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*Research article*

# A new approach on fractional calculus and probability density function

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**Abstract:** In statistical analysis, oftentimes a probability density function is used to describe the relationship between certain unknown parameters and measurements taken to learn about them. As soon as there is more than enough data collected to determine a unique solution for the parameters, an estimation technique needs to be applied such as "fractional calculus", for instance, which turns out to be optimal under a wide range of criteria. In this context, we aim to present some novel estimates based on the expectation and variance of a continuous random variable by employing generalized Riemann-Liouville fractional integral operators. Besides, we obtain a two-parameter extension of generalized Riemann-Liouville fractional integral inequalities, and provide several modifications in the Riemann-Liouville and classical sense. Our ideas and obtained results my stimulate further research in statistical analysis.

**Keywords:** generalized Riemann-Liouville fractional integral operator; integral inequality; expectation; variance
**Mathematics Subject Classification:** 26A51, 26D10, 26A33

## 1. Introduction

Integral inequalities exist in many branches of mathematics and physics [1–20], their roles in mathematics and its associated disciplines are priceless. Despite this, it was distinctly during the 1960s that the principal work [21] was distributed and it is this exemplary work changed the field of inequalities from an assortment of secluded equations into a precise and attractive discipline [22–32]. In recent years, the theory of inequalities has formed into a dynamic and autonomous area of research, requiring the development of new journals that gave exclusively to inequalities and their applications. Recently, several researchers have contributed to produce different results about fractional integral inequalities and their applications utilizing Riemann, Liouville, Caputo, Wely, and Hadamard fractional integral and differential operators. Specific consideration has been given to inequalities including special functions [33–39], fractional calculus [40–46] and probability density functions and this is the place the current work lies. We concentrate our attention around variants including the fractional calculus and continuous random variables.

A complete description of the distribution of a probability for a given random variable can be obtained by distribution function and density functions. Interestingly, they don't permit us to do comparisons between two distinct distributions. The random variables about mean that particularly portray the appropriation under reasonable conditions is helpful in making comparisons. Knowing the probability function, we can determine the expectation and variance. There are, however, applications wherein the exact forms of probability distributions are not known or are mathematically intractable so that the moments cannot be calculated–as an example, an application in insurance in connection with the insurer's payout on a given contract or group of contracts that follows a mixture or compound probability distribution. It is this problem that motivates researchers to obtain alternative estimations for the expectations and variances of a probability distribution. Applying the mathematical inequalities, some estimations for the expectation and variance of random variables were studied in [47, 48].

In 2001, Cerone and Dargomir [49] estimated the bounds of a continuous random variable whose probability density function for the expectation and variance is defined on a finite interval, some integral inequalities have been contemplated for the expectation and variance of a random variable having a probability density function. Kumar [50] derived certain variants for the moments and higher-order moments of a continuous random variable.

The main purpose of the article is to establish some novel estimates for the expectation and variance of the continuous random variables by use of the generalized Riemann-Liouville fractional integral operator, and provide new bounds for certain consequences of the Riemann-Liouville fractional integral, Katugampola fractional integral, conformable fractional integral and Hadamard fractional integral operators by varying the domain as special cases.

## 2. Prelude

In this section, we give some basic notions for the generalized Riemann-Liouville fractional integral operators.

**Definition 2.1.** *(See [51]) Let $p \geq 1$, $r \geq 0$ and $\upsilon_1 < \upsilon_2$. Then the function $\mathcal{F}(\xi)$ is said to be in*

*$L_{p,r}(\upsilon_1,\upsilon_2)$-space if*

$$\|\mathcal{F}\|_{L_{p,r}(\upsilon_1,\upsilon_2)} = \Big(\int_{\upsilon_1}^{\upsilon_2} |\mathcal{F}(\xi)|^p \xi^r d\xi\Big)^{\frac{1}{p}} < \infty.$$

*If $r = 0$, then we denote*

$$L_p(\upsilon_1,\upsilon_2) = L_{p,0}(\upsilon_1,\upsilon_2) = \Big\{\mathcal{F} : \|\mathcal{F}\|_{L_p(\upsilon_1,\upsilon_2)} = \Big(\int_{\upsilon_1}^{\upsilon_2} |\mathcal{F}(\xi)|^p d\xi\Big)^{\frac{1}{p}} < \infty\Big\}.$$

**Definition 2.2.** *(See [52]) Let $\mathcal{F} \in L_1[0,\infty)$ and $\mathfrak{u}$ be an increasing and positive function defined on $[0,\infty)$ such that $\mathfrak{u}'$ is continuous on $[0,\infty)$ and $\mathfrak{u}(0) = 0$. Then the space $\chi^p_{\mathfrak{u}}(0,\infty)$ $(1 \le p < \infty)$ is all the real-valued Lebesgue measureable functions $\mathcal{F}$ defined on $[0,\infty)$ such that*

$$\|\mathcal{F}\|_{\chi^p_{\mathfrak{u}}} = \Big(\int_0^{\infty} |\mathcal{F}(\xi)|^p \mathfrak{u}'(\xi) d\xi\Big)^{\frac{1}{p}} < \infty \quad (1 \le p < \infty).$$

*If $p = \infty$, then $\|\mathcal{F}\|_{\chi^\infty_{\mathfrak{u}}}$ is defined by*

$$\|\mathcal{F}\|_{\chi^\infty_{\mathfrak{u}}} = ess \sup_{0 \le \xi < \infty} [\mathfrak{u}'(\xi)\mathcal{F}(\xi)].$$

In particular, if $\mathfrak{u}(\varsigma) = \varsigma \quad (1 \le p < \infty)$, then the space $\chi^p_{\mathfrak{u}}(0,\infty)$ coincides with the $L_p[0,\infty)$-space; if $\mathfrak{u}(\varsigma) = \frac{\varsigma^{r+1}}{r+1} \quad (1 \le p < \infty,\ r \ge 0)$, the the space $\chi^p_{\mathfrak{u}}(0,\infty)$ reduces to the $L_{p,u}[0,\infty)$-space.

**Definition 2.3.** *(See [51]) Let $\mathcal{F} \in L_1([\eta_1,\eta_2])$. Then the left-sided and right-sided Riemann-Liouville fractional integrals of order $\delta > 0$ are defined by*

$$\mathfrak{I}^{\delta}_{\eta_1^+}\mathcal{F}(\varsigma) = \frac{1}{\Gamma(\delta)} \int_{\eta_1}^{\varsigma} (\varsigma - \xi)^{\delta-1} \mathcal{F}(\xi) d\xi \quad \varsigma > \eta_1$$

*and*

$$\mathfrak{I}^{\delta}_{\eta_2^-}\mathcal{F}(\varsigma) = \frac{1}{\Gamma(\delta)} \int_{\varsigma}^{\eta_2} (\xi - \varsigma)^{\delta-1} \mathcal{F}(\xi) d\xi \quad \varsigma < \eta_2,$$

*where $\Gamma(\delta) = \int_0^{\infty} e^{-w} w^{\delta-1} dw$ is the Gamma function.*

A generalization of the Riemann-Liouville fractional integrals with respect to another function can be found in [51].

**Definition 2.4.** *(See [51]) Let $\delta > 0$, $(\eta_1,\eta_2)$ $(-\infty \le \eta_1 < \eta_2 \le \infty)$ be a finite or infinite real interval, and $\mathfrak{u}(\xi)$ be an increasing and positive function defined on $(\eta_1,\eta_2]$ such that $\mathfrak{u}'$ is continuous on $[0,\infty)$ and $\mathfrak{u}(0) = 0$. Then the left-sided and right-sided generalized Riemann-Liouville fractional integrals of a function $\mathcal{F}$ with respect to another function $\mathfrak{u}$ of order $\delta > 0$ are defined by*

$$\mathfrak{I}^{\delta}_{\mathfrak{u},\eta_1^+}\mathcal{F}(\varsigma) = \frac{1}{\Gamma(\delta)} \int_{\eta_1}^{\varsigma} \mathfrak{u}'(\xi)(\mathfrak{u}(\varsigma) - \mathfrak{u}(\xi))^{\delta-1} \mathcal{F}(\xi) d\xi \tag{2.1}$$

*and*

$$\Im^{\delta}_{\mathrm{u},\eta_2^-}\mathcal{F}(\varsigma) = \frac{1}{\Gamma(\delta)}\int_{\varsigma}^{\eta_2} \mathrm{u}'(\xi)(\mathrm{u}(\xi) - \mathrm{u}(\varsigma))^{\delta-1}\mathcal{F}(\xi)d\xi. \tag{2.2}$$

**Remark 2.1.** *From Definition 2.4 we clearly see that*

*(1) If* $\mathrm{u}(\varsigma) = \varsigma$*, then we get Definition 2.3.*

*(2) If* $\mathrm{u}(\varsigma) = \log\varsigma$*, then Definition 2.4 reduces to the Hadamard fractional integral operator given in [51].*

*(3) If* $\mathrm{u}(\varsigma) = \frac{\varsigma^\beta}{\beta}$ $(\beta > 0)$*, then Definition 2.4 becomes the Katugampola fractional integrals operators [53].*

*(4) If* $\mathrm{u}(\varsigma) = \frac{(\varsigma-a)^\beta}{\beta}$ $(\beta > 0)$*, then it reduces to the conformable fractional integrals operator defined by Jarad et al. [54].*

*(5) If* $\mathrm{u}(\varsigma) = \frac{\varsigma^{u+v}}{u+v}$*, then it becomes the generalized conformable fractional integrals defined by Khan et al. [55].*

**Definition 2.5.** *Let Y be a random variable with a positive probability density function* $\mathcal{F}$ *defined on* $[\eta_1, \eta_2]$ *and* $\mathrm{u}(\xi)$ *be an increasing and positive function defined on* $(\eta_1, \eta_2]$*. Then the fractional expectation function* $E_{Y,\delta}(\varsigma)$ *of order* $\delta \geq 0$ *is defined by*

$$E_{Y,\delta}(\varsigma) = \Im^{\mathrm{u},\delta}_{\eta_1^+}[\varsigma\mathcal{F}(\varsigma)] = \frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\varsigma} \mathrm{u}'(\xi)[\mathrm{u}(\varsigma) - \mathrm{u}(\xi)]^{\delta-1}\xi\mathcal{F}(\xi)d\xi \quad (\eta_1 < \xi \leq \eta_2). \tag{2.3}$$

Similarly, we define the fractional expectation function of $Y - E(Y)$ as follows.

**Definition 2.6.** *Let* $\mathrm{u}(\xi)$ *be an increasing and positive function defined on* $(\eta_1, \eta_2]$*. Then the fractional expectation function* $E_{Y-E(Y),\delta}(\varsigma)$ *of order* $\delta \geq 0$ *for a random variable* $Y - E(Y)$ *with a positive probability density function* $\mathcal{F}$ *defined on* $[\eta_1, \eta_2]$ *is defined by*

$$E_{Y-E(Y),\delta}(\varsigma) = \Im^{\mathrm{u},\delta}_{\eta_1^+}[\varsigma\mathcal{F}(\varsigma)] = \frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\xi} \mathrm{u}'(\xi)[\mathrm{u}(\varsigma) - \mathrm{u}(\xi)]^{\delta-1}(\xi - E(Y))\mathcal{F}(\xi)d\xi, \quad \eta_1 < \xi \leq \eta_2, \tag{2.4}$$

*where* $\mathcal{F} : [\eta_1, \eta_2] \to \mathbb{R}^+$ *is the probability density function.*

If $\xi = \eta_2$, then we present the following definitions.

**Definition 2.7.** *Let* $\eta_1 \geq 0$ *and* $\mathrm{u}(\xi)$ *be an increasing and positive function defined on* $(\eta_1, \eta_2]$*. Then the fractional expectation function of order* $\delta \geq 0$ *for a random variable Y with a positive probability density function* $\mathcal{F}$ *defined on* $[\eta_1, \eta_2]$ *is defined by*

$$E_{Y,\delta}(\varsigma) := \Im^{\mathrm{u},\delta}_{\eta_1^+}[\varsigma\mathcal{F}(\varsigma)] = \frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\eta_2} \mathrm{u}'(\xi)[\mathrm{u}(\eta_2) - \mathrm{u}(\xi)]^{\delta-1}\xi\mathcal{F}(\xi)d\xi \quad (\eta_1 < \xi \leq \eta_2). \tag{2.5}$$

**Definition 2.8.** *Let $\eta_1 \geq 0$ and $\mathfrak{u}(\xi)$ be an increasing and positive function defined on $(\eta_1, \eta_2]$. Then the generalized fractional variance function of order $\delta \geq 0$ for a random variable $Y$ with a positive probability density function $\mathcal{F}$ defined on $[\eta_1, \eta_2]$ is defined by*

$$\sigma^2_{Y,\delta}(\xi) := \mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}[(\varsigma - E(Y))^2\mathcal{F}(\varsigma)] = \frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\varsigma} \mathfrak{u}'(\xi)[\mathfrak{u}(\varsigma) - \mathfrak{u}(\xi)]^{\delta-1}(\xi - E(Y))^2\mathcal{F}(\xi)d\xi \quad (\eta_1 < \xi \leq \eta_2), \tag{2.6}$$

*where $E(Y) = \int_{\eta_1}^{\eta_2} \xi\mathcal{F}(\xi)d\xi$ represents the classical expectation of $Y$.*

If $\xi = \eta_2$, then we have the following definition.

**Definition 2.9.** *Let $\eta_1 \geq 0$ and $\mathfrak{u}(\xi)$ be an increasing and positive function defined on $(\eta_1, \eta_2]$. Then the generalized fractional variance function of order $\delta \geq 0$ for a random variable $Y$ with a positive probability density function $\mathcal{F} : [\eta_1, \eta_2] \to \mathbb{R}^+$ is defined by*

$$\sigma^2_{Y,\delta}(\xi) := \frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\eta_2} \mathfrak{u}'(\xi)[\mathfrak{u}(\eta_2) - \mathfrak{u}(\xi)]^{\delta-1}(\xi - E(Y))^2\mathcal{F}(\xi)d\xi. \tag{2.7}$$

**Remark 2.2.** *Definitions 2.5–2.9 lead to the conclusions that*

*(1) It $\delta = 1$ and $\mathfrak{u}(\varsigma) = \varsigma$, then Definition 2.5 leads to definition of the classical expectation.*

*(2) If $\delta = 1$ and $\mathfrak{u}(\varsigma) = \varsigma$, then Definition 2.8 becomes the definition of the classical variance.*

*(3) If $\mathfrak{u}(\varsigma) = \varsigma$, then from Definitions 2.5–2.9 we obtain Definitions 2.2–2.6 in [56].*

## 3. Main results

The key aim of this section is to establish several results for the continuous random variable having probability density functions via generalized Riemann-Liouville fractional integral operator. Throughout this paper, we assume that $\mathfrak{u}(\xi)$ is an increasing and positive function defined defined on $[0, \infty)$ such that $\mathfrak{u}(0) = 0$ and $\mathfrak{u}'(\xi)$ is continuous on $[0, \infty)$.

**Lemma 3.1.** *Let $Y$ be a continuous random variable with probability density function $\mathcal{F} : [\eta_1, \eta_2] \to \mathbb{R}^+$. Then*

$$\sigma^2_{Y,\delta} = E_{Y^2,\delta} - 2E(Y)E_{Y,\delta} + E(Y)^2\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}[\mathcal{F}(\eta_2)] \tag{3.1}$$

*for all $\delta \geq 0$.*

*Proof.* It follows from Definition (2.9) that

$$\sigma^2_{Y,\delta} = \frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\eta_2} \Big[\mathfrak{u}(\eta_2) - \mathfrak{u}(\xi)\Big]^{\delta-1}\mathfrak{u}'(\xi)\Big[\xi^2 + E(Y)^2 - 2\xi E(Y)\Big]\mathcal{F}(\xi)d\xi$$

and

$$\sigma^2_{Y,\delta} = \frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\eta_2} \Big[\mathfrak{u}(\eta_2) - \mathfrak{u}(\xi)\Big]^{\delta-1}\mathfrak{u}'(\xi)\xi^2\mathcal{F}(\xi)d\xi + \frac{E(Y)^2}{\Gamma(\delta)}\int_{\eta_1}^{\eta_2} \Big[\mathfrak{u}(\eta_2) - \mathfrak{u}(\xi)\Big]^{\delta-1}\mathfrak{u}'(\xi)\mathcal{F}(\xi)d\xi$$

$$-\frac{2E(Y)}{\Gamma(\delta)}\int_{\eta_1}^{\eta_2}\left[\mathfrak{u}(\eta_2)-\mathfrak{u}(\xi)\right]^{\delta-1}\mathfrak{u}'(\xi)\xi\mathcal{F}(\xi)d\xi.$$

Therefore,

$$\sigma^2_{Y,\delta}=E_{Y^2,\delta}-2E(Y)E_{Y,\delta}+E(Y)^2\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}[\mathcal{F}(\eta_2)].$$

□

**Theorem 3.2.** *Let $Y$ be a continuous random variable wtih probability density function $\mathcal{F}:[\eta_1,\eta_2]\to\mathbb{R}^+$. Then the following statements are true:*

*(1) For all $\delta\geq 0$ and $\eta_1<\xi\leq\eta_2$, one has*

$$\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}[\mathcal{F}(\xi)]\sigma^2_{Y,\delta}-\left(E_{Y-E(Y),\delta}(\xi)\right)^2\leq\|\mathcal{F}\|^2_\infty\left[2\frac{(\mathfrak{u}(\xi)-\mathfrak{u}(\eta_1))^\delta}{\Gamma(\delta+1)}\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}[\xi^2]-2(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}[\xi])^2\right]\tag{3.2}$$

*if $\mathcal{F}\in L_\infty([\eta_1,\eta_2])$.*

*(2) The inequality*

$$\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}[\mathcal{F}(\xi)]\sigma^2_{Y,\delta}-\left(E_{Y-E(Y),\delta}(\xi)\right)^2\leq\frac{1}{2}(\mathfrak{u}(\xi)-\mathfrak{u}(\eta_1))^2(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{F}(\xi))^2\tag{3.3}$$

*holds for all $\delta\geq 0$ and $\eta_1<\xi\leq\eta_2$.*

*Proof.* Let $\eta_1<\xi\leq\eta_2$ and $x,y\in(\eta_1,\xi)$. Then

$$\begin{aligned}\mathfrak{I}(x,y)&=(\mathcal{G}(x)-\mathcal{G}(y))(\mathcal{H}(x)-\mathcal{H}(y))\\&=\mathcal{G}(x)\mathcal{H}(x)+\mathcal{G}(y)\mathcal{H}(y)-\mathcal{G}(x)\mathcal{H}(y)-\mathcal{G}(y)\mathcal{H}(x).\end{aligned}\tag{3.4}$$

Multiplying both sides of (3.4) by $\frac{\left[\mathfrak{u}(\xi)-\mathfrak{u}(x)\right]^{\delta-1}\mathfrak{u}'(x)\mathcal{P}(x)}{\Gamma(\delta)}$ $(x\in(\eta_1,\xi))$ and then integrating the obtained result with respect to $x$ from $(\eta_1,\xi)$ leads to

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}\mathfrak{u}'(x)\mathcal{P}(x)\mathfrak{I}(x,y)dx\\&=\frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}\mathfrak{u}'(x)\mathcal{P}(x)\mathcal{G}(x)\mathcal{H}(x)dx+\frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}\mathfrak{u}'(x)\mathcal{P}(x)\mathcal{G}(y)\mathcal{H}(y)\\&\quad-\frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}\mathfrak{u}'(x)\mathcal{P}(x)\mathcal{G}(x)\mathcal{H}(y)-\frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}\mathfrak{u}'(x)\mathcal{P}(x)\mathcal{G}(y)\mathcal{H}(x)dx.\end{aligned}$$

Therefore, we obtain

$$\frac{1}{\Gamma(\delta)}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}\mathfrak{u}'(x)\mathcal{P}(x)\mathfrak{I}(x,y)dx$$

$$= \left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{P}\mathcal{G}\mathcal{H}(\xi)\right) + \left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{P}(\xi)\right)\mathcal{G}(y)\mathcal{H}(y) - \left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{P}\mathcal{G}(\xi)\right)\mathcal{H}(y) - \left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{P}\mathcal{H}(\xi)\right)\mathcal{G}(y). \tag{3.5}$$

Again, multiplying both sides of (3.5) by $\frac{\left[\mathrm{u}(\xi)-\mathrm{u}(y)\right]^{\delta-1}\mathrm{u}'(y)\mathcal{P}(y)}{\Gamma(\delta)}$ $(y \in (\eta_1, \xi))$ and then integrating the obtained result with respect to $y$ from $(\eta_1, \xi)$ gives

$$\frac{1}{(\Gamma(\delta))^2}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}\left[\mathrm{u}(\xi)-\mathrm{u}(x)\right]^{\delta-1}\left[\mathrm{u}(\xi)-\mathrm{u}(y)\right]^{\delta-1}\mathrm{u}'(x)\mathcal{P}(x)\mathrm{u}'(y)\mathcal{P}(y)\mathfrak{I}(x,y)dxdy$$
$$= 2\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{P}(\xi)\right)\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{P}\mathcal{G}\mathcal{H}(\xi)\right) - 2\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{P}\mathcal{H}(\xi)\right)\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{P}\mathcal{G}(\xi)\right). \tag{3.6}$$

Substituting $\mathcal{P}(\xi) = \mathcal{F}(\xi)$ and $\mathcal{G}(\xi) = \mathcal{H}(\xi) = \mathrm{u}(\xi) - E(Y)$ $(\xi \in (\eta_1, \eta_2))$, we have

$$\frac{1}{(\Gamma(\delta))^2}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}\left[\mathrm{u}(\xi)-\mathrm{u}(x)\right]^{\delta-1}\left[\mathrm{u}(\xi)-\mathrm{u}(y)\right]^{\delta-1}\mathrm{u}'(x)\mathcal{P}(x)\mathrm{u}'(y)\mathcal{P}(y)\mathcal{F}(x)\mathcal{F}(y)(\mathrm{u}(x)-\mathrm{u}(y))^2dxdy$$
$$= 2\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{F}(\xi)\right)\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{F}(\xi)(\mathrm{u}(\xi)-E(Y))^2\right) - 2\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{F}(\xi)(\mathrm{u}(\xi)-E(Y))^2\right). \tag{3.7}$$

Similarly, we have

$$\frac{1}{(\Gamma(\delta))^2}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}\left[\mathrm{u}(\xi)-\mathrm{u}(x)\right]^{\delta-1}\left[\mathrm{u}(\xi)-\mathrm{u}(y)\right]^{\delta-1}\mathrm{u}'(x)\mathcal{P}(x)\mathrm{u}'(y)\mathcal{P}(y)\mathcal{F}(x)\mathcal{F}(y)(\mathrm{u}(x)-\mathrm{u}(y))^2dxdy$$
$$\leq \|\mathcal{F}\|_\infty^2\frac{1}{(\Gamma(\delta))^2}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}\left[\mathrm{u}(\xi)-\mathrm{u}(x)\right]^{\delta-1}\left[\mathrm{u}(\xi)-\mathrm{u}(y)\right]^{\delta-1}\mathrm{u}'(x)\mathcal{P}(x)\mathrm{u}'(y)\mathcal{P}(y)(\mathrm{u}(x)-\mathrm{u}(y))^2dxdy$$
$$\leq \|\mathcal{F}\|_\infty^2\left[2\frac{(\mathrm{u}(\xi)-\mathrm{u}(\eta_1))^\delta}{\Gamma(\delta+1)}\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}[\xi^2] - 2(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}[\xi])^2\right]. \tag{3.8}$$

From (3.7) and (3.8) we get the first inequality of Theorem 3.2.

Next, we prove the the second part of Theorem 3.2. Note that

$$\frac{1}{(\Gamma(\delta))^2}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}\left[\mathrm{u}(\xi)-\mathrm{u}(x)\right]^{\delta-1}\left[\mathrm{u}(\xi)-\mathrm{u}(y)\right]^{\delta-1}\mathrm{u}'(x)\mathcal{P}(x)\mathrm{u}'(y)\mathcal{P}(y)\mathcal{F}(x)\mathcal{F}(y)(\mathrm{u}(x)-\mathrm{u}(y))^2dxdy$$
$$\leq \sup_{x,y\in[\eta_1,\xi]}\left|(\mathrm{u}(x)-\mathrm{u}(y))\right|^2(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{F}(\xi))^2$$
$$= (\mathrm{u}(\xi)-\mathrm{u}(\eta_1))^2(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\delta}\mathcal{F}(\xi))^2. \tag{3.9}$$

From (3.7) and (3.9) we derive the inequality (3.3). □

Theorem 3.2 leads to Corollary 3.3 immediately.

**Corollary 3.3.** *Let $Y$ be a continuous random variable with probability density function $\mathcal{F} : [\eta_1, \eta_2] \to \mathbb{R}^+$. Then one has the following two conclusion.*

*(1) For all $\delta \geq 0$ and $\eta_1 < \xi \leq \eta_2$, we have the inequlaity*

$$\frac{\left(\mathfrak{u}(\eta_2)-\mathfrak{u}(\eta_1)\right)^{\delta-1}}{\Gamma(\delta)}\sigma^2_{Y,\delta}-E^2_{Y,\delta} \leq \|\mathcal{F}\|^2_\infty\left[\frac{2\left(\mathfrak{u}(\eta_2)-\mathfrak{u}(\eta_1)\right)^{2\delta+2}}{\Gamma(\delta+1)\Gamma(\delta+3)}-\left(\frac{\left(\mathfrak{u}(\eta_2)-\mathfrak{u}(\eta_1)\right)^{\delta+1}}{\Gamma(\delta+2)}\right)^2\right].$$

*(2) The inequality*

$$\frac{\left(\mathfrak{u}(\eta_2)-\mathfrak{u}(\eta_1)\right)^{\delta-1}}{\Gamma(\delta)}\sigma^2_{Y,\delta}-\left(E_{Y-E(Y),\delta}(\xi)\right)^2 \leq \frac{1}{2}\frac{\left(\mathfrak{u}(\eta_2)-\mathfrak{u}(\eta_1)\right)^{2\delta}}{\Gamma^2(\delta)},$$

*holds for all $\delta \geq 0$ and $\eta_1 < \xi \leq \eta_2$.*

**Remark 3.1.** *We clearly see that*

*(1) If we choose $\Psi(\varsigma) = \varsigma$, then Theorem 3.2 reduces to Theorem 3.1 of [56].*

*(2) If we choose $\Psi(\varsigma) = \varsigma$, then Corollary 3.3 becomes Corollary 3.1 of [56].*

*(3) If we choose $\delta = 1$ and $\Psi(\varsigma) = \varsigma$ in part (1) of Corollary 3.3, then we get the first part of Theorem 1 in [48].*

*(4) If we choose $\delta = 1$ and $\Psi(\varsigma) = \varsigma$ in part (2) of Corollary 3.3, then we get the last part of Theorem 1 in [48].*

Next we provide more general form of Theorem 3.2 by proposing two fractional parameters.

**Theorem 3.4.** *Let $Y$ be a continuous random variable with probability density function $\mathcal{F} : [\eta_1, \eta_2] \to \mathbb{R}^+$. Then the following statements are true:*

*(1) For all $\delta, \gamma \geq 0$ and $\eta_1 < \xi \leq \eta_2$, one has*

$$\begin{aligned}&\left(\mathfrak{J}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{F}(\xi)\right)\sigma^2_{Y,\beta}+\left(\mathfrak{J}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{F}(\xi)\right)\sigma^2_{Y,\alpha}-\left(E_{Y-E(Y),\delta}(\xi)\right)\left(E_{Y-E(Y),\gamma}(\xi)\right)\\ &\leq \|\mathcal{F}\|^2_\infty\left[\frac{\left(\mathfrak{u}(\xi)-\mathfrak{u}(\eta_1)\right)^{\delta}}{\Gamma(\delta+1)}\left(\mathfrak{J}^{\mathfrak{u},\delta}_{\eta_1^+}\xi^2\right)+\frac{\left(\mathfrak{u}(\xi)-\mathfrak{u}(\eta_1)\right)^{\gamma}}{\Gamma(\gamma+1)}\left(\mathfrak{J}^{\mathfrak{u},\gamma}_{\eta_1^+}\xi^2\right)-\left(\mathfrak{J}^{\mathfrak{u},\delta}_{\eta_1^+}\xi\right)\left(\mathfrak{J}^{\mathfrak{u},\gamma}_{\eta_1^+}\xi\right)\right],\end{aligned} \tag{3.10}$$

*where $\mathcal{F} \in L_\infty([\delta, \gamma])$.*

*(2) The inequality*

$$\begin{aligned}&\left(\mathfrak{J}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{F}(\xi)\right)\sigma^2_{Y,\beta}+\left(\mathfrak{J}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{F}(\xi)\right)\sigma^2_{Y,\alpha}-\left(E_{Y-E(Y),\delta}(\xi)\right)\left(E_{Y-E(Y),\gamma}(\xi)\right)\\ &\leq \left(\mathfrak{u}(\xi)-\mathfrak{u}(\eta_1)\right)^2\left(\mathfrak{J}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{F}(\xi)\right)\left(\mathfrak{J}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{F}(\xi)\right)\end{aligned} \tag{3.11}$$

*holds for any $\eta_1 < \xi \leq \eta_2$, $\delta \geq 0$ and $\gamma \geq 0$.*

*Proof.* Taking product on both sides of (3.5) by $\frac{[\mathfrak{u}(\xi)-\mathfrak{u}(y)]^{\gamma-1}\mathfrak{u}'(y)\mathcal{P}(y)}{\Gamma(\gamma)}$, we obtain

$$\frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\gamma)}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}[\mathfrak{u}(\xi)-\mathfrak{u}(y)]^{\gamma-1}\mathfrak{u}'(y)\mathcal{P}(y)\mathfrak{u}'(x)\mathcal{P}(x)\mathfrak{J}(x,y)dxdy$$

$$= \left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{P}(\xi)\right)\left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{P}\mathcal{G}\mathcal{H}(\xi)\right) + \left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{P}(\xi)\right)\left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{P}\mathcal{G}\mathcal{H}(\xi)\right)$$
$$- \left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{P}\mathcal{G}(\xi)\right)\left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{P}\mathcal{H}(\xi)\right) - \left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{P}\mathcal{H}(\xi)\right)\left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{P}\mathcal{G}(\xi)\right). \tag{3.12}$$

Taking $\mathcal{P}(\xi) = \mathcal{F}(\xi)$ and $\mathcal{G}(\xi) = \mathcal{H}(\xi) = \mathfrak{u}(\xi) - E(Y)$ $(\xi \in (\eta_1, \eta_2))$ in (3.12), we get

$$\frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\gamma)}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}[\mathfrak{u}(\xi)-\mathfrak{u}(y)]^{\gamma-1}\mathfrak{u}'(y)\mathfrak{u}'(x)\big(\mathfrak{u}(x)-\mathfrak{u}(y)\big)^2\mathcal{F}(x)\mathcal{F}(y)dxdy$$
$$= \left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{F}(\xi)\right)\left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{F}(\xi)[\mathfrak{u}(\xi)-E(Y)]^2\right) + \left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{F}(\xi)\right)\left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{F}(\xi)[\mathfrak{u}(\xi)-E(Y)]^2\right)$$
$$- 2\left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{F}(\xi)[\mathfrak{u}(\xi)-E(Y)]\right)\left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{F}(\xi)[\mathfrak{u}(\xi)-E(Y)]\right). \tag{3.13}$$

Moreover, we have

$$\frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\gamma)}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}[\mathfrak{u}(\xi)-\mathfrak{u}(y)]^{\gamma-1}\mathfrak{u}'(y)\mathfrak{u}'(x)\big(\mathfrak{u}(x)-\mathfrak{u}(y)\big)^2\mathcal{F}(x)\mathcal{F}(y)dxdy$$
$$\leq \|\mathcal{F}\|_\infty^2\frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\gamma)}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}[\mathfrak{u}(\xi)-\mathfrak{u}(y)]^{\gamma-1}\mathfrak{u}'(y)\mathfrak{u}'(x)\big(\mathfrak{u}(x)-\mathfrak{u}(y)\big)^2dxdy$$
$$\leq \|\mathcal{F}\|_\infty^2\left[\frac{\big(\mathfrak{u}(\xi)-\mathfrak{u}(\eta_1)\big)^\delta}{\Gamma(\delta+1)}\left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\xi^2\right) + \frac{\big(\mathfrak{u}(\xi)-\mathfrak{u}(\eta_1)\big)^\gamma}{\Gamma(\gamma+1)}\left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\xi^2\right) - 2\left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\xi\right)\left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\xi\right)\right]. \tag{3.14}$$

Therefore, (3.10) follows from (3.12) and (3.14).

For inequality (3.11), using (3.12) and the fact that $\sup_{x,y\in[\eta_1,\xi]}|\mathfrak{u}(x)-\mathfrak{u}(y)|^2 = \big(\mathfrak{u}(\xi)-\mathfrak{u}(\eta_1)\big)^2$, we have

$$\frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\gamma)}\int_{\eta_1}^{\xi}\int_{\eta_1}^{\xi}[\mathfrak{u}(\xi)-\mathfrak{u}(x)]^{\delta-1}[\mathfrak{u}(\xi)-\mathfrak{u}(y)]^{\gamma-1}\mathfrak{u}'(y)\mathfrak{u}'(x)\big(\mathfrak{u}(x)-\mathfrak{u}(y)\big)^2\mathcal{F}(x)\mathcal{F}(y)dxdy$$
$$\leq \big(\mathfrak{u}(\xi)-\mathfrak{u}(\eta_1)\big)^2\left(\mathfrak{I}^{\mathfrak{u},\delta}_{\eta_1^+}\mathcal{F}(\xi)\right)\left(\mathfrak{I}^{\mathfrak{u},\gamma}_{\eta_1^+}\mathcal{F}(\xi)\right), \tag{3.15}$$

which gives the desired inequality (3.11). □

**Remark 3.2.** *We clearly see that*

*(1) If we choose $\delta = \gamma$, then Theorem 3.4 reduces to Theorem 3.2.*

*(2) If we choose $\mathfrak{u}(\varsigma) = \varsigma$, then Theorem 3.4 reduces to Theorem 3.2 of [56].*

*(3) If we choose $\mathfrak{u}(\varsigma) = \varsigma$ and $\delta = \gamma = 1$, then Theorem 3.4 becomes the first inequality given in [48].*

*(4) If we choose $\mathfrak{u}(\varsigma) = \varsigma$ and $\delta = \gamma = 1$, then Theorem 3.4 reduces to the first inequality of Theorem 1 given in [48].*

*(5) If we choose $\mathfrak{u}(\varsigma) = \varsigma$ and $\delta = \gamma = 1$, then Theorem 3.4 becomes the last part of Theorem 1 in [48].*

**Theorem 3.5.** *Let $Y$ be a continuous random variable with probability density function $\mathcal{F} : [\eta_1, \eta_2] \to \mathbb{R}^+$. Then*

$$\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{F}(\xi)\right)\sigma^2_{Y,\delta}(\xi) - \left(E_{Y-E(Y),\delta}(\xi)\right)^2 \leq \frac{1}{4}\left(\mathrm{u}(\eta_2) - \mathrm{u}(\eta_1)\right)^2\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\xi\right)^2. \tag{3.16}$$

*Proof.* It follows from Theorem 1 of [57] that

$$\left|\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{P}(\xi)\right)\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{P}\mathcal{G}^2(\xi)\right) - \left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{P}\mathcal{G}(\xi)\right)^2\right| \leq \frac{1}{4}\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{P}(\xi)\right)^2(\Upsilon - \gamma)^2. \tag{3.17}$$

Substituting $\mathcal{P}(\xi) = \mathcal{F}(\xi)$ and $\mathcal{G}(\xi) = \mathrm{u}(\xi) - E(Y)$ ($\xi \in (\eta_1, \eta_2)$), then $\Upsilon = \mathrm{u}(\eta_2) - E(Y)$, $\gamma = \mathrm{u}(\eta_1) - E(Y)$ and (3.18) can be rewritten as

$$\begin{aligned} 0 &\leq \left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{F}(\xi)\right)\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{F}(\xi)\left(\mathrm{u}(\xi) - E(Y)\right)^2\right) - \left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{F}(\xi)(\mathrm{u}(\xi) - E(Y))\right)^2 \\ &\leq \frac{1}{4}\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{F}(\xi)\right)^2(\mathrm{u}(\eta_2) - \mathrm{u}(\eta_1))^2. \end{aligned} \tag{3.18}$$

Therefore, we get

$$\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\mathcal{F}(\xi)\right)\sigma^2_{Y,\delta}(\xi) - \left(E_{Y-E(Y),\delta}(\xi)\right)^2 \leq \frac{1}{4}\left(\mathrm{u}(\eta_2) - \mathrm{u}(\eta_1)\right)^2\left(\mathfrak{I}_{\eta_1^+}^{\mathrm{u},\gamma}\xi\right)^2,$$

which is the required result. □

Let $\xi = \eta_2$. Then Theorem 3.5 leads to Corollary 3.6.

**Corollary 3.6.** *Let $Y$ be a continuous random variable with probability density function $\mathcal{F} : [\eta_1, \eta_2] \to \mathbb{R}^+$. Then*

$$\frac{\left(\mathrm{u}(\eta_2) - \mathrm{u}(\eta_1)\right)^{\delta-1}}{\Gamma(\delta)}\sigma^2_{Y,\delta}(\xi) - \left(E_{Y-E(Y),\delta}(\xi)\right)^2 \leq \frac{1}{4}\frac{\left(\mathrm{u}(\eta_2) - \mathrm{u}(\eta_1)\right)^{2\alpha}}{\Gamma^2(\delta)}.$$

**Remark 3.3.** *We clearly see that*

*(1) If we choose $\mathrm{u}(\varsigma) = \varsigma$, then Theorem 3.5 reduces to Theorem 3.3 of [56].*

*(2) If we choose $\mathrm{u}(\varsigma) = \varsigma$, then Corollary 3.6 reduces to Corollary 3.2 of [56].*

*(3) If we choose $\mathrm{u}(\varsigma) = \varsigma$ and $\delta = 1$, then Corollary 3.6 becomes Theorem 2 of [48].*

## 4. Conclusions

In the aritcle, we have derived numerous new inequalities in the frame of generalized Riemann-Liouville fractional integral operators via a continuous random variable, our obtained results are the generalizations and refinements of the known results given in [47] and [56]. In the special case of $\delta = 1$, it is worth mentioning that our results can recapture many previously existing operators. Adopting our ideas and approach, researchers can also generate several variants by use of the Hadamard and conformable fractional integral operators and obtain many new inequalities for the probability density functions using different parameters and random variables.

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## Conflict of interest

The authors declare that they have no competing interests.

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