas for π_3 and π_4. Moreover, the following formulas for the complete p-elliptic integrals in terms of the Gaussian hypergeometric function can be found in the literature [48]:

$$\mathcal{K}_p(r) = \frac{\pi_p}{2}F\left(1-\frac{1}{p}, \frac{1}{p}; 1; r^p\right), \quad \mathcal{E}_p(r) = \frac{\pi_p}{2}F\left(-\frac{1}{p}, \frac{1}{p}; 1; r^p\right), \tag{1.9}$$

where

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \quad (|x|<1)$$

is the Gaussian hypergeometric function [51, 52] for real parameters a, b, c with $c \neq 0, -1, -2, \cdots$, and $(a)_0 = 1$ for $a \neq 0$ and $(a)_n$ denotes the Pochhammer function $(a)_n = a(a+1)(a+2)(a+3)\cdots(a+n-1)$ for $n = 1, 2, \cdots$. If $a + b = c$, then $F(a, b; c; x)$ is called zero-balanced, which has the following asymptotic formula [53]:

$$B(a,b)F(a,b;a+b;x) + \log(1-x) = R(a,b) + O((1-x)\log(1-x)), \tag{1.10}$$

where $B(z,w) = \Gamma(z)\Gamma(w)/[\Gamma(z+w)]$ is the classical Beta function for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$,

$$R(a,b) = -\psi(a) - \psi(b) - 2\gamma, \quad R(a) = R(a, 1-a),$$

$\psi(z) = \Gamma'(z)/\Gamma(z)$ for $\mathrm{Re}(z) > 0$ and γ is the Euler-Mascheroni constant.

The main purpose of this paper is to extend inequalities (1.4)–(1.6) to the case of the complete p-elliptic integrals. Our main results are the following Theorems 1.1–1.3.

Theorem 1.1. *Let $p \in (1, +\infty)$, $p_0 = 2.523\cdots$ be the unique solution of the equation $2p^2 - p^2\pi_p + 4 = 0$ and the function f be defined on $(0, 1)$ by*

$$f_p(r) = \frac{\pi_p/2 - \mathcal{E}_p(r)}{1 - r'^p[\operatorname{arctanh}(r^{p/2})]/r^{p/2}}.$$

Then the following statements are true:

(1) *If* $p \in (1, \sqrt{5}]$, *then* $f_p(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(3\pi_p/(4p^2), \pi_p/2-1)$, *and the double inequality*

$$\frac{\pi_p}{2} - \frac{3\pi_p}{4p^2}\left[1 - r'^p \frac{\text{arctanh}(r^{p/2})}{r^{p/2}}\right] < \mathcal{E}_p(r) < \frac{\pi_p}{2} - \left(\frac{\pi_p}{2} - 1\right)\left[1 - r'^p \frac{\text{arctanh}(r^{p/2})}{r^{p/2}}\right] \tag{1.11}$$

holds for all $r \in (0,1)$;

(2) *If* $p \in (\sqrt{5}, p_0)$, *then there exists unique* $r_0 \in (0,1)$ *such that* $f_p(r)$ *is strictly increasing on* $(0, r_0)$, *and strictly decreasing on* $(r_0, 1)$. *Consequently, for* $r \in (0,1)$, *one has*

$$\mathcal{E}_p(r) < \frac{\pi}{2} - \min\left\{\frac{\pi_p}{2} - 1, \frac{3\pi_p}{4p^2}\right\}\left[1 - r'^p \frac{\text{arctanh}(r^{p/2})}{r^{p/2}}\right]; \tag{1.12}$$

(3) *If* $p \in [p_0, +\infty)$, *then* $f_p(r)$ *is strictly increasing from* $(0,1)$ *onto* $(\pi_p/2-1, 3\pi_p/(4p^2))$, *and the reverse inequality of* (1.11) *holds for all* $r \in (0,1)$.

Theorem 1.2. *Let* $p \in [2, +\infty)$, $\alpha, \beta \in \mathbb{R}$ *and the function* F *be defined on* $(0,1)$ *by*

$$F(r) = \frac{\mathcal{K}_p(r') - (\pi_p/2 - \log 2) - \log(1+1/r)}{1-r}.$$

Then $F(r)$ *is strictly decreasing from* $(0,1)$ *onto* $((p-1)\pi_p/(2p) - 1/2, R(1/p)/p - (\pi_p/2 - \log 2))$, *and the double inequality*

$$\frac{\pi_p}{2} - \log 2 + \log\left(1 + \frac{1}{r'}\right) + \alpha(1-r') < \mathcal{K}_p(r) < \frac{\pi_p}{2} - \log 2 + \log\left(1 + \frac{1}{r'}\right) + \beta(1-r') \tag{1.13}$$

holds for all $r \in (0,1)$ *with the best possible constants* $\alpha = (p-1)\pi_p/(2p) - 1/2$ *and* $\beta = R(1/p)/p - (\pi_p/2 - \log 2)$.

Theorem 1.3. *Let* $p \in [2, +\infty)$, $c \in \mathbb{R}$ *and the function* G_c *be defined on* $(0,1)$ *by*

$$G_c(r) = e^{\mathcal{K}_p(r)} - \frac{c}{r'}, \quad r \in (0,1).$$

Then the following statements are true:

(1) *The function* $G_c(r)$ *is strictly increasing on* $(0,1)$ *if and only if* $c \le e^{\pi_p/2}(p-1)\pi_p/(2p)$, *in this case the range of* G_c *is* $(e^{\pi_p/2} - c, +\infty)$;

(2) *The function* $G_c(r)$ *is strictly decreasing* $(0,1)$ *if and only if* $c \ge e^{R(1/p)/p}$, *in this case the range of* G_c *is* $(-\infty, e^{\pi_p/2} - c)$ *if* $c > e^{R(1/p)/p}$, *while the range of* G_c *is* $(0, e^{\pi_p/2} - c)$ *if* $c = e^{R(1/p)/p}$. *Furthermore, for all* $r \in (0,1)$, *we have*

$$\log\left(\frac{e^{R(1/p)/p}}{r'}\right) < \mathcal{K}_p(r) < \log\left(\frac{e^{R(1/p)/p}}{r'} + e^{\pi_p/2} - e^{R(1/p)/p}\right); \tag{1.14}$$

(3) *If* $e^{\pi_p/2}(p-1)\pi_p/(2p) < c < e^{R(1/p)/p}$, *then there exists* $r_0^* \in (0,1)$ *such that* $G_c(r)$ *is strictly decreasing on* $(0, r_0^*)$ *and strictly increasing on* $(r_0^*, 1)$;

(4) *The double inequality*

$$\log\left(e^{\pi/2} - s^* + \frac{s^*}{r'}\right) < \mathcal{K}_p(r) < \log\left(e^{\pi/2} - t^* + \frac{t^*}{r'}\right) \tag{1.15}$$

holds for all $r \in (0,1)$ *if and only if* $s^* \le e^{\pi_p/2}(p-1)\pi_p/(2p)$ *and* $t^* \ge e^{R(1/p)/p}$.

2. Lemmas

In order to prove our main results, we need several lemmas, which we present in this section.

Lemma 2.1. *(See [54, Theorem 1.25])Let* $-\infty < a < b < \infty$, $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and be differentiable on* (a, b) *such that* $g'(x) \neq 0$ *on* (a, b). *Then both the functions* $[f(x) - f(a)]/[g(x) - g(a)]$ *and* $[f(x) - f(b)]/[g(x) - g(b)]$ *are (strictly) increasing (decreasing) on* (a, b) *if* $f'(x)/g'(x)$ *is (strictly) increasing (decreasing) on* (a, b).

Lemma 2.2. *(See [55, Theorem 2.1])Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *have the radius of convergence* $r > 0$ *with* $b_n > 0$ *for all* $n \in \{0, 1, 2, \cdots\}$. *Let* $h(x) = f(x)/g(x)$ *and* $H_{f,g} = (f'/g')g - f$, *then the following statements are true:*

(1) *If the non-constant sequence* $\{a_n/b_n\}_{n=0}^{\infty}$ *is increasing (decreasing), then* $h(x)$ *is strictly increasing (decreasing) on* $(0, r)$;

(2) *If the non-constant sequence* $\{a_n/b_n\}$ *is increasing (decreasing) for* $0 < n \leq n_0$ *and decreasing (increasing) for* $n > n_0$, *then the function* h *is strictly increasing (decreasing) on* $(0, r)$ *if and only if* $H_{f,g}(r^-) \geq (\leq) 0$. *While if* $H_{f,g}(r^-) < (>) 0$, *then there exists* $\delta \in (0, r)$ *such that* $h(x)$ *is strictly increasing (decreasing) on* $(0, \delta)$ *and strictly decreasing (increasing) on* (δ, r).

The following Lemma can be found in the literature [56, 57].

Lemma 2.3. *Let* $p \in (1, +\infty)$. *Then we have the following five conclusions:*

(1) *The function* $r \mapsto (\mathcal{E}_p - r'^p \mathcal{K}_p)/r^p$ *is strictly increasing from* $(0, 1)$ *onto* $((p-1)\pi_p/(2p), 1)$;

(2) *The function* $r \mapsto \mathcal{K}_p(r) + \log r'$ *is strictly decreasing from* $(0, 1)$ *onto* $(R(1/p)/p, \pi_p/2)$;

(3) *The function* $r \mapsto r'^c \mathcal{K}_p$ *is strictly decreasing on* $(0, 1)$ *if and only if* $c \geq (p-1)/p$ *with the range* $(0, \pi_p/2)$;

(4) *The function* $r \mapsto r'^c \mathcal{E}_p$ *is strictly increasing on* $(0, 1)$ *if and only if* $c \leq -1/p$ *with the range* $(\pi_p/2, \infty)$;

(5) *The function* $r \mapsto \mathcal{E}_p + [r'^p(\mathcal{K}_p - \mathcal{E}_p)^2]/(r^p \mathcal{E}_p)$ *is strictly decreasing from* $(0, 1)$ *onto* $(1, \pi_p/2)$.

Lemma 2.4. *Let*

$$f(x) = \frac{1}{\sin x} - \frac{1}{x} - \frac{2x}{\pi^2}, \quad x \in (0, 1).$$

Then there exists unique $x_0 = 1.244\cdots \in (0, \pi)$, *such that* $f(x) < 0$ *for* $x \in (0, x_0)$, *and* $f(x) > 0$ *for* $x \in (x_0, \pi)$.

Proof. Since

$$\frac{1}{\sin x} - \frac{1}{x} = \sum_{k=1}^{\infty} \frac{2^{2k} - 2}{(2k)!} |B_{2k}| x^{2k-1}, \quad |x| < \pi,$$

where B_k are the Bernoulli numbers, one has

$$f(x) = -\frac{2x}{\pi^2} + |B_2| x + \sum_{k=2}^{\infty} \frac{2^{2k} - 2}{(2k)!} |B_{2k}| x^{2k-1} = \left(\frac{1}{6} - \frac{2}{\pi^2}\right) x + \sum_{k=2}^{\infty} \frac{2^{2k} - 2}{(2k)!} |B_{2k}| x^{2k-1}. \tag{2.1}$$

Differentiating f leas to

$$f'(x) = \left(\frac{1}{6} - \frac{2}{\pi^2}\right) + \sum_{k=2}^{\infty} \frac{2^{2k} - 2}{(2k)!} (2k-1) |B_{2k}| x^{2k-2}. \tag{2.2}$$

It is easy to check that $f'(x)$ is strictly increasing on $(0,\pi)$, $f'(0) = 1/6 - 2/\pi^2 < 0$ and $f'(\pi) = +\infty$. Hence there exists unique x_0^*, such that $f(x)$ is strictly decreasing on $(0, x_0^*)$ and strictly increasing on (x_0^*, π). This, together with the limiting values

$$f(0^+) = 0, \quad f(\pi^-) = +\infty, \tag{2.3}$$

implies that there exists a unique zero point $x_0 \in (0,\pi)$, such that $f(x_0) = 0$, $f(x)$ is negative on $(0, x_0)$, and $f(x)$ is positive on (x_0, π). By the mathematical software *Maple 13*, we compute that $x_0 = 1.244\cdots$. This completes the proof. □

Corollary 2.5. *Let $p \in (1,+\infty)$ and $\lambda(p) = 1 - \pi_p/2 + 2/p^2$. Then there exists unique $p_0 = 2.523\cdots \in (1,+\infty)$ such that $\lambda(p_0) = 0$, $\lambda(p) < 0$ for $p \in (1, p_0)$ and $\lambda(p) > 0$ for $p \in (p_0,+\infty)$.*

Proof. Let $x = \pi/p \in (0,\pi)$. Then

$$1 - \frac{\pi_p}{2} + \frac{2}{p^2} = 1 - \frac{x}{\sin x} + \frac{2x^2}{\pi^2} = -x\left(\frac{1}{\sin x} - \frac{1}{x} - \frac{2}{\pi}x\right).$$

Therefore, Corollary 2.5 follows from Lemma 2.4. □

Lemma 2.6. *Let $p \in [2,+\infty)$. Then one has*

(1) *The function*

$$g(r) = \frac{(2p+2)r' + 2p - 4}{(1+r')^4 r'^{2p-2}} + \frac{\pi_p}{2}\frac{(p^2-1)(2p-1)(2p^2-7p+6-1/p)}{24p^3}$$

is strictly increasing and positive on $(0, 1)$.

(2) *The inequality*

$$\frac{1}{4} + \frac{(1-1/p)(p-5+5/p-1/p^2)\pi_p}{12} > 0$$

holds for each $p \in [2,+\infty)$.

Proof. It is clear to see that

$$\begin{aligned} g(r) =& \frac{(2p+2)r' + 2p - 4}{(1+r')^4 r'^{2p-2}} + \frac{\pi_p}{2}\frac{(p^2-1)(2p-1)(2p^2-7p+6-1/p)}{24p^3} \\ =& \frac{2p+2}{(1+r')^4 r'^{2p-3}} + \frac{2p-4}{(1+r')^4 r'^{2p-2}} + \frac{\pi_p}{2}\frac{(p^2-1)(2p-1)(2p^2-7p+6-1/p)}{24p^3} \end{aligned}$$

is strictly increasing on $(0, 1)$. Since $\pi_p = 2\pi/[p\sin(\pi/p)] > 2$ for $p \in [2,+\infty)$, one has

$$\begin{aligned} \lim_{r\to 0^+} g(r) &= \frac{4p-2}{16} + \frac{\pi_p}{2}\frac{(p^2-1)(2p-1)(p-1)(2p^2-5p+1)}{24p^4} \\ &> \frac{4p-2}{16} + \frac{(p^2-1)(2p-1)(p-1)(2p^2-5p+1)}{24p^4} \\ &= \frac{(2p-1)(2p^5-4p^4+4p^3+6p^2-6p+1)}{24p^4} \end{aligned}$$

$$= \frac{(2p-1)[2p^3(p^2-2p+2)+6p(p-1)+1]}{24p^4} > 0.$$

Therefore, part (1) follows.

For part (2), employing inequality $\pi_p = 2\pi/[p\sin(\pi/p)] > 2$ for $p \in [2, +\infty)$ again, we derive that

$$\begin{aligned}
&\frac{1}{4} + \frac{(1-1/p)(p-5+5/p-1/p^2)\pi_p}{12} > \frac{1}{4} + \frac{(1-1/p)(p-5+5/p-1/p^2)}{6} \\
&= \frac{(2p-1)(p^3-4p^2+8p-2)}{12p^3} = \frac{(2p-1)[p(p^2-4p+8)-2]}{12p^3} > 0
\end{aligned}$$

immediately. □

Lemma 2.7. *Let* $p \in [2, +\infty)$*. Then the function*

$$h(r) = \frac{1}{(1+r')^2} + \frac{r^p r'^{p-1}\mathcal{K}_p - pr'^{p-1}(\mathcal{K}_p - \mathcal{E}_p)}{r^{2p}}$$

is strictly increasing and convex on $(0, 1)$.

Proof. Differentiating h gives

$$\begin{aligned}
h'(r) =& \frac{2r^{p-1}}{(1+r')^3 r'^{p-1}} + \frac{r^{p-1}}{r'} \frac{[-(p^2+2p+1)r^p + 2r^{2p} + 2p^2]\mathcal{K}_p - [2p^2 - (p^2+1)r^p]\mathcal{E}_p}{r^{3p}} \\
=& \frac{r^{p-1}}{r'} \left\{ \frac{2}{(1+r')^3 r'^{p-2}} + \frac{[-(p^2+2p+1)r^p + 2r^{2p} + 2p^2]\mathcal{K}_p - [2p^2 - (p^2+1)r^p]\mathcal{E}_p}{r^{3p}} \right\} \\
=& \frac{r^{p-1}}{r'} [h_1(r) + h_2(r)], \qquad (2.4)
\end{aligned}$$

where

$$h_1(r) = \frac{2}{(1+r')^3 r'^{p-2}} + \frac{\pi_p}{2} \frac{(p^2-1)(2p-1)(2p^2-7p+6-1/p)}{24p^4} r^p \qquad (2.5)$$

and

$$\begin{aligned}
h_2(r) =& \frac{[-(p^2+2p+1)r^p + 2r^{2p} + 2p^2]\mathcal{K}_p - [2p^2 - (p^2+1)r^p]\mathcal{E}_p}{r^{3p}} \\
&- \frac{\pi_p}{2} \frac{(p^2-1)(2p-1)(2p^2-7p+6-1/p)}{24p^4} r^p. \qquad (2.6)
\end{aligned}$$

Simple computations lead to

$$\lim_{r \to 0^+} h_1(r) = \frac{1}{4}, \quad \lim_{r \to 1^-} h_1(r) = \infty$$

and

$$h_1'(r) = 2\frac{-[(p-2)(1+r')^3 r'^{p-3} + 3(1+r')^2 r'^{p-2}]}{(1+r')^6 r'^{2p-4}} \left(-\frac{r^{p-1}}{r'^{p-1}}\right)$$

$$+\frac{\pi_p}{2}\frac{(p^2-1)(2p-1)(2p^2-7p+6-1/p)}{24p^3}r^{p-1}$$
$$=r^{p-1}g(r),$$

where $g(r)$ is defined as in Lemma 2.6(1). By Lemma 2.6(1) we conclude that $h_1(r)$ is strictly increasing from $(0,1)$ onto $(1/4,+\infty)$.

Expanding the right side of (2.6) into power series, we have

$$h_2(r)=\frac{\pi_p}{2}\left\{\sum_{n=2}^{\infty}\frac{(1-1/p,n+1)(1/p,n+1)[(p-1)^2n+p^2-5p+5-1/p]}{n!(n+3)!}r^{pn}\right\}$$
$$+\frac{\pi_p}{2}\frac{(1-1/p)(p-5+5/p-1/p^2)}{6}.$$

Note that

$$\frac{(1-1/p,n+1)(1/p,n+1)[(p-1)^2n+p^2-5p+5-1/p]}{n!(n+3)!}>0$$

for $n\geq 2$, h_2 is strictly increasing on $(0,1)$, and the range is $((1-1/p)(p-5+5/p-1/p^2)\pi_p/12,\infty)$.

Combining Lemma 2.6(2) and monotonicity properties together with the ranges of h_1 and h_2, we know that the sum function $h_1(r)+h_2(r)$ is strictly increasing and positive on $(0,1)$,

Finally, according to equations (2.4)–(2.6) we obtain that $h(r)$ is strictly increasing and convex on $(0,1)$. □

Lemma 2.8. *Let $p\in[2,+\infty)$. Then the function*

$$\varphi(r)=\frac{\mathcal{K}_p(r)-\mathcal{E}_p(r)}{[\mathcal{E}_p(r)-r'^p\mathcal{K}_p(r)]\mathcal{K}_p(r)}$$

is strictly increasing from $(0,1)$ onto $(2/[(p-1)\pi_p],1)$.

Proof. Let $\varphi_1(r)=\mathcal{K}_p(r)-\mathcal{E}_p(r)$ and $\varphi_2(r)=[\mathcal{E}_p(r)-r'^p\mathcal{K}_p(r)]\mathcal{K}_p(r)$. Then $\varphi(r)=\varphi_1(r)/\varphi_2(r)$, $\varphi_1(0)=\varphi_2(0)=0$ and

$$\frac{\varphi_1'(r)}{\varphi_2'(r)}=\frac{r^p\mathcal{E}_p}{(p-1)r^pr'^p\mathcal{K}_p{}^2+(\mathcal{E}_p-r'^p\mathcal{K}_p)^2}=\frac{r^p\mathcal{E}_p}{(p-2)r^pr'^p\mathcal{K}_p{}^2+r^p\mathcal{E}_p{}^2+r'^p(\mathcal{K}_p-\mathcal{E}_p)^2}$$
$$=\frac{1}{(p-2)(r'^p\mathcal{K}_p{}^2/\mathcal{E}_p)+[\mathcal{E}_p+r'^p(\mathcal{K}_p-\mathcal{E}_p)^2/(r^p\mathcal{E}_p)]}. \tag{2.7}$$

Since $p\geq 2$, the function $r\mapsto r'^p\mathcal{K}_p{}^2/\mathcal{E}_p$ is strictly decreasing on from $(0,1)$ onto $(0,\pi_p/2)$. This together with Lemma 2.3(5) leads to the conclusion that the function $\varphi_1'(r)/\varphi_2'(r)$ is strictly increasing from $(0,1)$ onto $(2/[(p-1)\pi_p],1)$. Applying Lemma 2.1, we obtain that $\varphi(r)$ is also strictly increasing on $(0,1)$. Moreover, $\lim_{r\to 1^-}\varphi(r)=1$, and

$$\lim_{r\to 0^+}\varphi(r)=\lim_{r\to 0^+}\frac{\varphi_1'(r)}{\varphi_2'(r)}=\frac{2}{(p-1)\pi_p}.$$

□

Lemma 2.9. *Let $p \in [2, +\infty)$. Then the function*

$$\phi(r) = \frac{\mathcal{E}_p(r) - r'^p \mathcal{K}_p(r) - (p-1) r'^p [\mathcal{K}_p(r) - \mathcal{E}_p(r)]}{(\mathcal{E}_p - r'^p \mathcal{K}_p)^2}$$

is strictly increasing from $(0, 1)$ *onto* $((p+1)/[(p-1)\pi_p], 1)$.

Proof. Let $\phi_1(r) = \mathcal{E}_p(r) - r'^p \mathcal{K}_p(r) - (p-1) r'^p (\mathcal{K}_p - \mathcal{E}_p)$ and $\phi_2(r) = (\mathcal{E}_p - r'^p \mathcal{K}_p)^2$. Then $\phi(r) = \phi_1(r)/\phi_2(r)$, $\phi_1(0) = \phi_2(0 = 0$ and

$$\frac{\phi_1'(r)}{\phi_2'(r)} = \frac{(p^2-1) r^{p-1} (\mathcal{K}_p - \mathcal{E}_p)}{2(p-1) r^{p-1} \mathcal{K}_p (\mathcal{E}_p - r'^p \mathcal{K}_p)} = \frac{(p+1)(\mathcal{K}_p - \mathcal{E}_p)}{2\mathcal{K}_p (\mathcal{E}_p - r'^p \mathcal{K}_p)}. \tag{2.8}$$

Eq (2.8) and Lemma 2.8 show that $\phi_1'(r)/\phi_2'(r)$ is strictly increasing on $(0, 1)$. By application of Lemma 2.1, the monotonicity of $\phi(r)$ follows. Clearly $\phi(1^-) = 1$, and by l'Hôpital's rule we get

$$\lim_{r \to 0^+} \phi(r) = \lim_{r \to 0^+} \frac{\phi_1'(r)}{\phi_2'(r)} = \frac{p+1}{2} \cdot \frac{2}{(p-1)\pi_p} = \frac{p+1}{(p-1)\pi_p}.$$

□

Lemma 2.10. *Let $p \in [2, +\infty)$. Then the function*

$$\omega(r) = e^{\mathcal{K}_p(r)} \frac{r'(\mathcal{E}_p - r'^p \mathcal{K}_p)}{r^p}$$

is strictly increasing from $(0, 1)$ *onto* $(e^{\pi_p/2}(p-1)\pi_p/(2p), e^{R(1/p)/p})$.

Proof. By differentiation, we have

$$\begin{aligned}
\omega'(r) =& e^{\mathcal{K}_p(r)} \frac{\left[(-r^{p-1}/r'^{p-1})(\mathcal{E}_p - r'^p \mathcal{K}_p) + (p-1) r' r^{p-1} \mathcal{K}_p\right] r^p - r'(\mathcal{E}_p - r'^p \mathcal{K}_p) p r^{p-1}}{r^{2p}} \\
&+ e^{\mathcal{K}_p(r)} \left(\frac{\mathcal{E}_p - r'^p \mathcal{K}_p}{r r'^p}\right) \frac{r'(\mathcal{E}_p - r'^p \mathcal{K}_p)}{r^p} \\
=& \frac{e^{\mathcal{K}_p(r)}}{r'^{p-1} r^{p+1}} \left[-r^p (\mathcal{E}_p - r'^p \mathcal{K}_p) + (p-1) r^p r'^p \mathcal{K}_p - p r'^p (\mathcal{E}_p - r'^p \mathcal{K}_p) + (\mathcal{E}_p - r'^p \mathcal{K}_p)^2\right] \\
=& \frac{e^{\mathcal{K}_p(r)}}{r'^{p-1} r^{p+1}} \left\{(\mathcal{E}_p - r'^p \mathcal{K}_p)^2 - [(\mathcal{E}_p - r'^p \mathcal{K}_p) - (p-1) r'^p \left(\mathcal{K}_p - \mathcal{E}_p\right)]\right\} \\
=& \frac{e^{\mathcal{K}_p(r)} (\mathcal{E}_p - r'^p \mathcal{K}_p)^2}{r'^{p-1} r^{p+1}} [1 - \phi(r)],
\end{aligned} \tag{2.9}$$

where $\phi(r)$ is defined as in Lemma 2.9.

The monotonicity of $\omega(r)$ on $(0, 1)$ directly follows from (2.9) and Lemma 2.9. By Lemma 2.3(1) and (3), one has $\omega(0^+) = e^{\pi_p/2}(p-1)\pi_p/(2p)$ and $\omega(1^-) = e^{R(1/p)/p}$. □

3. Proofs of theorems 1.1–1.3

Proof of Theorem 1.1. Let $A(r) = \pi_p/2 - \mathcal{E}_p(r)$ and $B(r) = 1 - r'^p\text{arctanh}(r^{p/2})/r^{p/2}$. Then using the series expansion (1.9) we get

$$A(r) = \frac{\pi_p}{2} - \frac{\pi_p}{2}\sum_{n=0}^{\infty}\frac{(1/p,n)(-1/p,n)}{(1,n)}\frac{r^{pn}}{n!} = \frac{\pi_p}{2p}\sum_{n=1}^{\infty}\frac{(1/p,n)(1-1/p,n)}{(n!)^2)}r^{pn},$$

$$B(r) = 1 - (1-r^p)\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{pn} = 2\sum_{n=1}^{\infty}\frac{1}{4n^2-1}r^{pn}.$$

Thus

$$f_p(r) = \frac{A(r)}{B(r)} = \frac{\pi_p}{4p}\frac{\sum_{n=1}^{\infty}R_n r^{pn}}{\sum_{n=1}^{\infty}S_n r^{pn}}, \tag{3.1}$$

where

$$R_n = \frac{(1/p,n)(1-1/p,n)}{(n!)^2)}, \quad S_n = \frac{1}{4n^2-1}. \tag{3.2}$$

Let $T_n = R_n/S_n$. Then

$$\begin{aligned}\frac{T_{n+1}}{T_n} - 1 &= \frac{(n-1/p)(n+1/p)[4(n+1)^2-1]}{(n+1)^2(4n^2-1)} - 1\\ &= \frac{(2n+3)(n^2-1/p^2)}{(n+1)^2(2n-1)} - 1 = \frac{(2n+3)/p^2-1}{(n+1)^2(2n-1)}.\end{aligned} \tag{3.3}$$

Next, we divide the proof into two cases.

Case 1. $p \in (1, \sqrt{5}]$. Then from (3.3) we obtain that $T_{n+1}/T_n \leq 1$ for $n \geq 1$, and thereby $\{T_n\}$ is decreasing with respect to n. With an application of Lemma 2.2 and Eqs (3.1) and (3.2), the monotonicity of f_p on $(0,1)$ in this case follows. Moreover, clearly $f_p(1^-) = \pi_p/2 - 1$, and by l'Hospital's rule, one has

$$\lim_{r\to 0^+} f_p(r) = \frac{\pi_p}{4p}\frac{R_1}{S_1} = \frac{3\pi_p}{4p^2}.$$

Therefore, inequality (1.11) takes place. □

Case 2. $p \in (\sqrt{5}, +\infty)$. Then Eq (3.3) implies that there exists $n_0 > 1$ such that sequence $\{R_n/S_n\}$ is increasing for $1 < n \leq n_0$ and decreasing for $n > n_0$. For the limiting value of $H_{A,B}(r)$ at 1, by differentiation we get

$$A'(r) = \frac{\mathcal{K}_p(r) - \mathcal{E}_p(r)}{r},$$

$$\begin{aligned}B'(r) &= -\frac{\left[-pr^{p-1}\text{arctanh}(r^{p/2}) + pr^{p/2-1}/2\right] - pr^{p/2-1}r'^p\text{arctanh}(r^{p/2})/2}{r^p}\\ &= \frac{p}{2r}\left(r^{p/2} + r^{-p/2}\right)\text{arctanh}(r^{p/2}) - \frac{p}{2r},\end{aligned}$$

so that

$$H_{A,B}(r) = \frac{A'(r)}{B'(r)}B(r) - A(r) = \mathcal{E}_p - \frac{\pi_p}{2} + \frac{2}{p}\frac{(\mathcal{K}_p - \mathcal{E}_p)[r^{p/2} - r'^p\text{arctanh}(r^{p/2})]}{(1+r^p)\text{arctanh}(r^{p/2}) - r^{p/2}}.$$

It is not difficult to verify that

$$\lim_{r\to 1^-}\left[r^{p/2} - r'^p \text{arctanh}(r^{p/2})\right] = 1,$$

$$\lim_{r\to 1^-}\frac{\log(1/r')}{\text{arctanh}(r^{p/2})} = \lim_{r\to 1^-}\frac{r^{p-1}/r'^p}{pr^{p/2-1}/(2r'^p)} = \frac{2}{p},$$

$$\lim_{r\to 1^-}\frac{\mathcal{K}_p - E_p}{\log(1/r')} = \lim_{r\to 1^-}\frac{r^{p-1}\mathcal{E}_p/r'^p}{r^{p-1}/r'^p} = 1,$$

thus

$$\begin{aligned} H_{A,B}(1^-) &= \lim_{r\to 1^-}\left[\mathcal{E}_p - \frac{\pi_p}{2} + \frac{2}{p}\frac{(\mathcal{K}_p - \mathcal{E}_p)[r^{p/2} - r'^p \text{arctanh}(r^{p/2})]}{(1+r^p)\text{arctanh}(r^{p/2}) - r^{p/2}}\right] \\ &= 1 - \frac{\pi_p}{2} + \lim_{r\to 1^-}\frac{2}{p}\frac{\log(1/r')}{(1+r^p)\text{arctanh}(r^{p/2}) - r^{p/2}} \\ &= 1 - \frac{\pi_p}{2} + \frac{2}{p^2}. \end{aligned} \tag{3.4}$$

It follows from (3.4) and Corollary 2.4 that $H_{A,B}(1^-) < 0$ for $p \in (\sqrt{5}, p_0)$, and $H_{A,B}(1^-) \geq 0$ for $p \in [p_0, \infty)$. Applying Lemma 2.2(2), $f_p(r)$ is strictly increasing from $(0,1)$ onto $(\pi_p/2 - 1, 3\pi_p/(4p^2))$ if and only if $p \geq p_0$, so that the reverse inequality of (1.11) holds. While $p \in (\sqrt{5}, p_0)$, $f_p(r)$ is piecewise monotone on $(0,1)$, and therefore the inequality

$$\frac{\pi_p/2 - \mathcal{E}_p(r)}{1 - r'^p[\text{arctanh}(r^{p/2})]/r^{p/2}} > \min\left\{\frac{\pi_p}{2} - 1, \frac{3\pi_p}{4p^2}\right\} \tag{3.5}$$

takes place for each $r \in (0,1)$. Finally, by exchanging the terms of inequality (3.5), we obtain (1.12).

Proof of Theorem 1.2. Let

$$F_1(r) = \mathcal{K}_p(r') - (\pi_p/2 - \log 2) - \log(1 + 1/r), \quad F_2(r) = 1 - r,$$

$$F_3(r) = \frac{\mathcal{E}_p(r') - r^p\mathcal{K}_p(r')}{r'^p} - \frac{1}{1+r}, \quad F_4(r) = r.$$

Then $F(1^-) = F_2(1^-) = 0$, $F_3(0^+) = F_4(0^+) = 0$, and

$$\frac{F_1'(r)}{F_2'(r)} = \frac{F_3(r)}{F_4(r)}, \quad \frac{F_3'(r)}{F_4'(r)} = h(r'),$$

where $h(r)$ is defined as in Lemma 2.7, is a increasing function on $(0,1)$. Applying Lemma 2.1 twice, the monotonicity of F follows. Moreover, by Lemma 2.3(2) and Lemma 2.7 we have $F(0^+) = R(1/p)/p - (\pi_p/2 - \log 2)$ and

$$\lim_{r\to 1^-}F(r) = \lim_{r\to 1^-}\frac{F_1'(r)}{F_2'(r)} = \lim_{r\to 1^-}\frac{F_3(r)}{F_4(r)} = \frac{(p-1)\pi_p}{2p} - \frac{1}{2}.$$

Inequality (1.13) can be derived from the monotonicity of $F(r)$ on $(0,1)$ and the above limiting values immediately. The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. Clearly,
$$G_c(0^+) = e^{\pi_p/2} - c. \tag{3.6}$$

Substituting $1 - 1/p$ and $1/p$ respectively for a and b in (1.10), we get

$$\mathcal{K}_p(r) = \frac{R(1/p)}{p} + \log\frac{1}{r'} + O((1-r^p)\log(1-r^p)), \quad r \to 1.$$

Thus
$$G_c(1^-) = \begin{cases} +\infty, & c < e^{R(1/p)/p}, \\ 0, & c = e^{R(1/p)/p}, \\ -\infty, & e^{R(1/p)/p}. \end{cases} \tag{3.7}$$

Differentiating G_c yields

$$G_c'(r) = e^{\mathcal{K}_p(r)}\frac{(\mathcal{E}_p - r'^p\mathcal{K}_p)}{rr'^p} - \frac{cr^{p-1}}{r'^{p+1}} = \frac{r^{p-1}}{r'^{p+1}}[\omega(r) - c], \tag{3.8}$$

where $\omega(r)$ is defined as in Lemma 2.10. According to Lemma 2.10 and (3.8), the assertion of the monotonicity of $G_c(r)$ for any $c \in \mathbb{R}$ follows. Combining with (3.6) and (3.7), parts (1)–(3) hold.

For part (4), it follows from parts (1)-(3) that inequality $G_c(r) > G_c(0^+) = e^{\pi_p/2} - c$ holds for all $r \in (0,1)$ if and only if $c \leq e^{\pi_p/2}(p-1)\pi_p/(2p)$, and $G_c(r) < G_c(0^+) = e^{\pi_p/2} - c$ holds for all $r \in (0,1)$ if and only if $c \geq e^{R(1/p)/p}$. Therefore, the inequality

$$e^{\pi_p/2} - s^* + \frac{s^*}{r'} < e^{\mathcal{K}_p(r)} < e^{\pi_p/2} - t^* + \frac{t^*}{r'},$$

namely,
$$\log\left(e^{\pi_p/2} - s^* + \frac{s^*}{r'}\right) < \mathcal{K}_p(r) < \log\left(e^{\pi_p/2} - t^* + \frac{t^*}{r'}\right),$$

holds for all $r \in (0,1)$ if and only if $s^* \leq e^{\pi_p/2}(p-1)\pi_p/(2p)$ and $t^* \geq e^{R(1/p)/p}$. □

4. Conclusions

In the article, we have found some new monotonicity properties for the functions involving the complete p-elliptic integrals of the first and second kinds, and provided several optimal upper and lower bounds for the p-elliptic integrals. Our ideas and approach may lead to a lot of follow-up research.

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Conflict of interest

The authors declare that they have no competing interests.

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