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Research article

First and second critical exponents for an inhomogeneous damped wave equation with mixed nonlinearities

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Abstract: We investigate the Cauchy problem for the nonlinear damped wave equation $u_{tt} - \Delta u + u_t = |u|^p + |\nabla u|^q + w(x)$, where $N \geq 1$, $p, q > 1$, $w \in L^1_{loc}(\mathbb{R}^N)$, $w \geq 0$ and $w \not\equiv 0$. Namely, we first obtain the Fujita critical exponent for the considered problem. Next, we determine its second critical exponent in the sense of Lee and Ni. In particular, we show that the nonlinear gradient term $|\nabla u|^q$ induces a phenomenon of discontinuity of the Fujita critical exponent.

Keywords: damped wave equation; mixed nonlinearity; global weak solution; critical exponent
Mathematics Subject Classification: 35L05, 35B44, 35B33

1. Introduction

We consider the Cauchy problem for the nonlinear damped wave equation with linear damping

$$u_{tt} - \Delta u + u_t = |u|^p + |\nabla u|^q + w(x), \quad t > 0,\ x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$, $p, q > 1$, $w \in L^1_{loc}(\mathbb{R}^N)$, $w \geq 0$ and $w \not\equiv 0$. Namely, we are concerned with the existence and nonexistence of global weak solutions to (1.1). We mention below some motivations for studying problems of type (1.1).

Consider the semilinear damped wave equation

$$u_{tt} - \Delta u + u_t = |u|^p, \quad t > 0,\ x \in \mathbb{R}^N \tag{1.2}$$

under the initial conditions

$$(u(0, x), u_t(0, x)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}^N. \tag{1.3}$$

In [12], a Fujita-type result was obtained for the problem (1.2) and (1.3). Namely, it was shown that,

(i) if $1 < p < 1 + \frac{2}{N}$ and $\int_{\mathbb{R}^N} u_j(x)\,dx > 0$, $j = 0, 1$, then the problem (1.2) and (1.3) admits no global solution;
(ii) if $1 + \frac{2}{N} < p < N$ for $N \geq 3$, and $1 + \frac{2}{N} < p < \infty$ for $N \in \{1, 2\}$, then the problem (1.2) and (1.3) admits a unique global solution for suitable initial values.

The proof of part (i) makes use of the fundamental solution of the operator $(\partial_{tt} - \Delta + \partial_t)^k$. In [15], it was shown that the exponent $1 + \frac{2}{N}$ belongs to the blow-up case (i). Notice that $1 + \frac{2}{N}$ is also the Fujita critical exponent for the semilinear heat equation $u_t - \Delta u = |u|^p$ (see [3]). In [6], the authors considered the problem

$$u_{tt} + (-1)^m |x|^\alpha \Delta^m u + u_t = f(t, x)|u|^p + w(t, x), \quad t > 0,\ x \in \mathbb{R}^N \tag{1.4}$$

under the initial conditions (1.3), where m is a positive natural number, $\alpha \geq 0$, and $f(t, x) \geq 0$ is a given function satisfying a certain condition. Using the test function method (see e.g. [11]), sufficient conditions for the nonexistence of a global weak solution to the problem (1.4) and (1.3) are obtained. Notice that in [6], the influence of the inhomogeneous term $w(t, x)$ on the critical behavior of the problem (1.4) and (1.3) was not investigated. Recently, in [5], the authors investigated the inhomogeneous problem

$$u_{tt} - \Delta u + u_t = |u|^p + w(x), \quad t > 0,\ x \in \mathbb{R}^N, \tag{1.5}$$

where $w \in L^1_{loc}(\mathbb{R}^N)$ and $w \not\equiv 0$. It was shown that in the case $N \geq 3$, the critical exponent for the problem (1.5) jumps from $1 + \frac{2}{N}$ (the critical exponent for the problem (1.2)) to the bigger exponent $1 + \frac{2}{N-2}$. Notice that a similar phenomenon was observed for the inhomogeneous semilnear heat equation $u_t - \Delta u = |u|^p + w(x)$ (see [14]). For other results related to the blow-up of solutions to damped wave equations, see, for example [1, 2, 4, 7–9, 13] and the references therein.

In this paper, motivated by the above mentioned works, our aim is to study the effect of the gradient term $|\nabla u|^q$ on the critical behavior of problem (1.5). We first obtain the Fujita critical exponent for the problem (1.1). Next, we determine its second critical exponent in the sense of Lee and Ni [10].

Before stating the main results, let us provide the notion of solutions to problem (1.1). Let $Q = (0, \infty) \times \mathbb{R}^N$. We denote by $C^2_c(Q)$ the space of C^2 real valued functions compactly supported in Q.

Definition 1.1. *We say that $u = u(t, x)$ is a global weak solution to* (1.1)*, if*

$$(u, \nabla u) \in L^p_{loc}(Q) \times L^q_{loc}(Q)$$

and

$$\int_Q (|u|^p + |\nabla u|^q)\,\varphi\,dx\,dt + \int_Q w(x)\varphi\ dx\,dt = \int_Q u\varphi_{tt}\,dx\,dt - \int_Q u\Delta\varphi\,dx\,dt - \int_Q u\varphi_t\,dx\,dt, \tag{1.6}$$

for all $\varphi \in C^2_c(Q)$.

The first main result is the following Fujita-type theorem.

Theorem 1.1. *Let $N \geq 1$, $p, q > 1$, $w \in L^1_{loc}(\mathbb{R}^N)$, $w \geq 0$ and $w \not\equiv 0$.*

(i) If $N \in \{1, 2\}$, for all $p, q > 1$, problem (1.1) *admits no global weak solution.*
(ii) Let $N \geq 3$. If

$$1 < p < 1 + \frac{2}{N-2} \quad or \quad 1 < q < 1 + \frac{1}{N-1},$$

then problem (1.1) *admits no global weak solution.*

(iii) Let $N \geq 3$. If

$$p > 1 + \frac{2}{N-2} \quad \textit{and} \quad q > 1 + \frac{1}{N-1},$$

then problem (1.1) *admits global solutions for some* $w > 0$.

We mention below some remarks related to Theorem 1.1. For $N \geq 3$, let

$$p_c(N,q) = \begin{cases} 1 + \frac{2}{N-2} & \text{if} \quad q > 1 + \frac{1}{N-1}, \\ \infty & \text{if} \quad q < 1 + \frac{1}{N-1}. \end{cases}$$

From Theorem 1.1, if $1 < p < p_c(N,q)$, then problem (1.1) admits no global weak solution, while if $p > p_c(N,q)$, global weak solutions exist for some $w > 0$. This shows that $p_c(N,q)$ is the Fujita critical exponent for the problem (1.1). Notice that in the case $N \geq 3$ and $q > 1 + \frac{1}{N-1}$, $p_c(N,q)$ is also the critical exponent for $u_{tt} - \Delta u = |u|^p + w(x)$ (see [5]). This shows that in the range $q > 1 + \frac{1}{N-1}$, the gradient term $|\nabla u|^q$ has no influence on the critical behavior of $u_{tt} - \Delta u = |u|^p + w(x)$.

It is interesting to note that the gradient term $|\nabla u|^q$ induces an interesting phenomenon of discontinuity of the Fujita critical exponent $p_c(N,q)$ jumping from $1 + \frac{2}{N-2}$ to ∞ as q reaches the value $1 + \frac{1}{N-1}$ from above.

Next, for $N \geq 3$ and $\sigma < N$, we introduce the sets

$$I_\sigma^+ = \left\{ w \in C(\mathbb{R}^N) : \ w \geq 0, \ |x|^{-\sigma} = O(w(x)) \text{ as } |x| \to \infty \right\}$$

and

$$I_\sigma^- = \left\{ w \in C(\mathbb{R}^N) : \ w > 0, \ w(x) = O(|x|^{-\sigma}) \text{ as } |x| \to \infty \right\}.$$

The next result provides the second critical exponent for the problem (1.1) in the sense of Lee and Ni [10].

Theorem 1.2. *Let $N \geq 3$, $p > 1 + \frac{2}{N-2}$ and $q > 1 + \frac{1}{N-1}$.*

(i) If

$$\sigma < \max\left\{ \frac{2p}{p-1}, \frac{q}{q-1} \right\}$$

and $w \in I_\sigma^+$, then problem (1.1) *admits no global weak solution.*

(ii) If

$$\sigma \geq \max\left\{ \frac{2p}{p-1}, \frac{q}{q-1} \right\},$$

then problem (1.1) *admits global solutions for some $w \in I_\sigma^-$.*

From Theorem 1.2, if $N \geq 3$, $p > 1 + \frac{2}{N-2}$ and $q > 1 + \frac{1}{N-1}$, then problem (1.1) admits a second critical exponent, namely

$$\sigma^* = \max\left\{ \frac{2p}{p-1}, \frac{q}{q-1} \right\}.$$

The rest of the paper is organized as follows. In Section 2, some preliminary estimates are provided. Section 3 is devoted to the study of the Fujita critical exponent for the problem (1.1). Namely, Theorem 1.1 is proved. In Section 4, we study the second critical exponent for the problem (1.1) in the sense of Lee and Ni. Namely, we prove Theorem 1.2.

2. Preliminaries

Consider two cut-off functions $f, g \in C^\infty([0,\infty))$ satisfying

$$f \geq 0, \quad f \not\equiv 0, \quad \mathrm{supp}(f) \subset (0,1)$$

and

$$0 \leq g \leq 1, \quad g(s) = \begin{cases} 1 & \text{if} \quad 0 \leq s \leq 1, \\ 0 & \text{if} \quad s \geq 2. \end{cases}$$

For $T > 0$, we introduce the function

$$\xi_T(t,x) = f\left(\frac{t}{T}\right)^\ell g\left(\frac{|x|^2}{T^{2\rho}}\right)^\ell = F(t)G(x), \quad (t,x) \in Q, \tag{2.1}$$

where $\ell \geq 2$ and $\rho > 0$ are constants to be chosen. It can be easily seen that

$$\xi_T \geq 0 \quad \text{and} \quad \xi_T \in C_c^2(Q).$$

Throughout this paper, the letter C denotes various positive constants depending only on known quantities.

Lemma 2.1. *Let $\kappa > 1$ and $\ell > \frac{2\kappa}{\kappa-1}$. Then*

$$\int_Q |\xi_T|^{\frac{-1}{\kappa-1}} |\Delta \xi_T|^{\frac{\kappa}{\kappa-1}} \, dx \, dt \leq C T^{\frac{-2\rho\kappa}{\kappa-1}+N\rho+1}. \tag{2.2}$$

Proof. By (2.1), one has

$$\int_Q |\xi_T|^{\frac{-1}{\kappa-1}} |\Delta \xi_T|^{\frac{\kappa}{\kappa-1}} \, dx \, dt = \left(\int_0^\infty F(t) \, dt\right)\left(\int_{\mathbb{R}^N} |G|^{\frac{-1}{\kappa-1}} |\Delta G|^{\frac{\kappa}{\kappa-1}} \, dx\right). \tag{2.3}$$

Using the properties of the cut-off function f, one obtains

$$\begin{aligned} \int_0^\infty F(t) \, dt &= \int_0^\infty f\left(\frac{t}{T}\right)^\ell dt \\ &= \int_0^T f\left(\frac{t}{T}\right)^\ell dt \\ &= T \int_0^1 f(s)^\ell \, ds, \end{aligned}$$

which shows that

$$\int_0^\infty F(t) \, dt = CT. \tag{2.4}$$

Next, using the properties of the cut-off function g, one obtains

$$\int_{\mathbb{R}^N} |G|^{\frac{-1}{\kappa-1}} |\Delta G|^{\frac{\kappa}{\kappa-1}} \, dx = \int_{T^\rho < |x| < \sqrt{2} T^\rho} |G|^{\frac{-1}{\kappa-1}} |\Delta G|^{\frac{\kappa}{\kappa-1}} \, dx \tag{2.5}$$

On the other hand, for $T^\rho < |x| < \sqrt{2}T^\rho$, an elementary calculation yields

$$\begin{aligned}\Delta G(x) &= \Delta\left[g\left(\frac{r^2}{T^{2\rho}}\right)^\ell\right], \quad r=|x|\\ &= \left(\frac{d^2}{dr^2}+\frac{N-1}{r}\frac{d}{dr}\right)g\left(\frac{r^2}{T^{2\rho}}\right)^\ell\\ &= 2\ell T^{-2\rho}g\left(\frac{r^2}{T^{2\rho}}\right)^{\ell-2}\theta(x),\end{aligned}$$

where

$$\theta(x)=Ng\left(\frac{r^2}{T^{2\rho}}\right)g'\left(\frac{r^2}{T^{2\rho}}\right)+2(\ell-1)T^{-2\rho}r^2g\left(\frac{r^2}{T^{2\rho}}\right)g'\left(\frac{r^2}{T^{2\rho}}\right)^2+2T^{-2\rho}r^2g\left(\frac{r^2}{T^{2\rho}}\right)g''\left(\frac{r^2}{T^{2\rho}}\right).$$

Notice that since $T^\rho < r < \sqrt{2}T^\rho$, one deduces that

$$|\theta(x)| \le C.$$

Hence, it holds that

$$|\Delta G(x)| \le CT^{-2\rho}g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell-2}, \quad T^\rho < |x| < \sqrt{2}T^\rho$$

and

$$|G(x)|^{\frac{-1}{\kappa-1}}|\Delta G(x)|^{\frac{\kappa}{\kappa-1}} \le CT^{\frac{-2\rho\kappa}{\kappa-1}}g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell-\frac{2\kappa}{\kappa-1}}, \quad T^\rho < |x| < \sqrt{2}T^\rho.$$

Therefore, by (2.5) and using the change of variable $x = T^\rho y$, one obtains

$$\begin{aligned}\int_{\mathbb{R}^N}|G|^{\frac{-1}{\kappa-1}}|\Delta G|^{\frac{\kappa}{\kappa-1}}\,dx &\le CT^{\frac{-2\rho\kappa}{\kappa-1}}\int_{T^\rho<|x|<\sqrt{2}T^\rho}g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell-\frac{2\kappa}{\kappa-1}}\,dx\\ &= CT^{\frac{-2\rho\kappa}{\kappa-1}+N\rho}\int_{1<|y|<\sqrt{2}}g(|y|^2)^{\ell-\frac{2\kappa}{\kappa-1}}\,dy,\end{aligned}$$

which yields (notice that $\ell > \frac{2\kappa}{\kappa-1}$)

$$\int_{\mathbb{R}^N}|G|^{\frac{-1}{\kappa-1}}|\Delta G|^{\frac{\kappa}{\kappa-1}}\,dx \le CT^{\frac{-2\rho\kappa}{\kappa-1}+N\rho}. \tag{2.6}$$

Finally, using (2.3), (2.4) and (2.6), (2.2) follows. □

Lemma 2.2. *Let $\kappa > 1$ and $\ell > \frac{\kappa}{\kappa-1}$. Then*

$$\int_Q|\xi_T|^{\frac{-1}{\kappa-1}}|\nabla\xi_T|^{\frac{\kappa}{\kappa-1}}\,dx\,dt \le CT^{\frac{-\rho\kappa}{\kappa-1}+N\rho+1}. \tag{2.7}$$

Proof. By (2.1) and the properties of the cut-off function g, one has

$$\int_Q |\xi_T|^{\frac{-1}{\kappa-1}} |\nabla \xi_T|^{\frac{\kappa}{\kappa-1}}\, dx\, dt = \left(\int_0^\infty F(t)\, dt\right)\left(\int_{T^\rho<|x|<\sqrt{2}T^\rho} |G|^{\frac{-1}{\kappa-1}} |\nabla G|^{\frac{\kappa}{\kappa-1}}\, dx\right). \tag{2.8}$$

On the other hand, for $T^\rho < |x| < \sqrt{2}T^\rho$, an elementary calculation shows that

$$\begin{aligned} |\nabla G(x)| &= 2\ell T^{-2\rho}|x| \left|g'\left(\frac{|x|^2}{T^{2\rho}}\right)\right| g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell-1} \\ &\le 2\sqrt{2}\ell T^{-\rho} \left|g'\left(\frac{|x|^2}{T^{2\rho}}\right)\right| g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell-1} \\ &\le C T^{-\rho} g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell-1}, \end{aligned}$$

which yields

$$|G|^{\frac{-1}{\kappa-1}} |\nabla G|^{\frac{\kappa}{\kappa-1}} \le C T^{\frac{-\rho\kappa}{\kappa-1}} g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell-\frac{\kappa}{\kappa-1}}, \quad T^\rho < |x| < \sqrt{2}T^\rho.$$

Therefore, using the change of variable $x = T^\rho y$ and the fact that $\ell > \frac{\kappa}{\kappa-1}$, one obtains

$$\begin{aligned} \int_{T^\rho<|x|<\sqrt{2}T^\rho} |G|^{\frac{-1}{\kappa-1}} |\nabla G|^{\frac{\kappa}{\kappa-1}}\, dx &\le C T^{\frac{-\rho\kappa}{\kappa-1}} \int_{T^\rho<|x|<\sqrt{2}T^\rho} g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell-\frac{\kappa}{\kappa-1}} dx \\ &= C T^{\frac{-\rho\kappa}{\kappa-1}+N\rho} \int_{1<|y|<\sqrt{2}} g(|y|^2)^{\ell-\frac{\kappa}{\kappa-1}}\, dy, \end{aligned}$$

i.e.

$$\int_{T^\rho<|x|<\sqrt{2}T^\rho} |G|^{\frac{-1}{\kappa-1}} |\nabla G|^{\frac{\kappa}{\kappa-1}}\, dx \le C T^{\frac{-\rho\kappa}{\kappa-1}+N\rho}. \tag{2.9}$$

Using (2.4), (2.8) and (2.9), (2.7) follows. □

Lemma 2.3. *Let $\kappa > 1$ and $\ell > \frac{2\kappa}{\kappa-1}$. Then*

$$\int_Q |\xi_T|^{\frac{-1}{\kappa-1}} |\xi_{T\,tt}|^{\frac{\kappa}{\kappa-1}}\, dx\, dt \le C T^{\frac{-2\kappa}{\kappa-1}+1+N\rho}. \tag{2.10}$$

Proof. By (2.1) and the properties of the cut-off functions f and g, one has

$$\int_Q |\xi_T|^{\frac{-1}{\kappa-1}} |\xi_{T\,tt}|^{\frac{\kappa}{\kappa-1}}\, dx\, dt = \left(\int_0^T F(t)^{\frac{-1}{\kappa-1}} |F''(t)|^{\frac{\kappa}{\kappa-1}}\, dt\right)\left(\int_{0<|x|<\sqrt{2}T^\rho} G(x)\, dx\right). \tag{2.11}$$

An elementary calculation shows that

$$F(t)^{\frac{-1}{\kappa-1}} |F''(t)|^{\frac{\kappa}{\kappa-1}} \le C T^{\frac{-2\kappa}{\kappa-1}} f\left(\frac{t}{T}\right)^{\ell-\frac{2\kappa}{\kappa-1}}, \quad 0 < t < T,$$

which yields (since $\ell > \frac{2\kappa}{\kappa-1}$)

$$\int_0^T F(t)^{\frac{-1}{\kappa-1}} |F''(t)|^{\frac{\kappa}{\kappa-1}}\, dt \le C T^{\frac{-2\kappa}{\kappa-1}+1}. \tag{2.12}$$

On the other hand, using the change of variable $x = T^\rho y$, one obtains

$$\begin{aligned}\int_{0<|x|<\sqrt{2}T^\rho} G(x)\,dx &= \int_{0<|x|<\sqrt{2}T^\rho} g\left(\frac{|x|^2}{T^{2\rho}}\right)^\ell dx\\ &= T^{N\rho}\int_{0<|y|<\sqrt{2}} g(|y|^2)^\ell\,dy,\end{aligned}$$

which yields

$$\int_{0<|x|<\sqrt{2}T^\rho} G(x)\,dx \le CT^{N\rho}. \tag{2.13}$$

Therefore, using (2.11), (2.12) and (2.13), (2.10) follows. □

Lemma 2.4. *Let $\kappa > 1$ and $\ell > \frac{\kappa}{\kappa-1}$. Then*

$$\int_Q |\xi_T|^{\frac{-1}{\kappa-1}} |\xi_{T_t}|^{\frac{\kappa}{\kappa-1}}\,dx\,dt \le CT^{\frac{-\kappa}{\kappa-1}+1+N\rho}. \tag{2.14}$$

Proof. By (2.1) and the properties of the cut-off functions f and g, one has

$$\int_Q |\xi_T|^{\frac{-1}{\kappa-1}} |\xi_{T_t}|^{\frac{\kappa}{\kappa-1}}\,dx\,dt = \left(\int_0^T F(t)^{\frac{-1}{\kappa-1}}|F'(t)|^{\frac{\kappa}{\kappa-1}}\,dt\right)\left(\int_{0<|x|<\sqrt{2}T^\rho} G(x)\,dx\right). \tag{2.15}$$

An elementary calculation shows that

$$F(t)^{\frac{-1}{\kappa-1}}|F'(t)|^{\frac{\kappa}{\kappa-1}} \le CT^{\frac{-\kappa}{\kappa-1}} f\left(\frac{t}{T}\right)^{\ell-\frac{\kappa}{\kappa-1}}, \quad 0<t<T,$$

which yields (since $\ell > \frac{\kappa}{\kappa-1}$)

$$\int_0^T F(t)^{\frac{-1}{\kappa-1}}|F'(t)|^{\frac{\kappa}{\kappa-1}}\,dt \le CT^{1-\frac{\kappa}{\kappa-1}}. \tag{2.16}$$

Therefore, using (2.13), (2.15) and (2.16), (2.14) follows. □

3. Fujita critical exponent

3.1. A general estimate

Proposition 3.1. *Suppose that problem* (1.1) *admits a global weak solution u with $(u, \nabla u) \in L^p_{loc}(Q) \times L^q_{loc}(Q)$. Then there exists a constant $C > 0$ such that*

$$\int_Q w(x)\varphi\,dx\,dt \le C\min\{A(\varphi), B(\varphi)\}, \tag{3.1}$$

for all $\varphi \in C_c^2(Q)$, $\varphi \ge 0$, where

$$A(\varphi) = \int_Q \varphi^{\frac{-1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}\,dx\,dt + \int_Q \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,dx\,dt + \int_Q \varphi^{\frac{-1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}}\,dx\,dt$$

and

$$B(\varphi) = \int_Q \varphi^{\frac{-1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}\,dx\,dt + \int_Q \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,dx\,dt + \int_Q \varphi^{\frac{-1}{q-1}}|\nabla\varphi|^{\frac{q}{q-1}}\,dx\,dt.$$

Proof. Let u be a global weak solution to problem (1.1) with

$$(u, \nabla u) \in L^p_{loc}(Q) \times L^q_{loc}(Q).$$

Let $\varphi \in C_c^2(Q)$, $\varphi \geq 0$. Using (1.6), one obtains

$$\int_Q |u|^p \varphi \, dx\, dt + \int_Q w(x)\varphi \, dx\, dt \leq \int_Q |u||\varphi_{tt}| \, dx\, dt + \int_Q |u||\Delta\varphi| \, dx\, dt + \int_Q |u||\varphi_t| \, dx\, dt. \tag{3.2}$$

On the other hand, by ε-Young inequality, $0 < \varepsilon < \frac{1}{3}$, one has

$$\begin{aligned}\int_Q |u||\varphi_{tt}| \, dx\, dt &= \int_Q \left(|u|\varphi^{\frac{1}{p}}\right)\left(\varphi^{\frac{-1}{p}}|\varphi_{tt}|\right) \, dx\, dt \\ &\leq \varepsilon \int_Q |u|^p \varphi \, dx\, dt + C \int_Q \varphi^{\frac{-1}{p-1}} |\varphi_{tt}|^{\frac{p}{p-1}} \, dx\, dt. \end{aligned} \tag{3.3}$$

Here and below, C denotes a positive generic constant, whose value may change from line to line. Similarly, one has

$$\int_Q |u||\Delta\varphi| \, dx\, dt \leq \varepsilon \int_Q |u|^p \varphi \, dx\, dt + C \int_Q \varphi^{\frac{-1}{p-1}} |\Delta\varphi|^{\frac{p}{p-1}} \, dx\, dt \tag{3.4}$$

and

$$\int_Q |u||\varphi_t| \, dx\, dt \leq \varepsilon \int_Q |u|^p \varphi \, dx\, dt + C \int_Q \varphi^{\frac{-1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} \, dx\, dt \tag{3.5}$$

Therefore, using (3.2), (3.3), (3.4) and (3.5), it holds that

$$(1 - 3\varepsilon) \int_Q |u|^p \varphi \, dx\, dt + \int_Q w(x)\varphi \, dx\, dt \leq CA(\varphi).$$

Since $1 - 3\varepsilon > 0$, one deduces that

$$\int_Q w(x)\varphi \, dx\, dt \leq CA(\varphi). \tag{3.6}$$

Again, by (1.6), one has

$$\begin{aligned} &\int_Q (|u|^p + |\nabla u|^q)\, \varphi \, dx\, dt + \int_Q w(x)\varphi \, dx\, dt \\ &= \int_Q u\varphi_{tt} \, dx\, dt - \int_Q u\Delta\varphi \, dx\, dt - \int_Q u\varphi_t \, dx\, dt \\ &= \int_Q u\varphi_{tt} \, dx\, dt + \int_Q \nabla u \cdot \nabla\varphi \, dx\, dt - \int_Q u\varphi_t \, dx\, dt, \end{aligned}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^N$. Hence, it holds that

$$\begin{aligned} &\int_Q (|u|^p + |\nabla u|^q)\, \varphi \, dx\, dt + \int_Q w(x)\varphi \, dx\, dt \\ &\leq \int_Q |u||\varphi_{tt}| \, dx\, dt + \int_Q |\nabla u||\nabla\varphi| \, dx\, dt + \int_Q |u||\varphi_t| \, dx\, dt. \end{aligned} \tag{3.7}$$

By ε-Young inequality ($\varepsilon = \frac{1}{2}$), one obtains

$$\int_Q |u||\varphi_{tt}|\,dx\,dt \leq \frac{1}{2}\int_Q |u|^p\varphi\,dx\,dt + C\int_Q \varphi^{\frac{-1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}\,dx\,dt, \tag{3.8}$$

$$\int_Q |u||\varphi_t|\,dx\,dt \leq \frac{1}{2}\int_Q |u|^p\varphi\,dx\,dt + C\int_Q \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,dx\,dt, \tag{3.9}$$

$$\int_Q |\nabla u||\nabla\varphi|\,dx\,dt \leq \frac{1}{2}\int_Q |\nabla u|^q\varphi\,dx\,dt + C\int_Q \varphi^{\frac{-1}{q-1}}|\nabla\varphi|^{\frac{q}{q-1}}\,dx\,dt. \tag{3.10}$$

Next, using (3.7), (3.8), (3.9) and (3.10), one deduces that

$$\frac{1}{2}\int_Q |\nabla u|^q\varphi\,dx\,dt + \int_Q w(x)\varphi\;dx\,dt \leq CB(\varphi),$$

which yields

$$\int_Q w(x)\varphi\;dx\,dt \leq CB(\varphi). \tag{3.11}$$

Finally, combining (3.6) with (3.11), (3.1) follows. □

Proposition 3.2. *Suppose that problem* (1.1) *admits a global weak solution u with* $(u, \nabla u) \in L^p_{loc}(Q) \times L^q_{loc}(Q)$. *Then*

$$\int_Q w(x)\xi_T\,dx\,dt \leq C\min\{A(\xi_T), B(\xi_T)\}, \tag{3.12}$$

for all $T > 0$, where ξ_T is defined by (2.1).

Proof. Since for all $T > 0$, $\xi_T \in C^2_c(Q)$, $\xi_T \geq 0$, taking $\varphi = \xi_T$ in (3.1), (3.12) follows. □

3.2. An estimate related to the inhomogeneous term

Proposition 3.3. *Let $w \in L^1_{loc}(\mathbb{R}^N)$, $w \geq 0$ and $w \not\equiv 0$. Then, for sufficiently large T,*

$$\int_Q w(x)\xi_T\,dx\,dt \geq CT, \tag{3.13}$$

where ξ_T is defined by (2.1).

Proof. By (2.1) and the properties of the cut-off functions f and g, one has

$$\int_Q w(x)\xi_T\,dx\,dt = \left(\int_0^T f\left(\frac{t}{T}\right)^\ell\,dt\right)\left(\int_{0<|x|<\sqrt{2}T^\rho} w(x)g\left(\frac{|x|^2}{T^{2\rho}}\right)^\ell\,dx\right). \tag{3.14}$$

On the other hand,

$$\int_0^T f\left(\frac{t}{T}\right)^\ell\,dt = T\int_0^1 f(s)^\ell\,ds. \tag{3.15}$$

Moreover, for sufficiently large T (since $w, g \geq 0$),

$$\int_{0<|x|<\sqrt{2}T^\rho} w(x)g\left(\frac{|x|^2}{T^{2\rho}}\right)^\ell\,dx \geq \int_{0<|x|<1} w(x)g\left(\frac{|x|^2}{T^{2\rho}}\right)^\ell\,dx. \tag{3.16}$$

Notice that since $w \in L^1_{loc}(\mathbb{R}^N)$, $w \geq 0$, $w \not\equiv 0$, and using the properties of the cut-off function g, by the domianted convergence theorem, one has

$$\lim_{T\to\infty} \int_{0<|x|<1} w(x) g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell} dx = \int_{0<|x|<1} w(x)\, dx > 0.$$

Hence, for sufficiently large T,

$$\int_{0<|x|<1} w(x) g\left(\frac{|x|^2}{T^{2\rho}}\right)^{\ell} dx \geq C. \tag{3.17}$$

Using (3.14), (3.15), (3.16) and (3.17), (3.13) follows. □

3.3. Proof of Theorem 1.1

Now, we are ready to prove the Fujita-type result given by Theorem 1.1. The proof is by contradiction and makes use of the nonlinear capacity method, which was developed by Mitidieri and Pohozaev (see e.g. [11]).

Proof of Theorem 1.1. (i) Suppose that problem (1.1) admits a global weak solution u with $(u, \nabla u) \in L^p_{loc}(Q) \times L^q_{loc}(Q)$. By Propositions 3.2 and 3.3, for sufficiently large T, one has

$$CT \leq \min\{A(\xi_T), B(\xi_T)\}, \tag{3.18}$$

where ξ_T is defined by (2.1),

$$A(\xi_T) = \int_Q \xi_T^{\frac{-1}{p-1}} |\xi_{T\,tt}|^{\frac{p}{p-1}}\, dx\, dt + \int_Q \xi_T^{\frac{-1}{p-1}} |\xi_{T\,t}|^{\frac{p}{p-1}}\, dx\, dt + \int_Q \xi_T^{\frac{-1}{p-1}} |\Delta \xi_T|^{\frac{p}{p-1}}\, dx\, dt \tag{3.19}$$

and

$$B(\xi_T) = \int_Q \varphi^{\frac{-1}{p-1}} |\xi_{T\,tt}|^{\frac{p}{p-1}}\, dx\, dt + \int_Q \xi_T^{\frac{-1}{p-1}} |\xi_{T\,t}|^{\frac{p}{p-1}}\, dx\, dt + \int_Q \xi_T^{\frac{-1}{q-1}} |\nabla \xi_T|^{\frac{q}{q-1}}\, dx\, dt. \tag{3.20}$$

Taking $\ell > \max\left\{\frac{2p}{p-1}, \frac{q}{q-1}\right\}$, using Lemmas 2.1, 2.3, 2.4 with $\kappa = p$, and Lemma 2.2 with $\kappa = q$, one obtains the following estimates

$$\int_Q \xi_T^{\frac{-1}{p-1}} |\xi_{T\,tt}|^{\frac{p}{p-1}}\, dx\, dt \leq CT^{\frac{-2p}{p-1}+1+N\rho}, \tag{3.21}$$

$$\int_Q \xi_T^{\frac{-1}{p-1}} |\xi_{T\,t}|^{\frac{p}{p-1}}\, dx\, dt \leq CT^{\frac{-p}{p-1}+1+N\rho}, \tag{3.22}$$

$$\int_Q \xi_T^{\frac{-1}{p-1}} |\Delta \xi_T|^{\frac{p}{p-1}}\, dx\, dt \leq CT^{\frac{-2\rho p}{p-1}+N\rho+1}, \tag{3.23}$$

$$\int_Q \xi_T^{\frac{-1}{q-1}} |\nabla \xi_T|^{\frac{q}{q-1}}\, dx\, dt \leq CT^{\frac{-\rho q}{q-1}+N\rho+1}. \tag{3.24}$$

Hence, by (3.19), (3.21), (3.22) and (3.23), one deduces that

$$A(\xi_T) \leq C\left(T^{\frac{-2p}{p-1}+1+N\rho} + T^{\frac{-p}{p-1}+1+N\rho} + T^{\frac{-2\rho p}{p-1}+N\rho+1}\right).$$

Observe that
$$\frac{-2p}{p-1}+1+N\rho<\frac{-p}{p-1}+1+N\rho. \tag{3.25}$$
So, for sufficiently large T, one deduces that
$$A(\xi_T)\le C\left(T^{\frac{-p}{p-1}+1+N\rho}+T^{\frac{-2\rho p}{p-1}+N\rho+1}\right).$$
Notice that the above estimate holds for all $\rho>0$. In particular, when $\rho=\frac{1}{2}$, it holds that
$$A(\xi_T)\le CT^{\frac{-p}{p-1}+1+\frac{N}{2}} \tag{3.26}$$
Next, by (3.20), (3.21), (3.22), (3.24) and (3.25), one deduces that
$$B(\xi_T)\le C\left(T^{\frac{-p}{p-1}+1+N\rho}+T^{\frac{-\rho q}{q-1}+N\rho+1}\right).$$
Similarly, the above inequality holds for all $\rho>0$. In particular, when $\rho=\frac{p(q-1)}{q(p-1)}$, it holds that
$$B(\xi_T)\le CT^{\frac{-p}{p-1}+1+\frac{Np(q-1)}{q(p-1)}}. \tag{3.27}$$
Therefore, it follows from (3.18), (3.26) and (3.27) that
$$0<C\le\min\left\{T^{\frac{-p}{p-1}+\frac{N}{2}},T^{\frac{-p}{p-1}+\frac{Np(q-1)}{q(p-1)}}\right\},$$
which yields
$$0<C\le T^{\frac{-p}{p-1}+\frac{N}{2}}:=T^{\alpha(N)} \tag{3.28}$$
and
$$0<C\le T^{\frac{-p}{p-1}+\frac{Np(q-1)}{q(p-1)}}:=T^{\beta(N)}. \tag{3.29}$$
Notice that for $N\in\{1,2\}$, one has
$$\alpha(N)=\begin{cases}\frac{-p-1}{2(p-1)} & \text{if}\quad N=1,\\ \frac{-1}{p-1} & \text{if}\quad N=2,\end{cases}$$
which shows that
$$\alpha(N)<0,\quad N\in\{1,2\}.$$
Hence, for $N\in\{1,2\}$, passing to the limit as $T\to\infty$ in (3.28), a contradiction follows ($0<C\le 0$). This proves part (i) of Theorem 1.1.
(ii) Let $N\ge 3$. In this case, one has
$$\alpha(N)<0\Longleftrightarrow p<1+\frac{2}{N-2}.$$
Hence, if $p<1+\frac{2}{N-2}$, passing to the limit as $T\to\infty$ in (3.28), a contradiction follows. Furthermore, one has
$$\beta(N)<0\Longleftrightarrow q<1+\frac{1}{N-1}.$$

Hence, if $q < 1 + \frac{1}{N-1}$, passing to the limit as $T \to \infty$ in (3.29), a contradiction follows. Therefore, we proved part (ii) of Theorem 1.1.
(iii) Let

$$p > 1 + \frac{2}{N-2} \quad \text{and} \quad q > 1 + \frac{1}{N-1}. \tag{3.30}$$

Let

$$u(x) = \varepsilon(1 + r^2)^{-\delta}, \quad r = |x|, \quad x \in \mathbb{R}^N, \tag{3.31}$$

where

$$\max\left\{\frac{1}{p-1}, \frac{2-q}{2(q-1)}\right\} < \delta < \frac{N-2}{2} \tag{3.32}$$

and

$$0 < \varepsilon < \min\left\{1, \left(\frac{2\delta(N-2\delta-2)}{1+2^q\delta^q}\right)^{\frac{1}{\min\{p,q\}-1}}\right\}. \tag{3.33}$$

Note that due to (3.30), the set of δ satisfying (3.32) is nonempty. Let

$$w(x) = -\Delta u - |u|^p - |\nabla u|^q, \quad x \in \mathbb{R}^N.$$

Elementary calculations yield

$$\begin{aligned} w(x) &= 2\delta\varepsilon\left[(1+r^2)^{-\delta-1} - 2r^2(\delta+1)(1+r^2)^{-\delta-2} + (N-1)(1+r^2)^{-\delta-1}\right] \\ &\quad -\varepsilon^p(1+r^2)^{-\delta p} - 2^q\delta^q\varepsilon^q r^q(1+r^2)^{-(\delta+1)q}. \end{aligned} \tag{3.34}$$

Hence, one obtains

$$\begin{aligned} w(x) &\geq 2\delta\varepsilon\left[(1+r^2)^{-\delta-1} - 2(\delta+1)(1+r^2)^{-\delta-1} + (N-1)(1+r^2)^{-\delta-1}\right] \\ &\quad -\varepsilon^p(1+r^2)^{-\delta p} - 2^q\delta^q\varepsilon^q(1+r^2)^{-(\delta+1)q+\frac{q}{2}} \\ &= 2\delta\varepsilon(N-2\delta-2)(1+r^2)^{-\delta-1} - \varepsilon^p(1+r^2)^{-\delta p} - 2^q\varepsilon^q\delta^q(1+r^2)^{-\left(\delta+\frac{1}{2}\right)q}. \end{aligned}$$

Next, using (3.32) and (3.33), one deduces that

$$w(x) \geq \varepsilon\left[2\delta(N-2\delta-2) - \varepsilon^{p-1} - 2^q\varepsilon^{q-1}\delta^q\right](1+r^2)^{-\delta-1} > 0.$$

Therefore, for any δ and ε satisfying respectively (3.32) and (3.33), the function u defined by (3.31) is a stationary solution (then global solution) to (1.1) for some $w > 0$. This proves part (iii) of Theorem 1.1.

□

4. Second critical exponent

4.1. An estimate related to the inhomogeneous term

Proposition 4.1. *Let $N \geq 3$, $\sigma < N$ and $w \in I_\sigma^+$. Then, for sufficiently large T,*

$$\int_Q w(x)\xi_T \, dx \, dt \geq CT^{\rho(N-\sigma)+1}. \tag{4.1}$$

where ξ_T is defined by (2.1).

Proof. By (3.14) and (3.15), one has

$$\int_Q w(x)\xi_T\,dx\,dt = CT\int_{0<|x|<\sqrt{2}T^\rho} w(x)g\left(\frac{|x|^2}{T^{2\rho}}\right)^\ell dx. \tag{4.2}$$

On the other hand, by definition of I_σ^+, and using that $g(s)=1$, $0\le s\le 1$, for sufficiently large T, one obtains (since $w, g \ge 0$)

$$\begin{aligned}
\int_{0<|x|<\sqrt{2}T^\rho} w(x)g\left(\frac{|x|^2}{T^{2\rho}}\right)^\ell dx &\ge \int_{0<|x|<T^\rho} w(x)g\left(\frac{|x|^2}{T^{2\rho}}\right)^\ell dx\\
&= \int_{0<|x|<T^\rho} w(x)\,dx\\
&\ge \int_{\frac{T^\rho}{2}<|x|<T^\rho} w(x)\,dx\\
&\ge C\int_{\frac{T^\rho}{2}<|x|<T^\rho} |x|^{-\sigma}\,dx\\
&= CT^{\rho(N-\sigma)}.
\end{aligned}$$

Hence, using (4.2), (4.1) follows. $\square$

4.2. Proof of Theorem 1.2

Now, we are ready to prove the new critical behavior for the problem (1.1) stated by Theorem 1.2.

Proof of Theorem 1.2. (i) Suppose that problem (1.1) admits a global weak solution u with $(u,\nabla u)\in L^p_{loc}(Q)\times L^q_{loc}(Q)$. By Propositions 3.2 and 4.1, for sufficiently large T, one obtains

$$CT^{\rho(N-\sigma)+1} \le \min\{A(\xi_T), B(\xi_T)\}, \tag{4.3}$$

where ξ_T, $A(\xi_T)$ and $B(\xi_T)$ are defined respectively by (2.1), (3.19) and (3.20). Next, using (3.26) and (4.3) with $\rho=\frac{1}{2}$, one deduces that

$$0 < C \le T^{\frac{-p}{p-1}+\frac{\sigma}{2}}. \tag{4.4}$$

Observe that

$$\frac{-p}{p-1}+\frac{\sigma}{2} < 0 \Longleftrightarrow \sigma < \frac{2p}{p-1}.$$

Hence, if $\sigma < \frac{2p}{p-1}$, passing to the limit as $T\to\infty$ in (4.4), a contradiction follows. Furthermore, using (3.27) and (4.3) with $\rho = \frac{p(q-1)}{q(p-1)}$, one deduces that

$$0 < C \le T^{\frac{p}{p-1}\left(\frac{\sigma(q-1)}{q}-1\right)}. \tag{4.5}$$

Observe that

$$\frac{p}{p-1}\left(\frac{\sigma(q-1)}{q}-1\right) < 0 \Longleftrightarrow \sigma < \frac{q}{q-1}.$$

Hence, if $\sigma < \frac{q}{q-1}$, passing to the limit as $T \to \infty$ in (4.5), a contradiction follows. Therefore, part (i) of Theorem 1.2 is proved.
(ii) Let

$$\sigma \geq \max\left\{\frac{2p}{p-1}, \frac{q}{q-1}\right\}. \tag{4.6}$$

Let u be the function defined by (3.31), where

$$\frac{\sigma-2}{2} < \delta < \frac{N-2}{2} \tag{4.7}$$

and ε satisfies (3.33). Notice that since $\sigma < N$, the set of δ satisfying (4.7) is nonempty. Moreover, due to (4.6) and (4.7), δ satisfies also (3.32). Hence, from the proof of part (iii) of Theorem 1.2, one deduces that

$$w(x) = -\Delta u - |u|^p - |\nabla u|^q > 0, \quad x \in \mathbb{R}^N.$$

On the other hand, using (3.34) and (4.7), for $|x|$ large enough, one obtains

$$w(x) \leq C(1+|x|^2)^{-\delta-1} \leq C|x|^{-2\delta-2} \leq C|x|^{-\delta},$$

which proves that $w \in I_\sigma^-$. This proves part (ii) of Theorem 1.2. □

5. Conclusions

We investigated the large-time behavior of solutions to the nonlinear damped wave equation (1.1). In the case when $N \in \{1,2\}$, we proved that for all $p > 1$, problem (1.1) admits no global weak solution (in the sense of Definition 1.1). Notice that from [5], the same result holds for the problem without gradient term, namely problem (1.5). This shows that in the case $N \in \{1,2\}$, the nonlinearity $|\nabla u|^q$ has no influence on the critical behavior of problem (1.5). In the case when $N \geq 3$, we proved that, if $1 < p < 1 + \frac{2}{N-2}$ or $1 < q < 1 + \frac{1}{N-1}$, then problem (1.1) admits no global weak solution, while if $p > 1 + \frac{2}{N-2}$ and $q > 1 + \frac{1}{N-1}$, global solutions exist for some $w > 0$. This shows that in this case, the Fujita critical exponent for the problem (1.1) is given by

$$p_c(N,q) = \begin{cases} 1 + \frac{2}{N-2} & \text{if} \quad q > 1 + \frac{1}{N-1}, \\ \infty & \text{if} \quad q < 1 + \frac{1}{N-1}. \end{cases}$$

From this result, one observes two facts. First, in the range $q > 1 + \frac{1}{N-1}$, from [5], the critical exponent $p_c(N,q)$ is also equal to the critical exponent for the problem without gradient term, which means that in this range of q, the nonlinearity $|\nabla u|^q$ has no influence on the critical behavior of problem (1.5). Secondly, one observes that the gradient term induces an interesting phenomenon of discontinuity of the Fujita critical exponent $p_c(N,q)$ jumping from $1 + \frac{2}{N-2}$ to ∞ as q reaches the value $1 + \frac{1}{N-1}$ from above. In the same case $N \geq 3$, we determined also the second critical exponent for the problem (1.1) in the sense of Lee and Ni [10], when $p > 1 + \frac{2}{N-2}$ and $q > 1 + \frac{1}{N-1}$. Namely, we proved that in this case, if $\sigma < \max\left\{\frac{2p}{p-1}, \frac{q}{q-1}\right\}$ and $w \in I_\sigma^+$, then there is no global weak solution, while if $\max\left\{\frac{2p}{p-1}, \frac{q}{q-1}\right\} \leq \sigma < N$, global solutions exist for some $w \in I_\sigma^-$. This shows that the second critical exponent for the problem (1.1) in the sense of Lee and Ni is given by

$$\sigma^* = \max\left\{\frac{2p}{p-1}, \frac{q}{q-1}\right\}.$$

We end this section with the following open questions:

(Q1). Find the first and second critical exponents for the system of damped wave equations with mixed nonlinearities

$$\begin{cases} u_{tt} - \Delta u + u_t &= |v|^{p_1} + |\nabla v|^{q_1} + w_1(x) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N, \\ v_{tt} - \Delta v + v_t &= |u|^{p_2} + |\nabla u|^{q_2} + w_2(x) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N, \end{cases}$$

where $p_i, q_i > 1$, $w_i \in L^1_{loc}(\mathbb{R}^N)$, $w_i \geq 0$ and $w_i \not\equiv 0$, $i = 1,2$.

(Q2). Find the Fujita critical exponent for the problem (1.1) with $w \equiv 0$.

(Q3) Find the Fujita critical exponent for the problem

$$u_{tt} - \Delta u + u_t = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}|u(s,x)|^p\,ds + |\nabla u|^q + w(x), \quad t>0,\ x\in\mathbb{R}^N,$$

where $N \geq 1$, $p,q > 1$, $0 < \alpha < 1$ and $w \in L^1_{loc}(\mathbb{R}^N)$, $w \geq 0$. Notice that in the limit case $\alpha \to 1^-$, the above equation reduces to (1.1).

(Q4) Same question as above for the problems

$$u_{tt} - \Delta u + u_t = |u|^p + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}|\nabla u(s,x)|^q\,ds + w(x)$$

and

$$u_{tt} - \Delta u + u_t = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}|u(s,x)|^p\,ds + \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}|\nabla u(s,x)|^q + w(x),$$

where $0 < \alpha,\beta < 1$.

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Conflict of interest

The author declares no conflict of interest.

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