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Research article

Equilibrium investment and risk control for an insurer with non-Markovian regime-switching and no-shorting constraints

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Abstract: In this paper, we study the time-consistent equilibrium investment and risk control policies for an insurer under the mean-variance criterion. The dynamics of liability and assets are given by non-Markovian regime-switching models driven by a Brownian motion and a continuous time finite-state Markov chain. It is assumed that the insurer has a time-varying preference of risk depending dynamically on current wealth, and is not allowed to short sell the risky assets. With the aid of backward stochastic differential equations and bounded mean oscillation martingales, we obtain feedback representations of the open-loop equilibrium strategies. Finally, the result is also applied to solve an example of the Markovian regime-switching model.

Keywords: mean-variance criterion; non-Markovian regime-switching; open-loop equilibrium strategy; bounded mean oscillation martingale; backward stochastic differential equation
Mathematics Subject Classification: 62P05, 93E20

1. Introduction

Since the pioneer work of [11], the mean-variance portfolio selection has become one of the foundations of modern finance theory and has been further extended in various directions. It is well known that due to the presence of a quadratic term of the expected terminal state, the dynamic mean-variance problem is time-inconsistent in the sense that the Bellman optimality principle does not hold. One way to get around the time-inconsistency issue is to seek equilibrium, instead of optimal, controls. [1] firstly consider a fairly general class of time-inconsistent objective functionals under a controlled Markovian diffusion process. Within the class of feedback controls, the authors derive an extension of the standard HJB equation, and apply the theory to a dynamic mean-variance problem. Since then, there are many extensions along this line of research. Examples include, but are not limited to, the case with state-dependent risk aversion ([2]), the situation of regime-switching economy ([20]), the insurer's investment and reinsurance problem ([22, 24]), among others.

Open-loop equilibrium controls represent another important classes of time-consistent controls for time-inconsistent problems. The definition of an open-loop equilibrium control is given by [7], where the authors consider a time-inconsistent stochastic linear quadratic (LQ) control problem. They derive a sufficient condition by forward backward stochastic differential equations (FBSDEs) and then present two special cases including a mean-variance portfolio selection problem. [9] further prove that the open-loop equilibrium strategy obtained in their previous work is unique. [14] later study a similar mean-variance problem in a jump-diffusion model. [21] derive the open-loop equilibrium strategy for the mean-variance asset-liability management problem with random parameters.

In recent years, the class of regime-switching models has become popular in economics, finance and actuarial science. The basic idea of regime-switching is to modulate the model parameters with an exogenous finite-state Markov chain, where each state represents a regime of the system or level of the economic indicator. There is a large literature on mean-variance problems with regime-switching. Some earlier works are, for example, [3, 4, 25], amongst others. The regime-switching models in the aforementioned literature on mean-variance problems all assume that the model coefficients only depend on the current state of the Markov chain, for which they are of the same kind of Markovian regime-switching models. However, the adoption of these models sometimes may be questionable due to their failure of capturing the memory effect or path-dependent structure in the economy. In this regard, non-Markovian regime-switching models, in which the model coefficients may depend on the historical paths of asset prices as well as the historical paths of the Markov chain, cater for the increasingly complicated market environment. The difficulty for solving non-Markovian control problems is that the HJB dynamic programming approach cannot be applied. Over the last decade or so, backward stochastic differential equations (BSDEs) turn out to be an effective tool in solving stochastic control problems under non-Markovian frameworks. For examples of applications, see, [13, 15–17] and references therein. Under non-Markovian regime-switching models, [18] and [19] study respectively the pre-committed and equilibrium strategies for the mean-variance problem. [23] study the proportional reinsurance/new business and investment problem when strategies are constrained in the non-negative cone. In the above work, the model coefficients are assumed to be adapted to the filtration generated by the Markov chain. [12] consider a mean-variance asset-liability management problem under a more general non-Markovian regime-switching model, in which the model coefficients are adapted to the filtration generated jointly by the Brownian motion and the Markov chain.

In this paper, we study the open-loop equilibrium investment and risk control policies with state-dependent risk aversion under short-selling prohibition. We assume that an insurer can invest in a financial market and sell insurance polices. The dynamics of both insurer's liability and risky asset prices are given by non-Markovian regime-switching models in the sense that the model parameters are stochastic processes depending on the historical paths of a Brownian motion and a Markov chain. This model not only possesses some important features such as memory effect, path-dependent structure, and non-Markovian property, but also incorporates the classical Markovian regime-switching model as a special case. We first derive a sufficient condition for the equilibrium strategies through a system of FBSDEs. Then we present explicit solutions to equilibriums in terms of the unique solution to a relevant regime-switching BSDE. Finally, we apply our results to an example of the Markovian regime-switching model, under which the BSDE reduces to a system of backward ordinary differential equations (ODEs).

Compared with [12, 18, 19], the distinctive feature of this paper is that short-selling is prohibited. As a result, a major difficulty in the present case is that the controls (portfolio and total number of insurance policies) are constrained. Another recent paper of [23] employ the BSDE approach to study the mean-variance reinsurance and investment problem with non-negative constraints on the strategies. The current paper differentiates from [23] at least in two aspects. Firstly, the underlying models are different. The model coefficients in [23] depend on the historical information of the Markov chain, while those in this paper may depend on the historical information of both the Brownian motion and the Markov chain. This makes the mean-variance problem in the current paper more challenging. Secondly, the induced BSDE is different. The BSDE considered in [23] is a linear BSDE driven by the Markov chain and its solvability can be directly derived by existing results. However, the associated BSDE in the current paper is a quadratic-exponential BSDE driven by both the Brownian motion and the Markov chain, which is much more complicated than that in [23].

The rest of this paper is organized as follows. Section 2 introduces the basic notation and reviews some facts of bounded mean oscillation martingales (in short, BMO-martingales). In Section 3, we set up the model dynamics and formulate the mean-variance problem. In Section 4, the open-loop equilibrium strategy is derived relying on the solvability of a regime-switching BSDE. An example of the Markovian regime-switching model is considered in Section 5. Finally, the last section concludes the paper.

2. Preliminaries

Let $T > 0$ be a fixed finite time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space. Suppose that $W := \{W(t)\}_{t\in[0,T]} = \{(W_0(t), W_1(t), \dots, W_n(t))'\}_{t\in[0,T]}$ is an $(n+1)$-dimensional standard Brownian motion, and $\alpha := \{\alpha(t)\}_{t\in[0,T]}$ is a continuous time, finite-state (m-state), homogeneous Markov chain on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Here, the filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ is the augmentation under $\mathbb{P}$ of the natural filtration generated by W and α. We further assume that the Brownian motion W and the Markov chain α are stochastically independent under $\mathbb{P}$.

Throughout the paper, we identify the state space of the Markov chain with $S := \{e_1, e_2 \dots, e_m\}$, where $m \in \mathbb{N}$, $e_i \in \mathbb{R}^m$ and for $i, j = 1, \dots, m$, the j-th component of e_i is the Kronecker delta δ_{ij}. Here we denote by $\mathbb{N}$ the set of natural numbers and $\mathbb{R}$ the set of real numbers. The Markov chain α describes the evolution of the states of an economy over time attributed to structural changes in the economic conditions. In particular, "e_i" represents the i-th state of the economy, for each $i = 1, \dots, m$. Suppose that the generator matrix of the chain under $\mathbb{P}$ is denoted by $Q := [q_{ij}]_{i,j=1,\dots,m}$, where $q_{ij} \geq 0$ is the instantaneous intensity of a transition from state e_i to state e_j, for $i \neq j$. Now, for each fixed $j = 1, \dots, m$, let $\Phi_j(t)$ be the number of jumps into state e_j from other states up to time t and set $\lambda_j(t) := \sum_{i=1, i\neq j}^m q_{ij} \langle \alpha(t-), e_i \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. Then following from [5], we have that

$$\widetilde{\Phi}_j(t) := \Phi_j(t) - \int_0^t \lambda_j(s) ds, \quad j = 1, \dots, m, \tag{2.1}$$

is a set of jump martingales related to the Markov chain α under $\mathbb{P}$. To simplify our notation, we denote the vector of counting processes $\{\Phi(t)\}_{t\in[0,T]}$, intensity processes $\{\lambda(t)\}_{t\in[0,T]}$ and compensated counting processes $\{\widetilde{\Phi}(t)\}_{t\in[0,T]}$ by $\Phi(t) := (\Phi_1(t),\ldots,\Phi_m(t))'$, $\lambda(t) := (\lambda_1(t),\ldots,\lambda_m(t))'$ and $\widetilde{\Phi}(t) := (\widetilde{\Phi}_1(t),\ldots,\widetilde{\Phi}_m(t))'$, respectively.

Let $k \geq 1$. We now introduce the following spaces of stochastic processes.

- $\mathbb{L}^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^k)$: the set of $\mathcal{F}_t$-measurable, $\mathbb{R}^k$-valued random variables ξ such that $\mathbb{E}[|\xi|^2] < \infty$;
- $\mathbb{L}^\infty_{\mathbb{F}}(t,T;\mathbb{R}^k)$: the set of $\mathbb{F}$-adapted, uniformly bounded, $\mathbb{R}^k$-valued càdlàg processes on $[t,T]$;
- $\mathbb{L}^2_{\mathbb{F}}(\Omega;C(t,T;\mathbb{R}^k))$: the set of $\mathbb{F}$-adapted, $\mathbb{R}^k$-valued continuous processes $\{f(s)\}_{s\in[t,T]}$ such that
$$\mathbb{E}\Big[\sup_{s\in[t,T]}|f(s)|^2\Big] < \infty;$$
- $\mathbb{L}^2_{\mathbb{F}}(\Omega;D(t,T;\mathbb{R}^k))$: the set of $\mathbb{F}$-adapted, $\mathbb{R}^k$-valued càdlàg processes $\{f(s)\}_{s\in[t,T]}$ such that
$$\mathbb{E}\Big[\sup_{s\in[t,T]}|f(s)|^2\Big] < \infty;$$
- $\mathbb{H}^2_{\mathbb{F}}(t,T;\mathbb{R}^k)$: the set of $\mathbb{F}$-predictable, $\mathbb{R}^k$-valued processes $\{f(s)\}_{s\in[t,T]}$ such that
$$\mathbb{E}\Big[\int_t^T |f(s)|^2 ds\Big] < \infty;$$
- $\mathbb{J}^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$: the set of $\mathbb{F}$-predictable, $\mathbb{R}^m$-valued processes $\{f(s)\}_{s\in[t,T]}$ such that
$$\mathbb{E}\Big[\sum_{j=1}^m \int_t^T |f_j(s)|^2\lambda_j(s) ds\Big] < \infty.$$

For later use, we also introduce the notion of *BMO*-martingales and, for further details, one may consult [10].

Definition 2.1. *Let $\mathcal{T}$ denote the set of all $\mathbb{F}$-stopping times on $[0,T]$. Let M be a square integrable $(\mathbb{P},\mathbb{F})$-martingale. When it satisfies*
$$||M||_{BMO(\mathbb{P})} := \sup_{\tau\in\mathcal{T}}\Big\|\{\mathbb{E}[|M(T)-M(\tau-)|^2|\mathcal{F}_\tau]\}^{\frac{1}{2}}\Big\|_\infty < \infty,$$
then M is called a BMO-martingale on $(\mathbb{P},\mathbb{F})$ and denoted by $M \in BMO(\mathbb{P})$.

Following from Definition 2.1, it is easy to check that a local martingale with the form $Z'\cdot W := \int_0^\cdot Z(s)'dW(s)$ is a *BMO*-martingale if and only if
$$||Z'\cdot W||_{BMO(\mathbb{P})} = \sup_{\tau\in\mathcal{T}}\Big\|\Big\{\mathbb{E}\Big[\int_\tau^T |Z(s)|^2 ds\Big|\mathcal{F}_\tau\Big]\Big\}^{\frac{1}{2}}\Big\|_\infty < \infty.$$
Similarly, the process $U'\cdot\widetilde{\Phi} := \int_0^\cdot U(s)'d\widetilde{\Phi}(s)$ is a *BMO*-martingale if and only if
$$||U'\cdot\widetilde{\Phi}||_{BMO(\mathbb{P})} = \sup_{\tau\in\mathcal{T}}\Big\|\Big\{\mathbb{E}\Big[\sum_{j=1}^m\int_\tau^T |U_j(s)|^2 d\Phi_j(s)\Big|\mathcal{F}_\tau\Big]\Big\}^{\frac{1}{2}}\Big\|_\infty < \infty.$$

Let us introduce the following spaces.

- $\mathbb{H}^2_{BMO(\mathbb{P})}(0,T;\mathbb{R}^{n+1})$: the set of $\mathbb{F}$-predictable, $\mathbb{R}^{n+1}$-valued processes $\{Z(s)\}_{s\in[0,T]}$ such that $Z'\cdot W \in BMO(\mathbb{P})$;
- $\mathbb{J}^2_{BMO(\mathbb{P})}(0,T;\mathbb{R}^m)$: the set of $\mathbb{F}$-predictable, $\mathbb{R}^m$-valued processes $\{U(s)\}_{s\in[0,T]}$ such that $U'\cdot\widetilde{\Phi} \in BMO(\mathbb{P})$.

3. Problem formulation

In this section, we specify the model dynamics and formulate the mean-variance problem.

3.1. Risk control process

For an insurer, its main risk (liabilities) comes from writing insurance policies, and we denote by $q(t) \geq 0$ the $\mathbb{F}$-predictable total outstanding number of policies (liabilities) at time t. In the actuarial industry, a commonly used risk model for claims is the compound Poisson model. [6] proposed to approximate the claim process $\{C(t)\}_{t\in[0,T]}$ by a diffusion type model as follows:

$$dC(t) = adt - bdW_0(t), \tag{3.1}$$

where both a and b are positive constants.

Now we extend the diffusion approximated model (3.1) to one with time varying and random coefficients. More precisely, we assume that the claim process $\{C(t)\}_{t\in[0,T]}$ is described as as an extensive diffusion type model:

$$dC(t) = a(t)dt - b(t)dW_0(t), \tag{3.2}$$

where $a(\cdot)$ and $b(\cdot)$ are $\mathbb{F}$-predictable, positive and uniformly bounded processes on $[0, T]$. Furthermore, we assume that the average premium per policy is $p(\cdot)$ which is also $\mathbb{F}$-predictable, positive and uniformly bounded. As usual, the net profit condition, i.e., $p(t) > a(t)$ for $t \in [0, T]$, is imposed. Then, by [26], the total surplus $\{R^q(t)\}_{t\in[0,T]}$ of the insurer evolves according to

$$\begin{aligned} dR^q(t) &= q(t)(p(t)dt - dC(t)) \\ &= q(t)(p(t) - a(t))dt + q(t)b(t)dW_0(t). \end{aligned} \tag{3.3}$$

3.2. Market model

Suppose that the financial market consists of $k + 1$ tradable assets including one risk-free asset and k risky assets. Let $r(t) > 0$ be the instantaneous risk-free interest rate at time t. Then the price process $\{S_0(t)\}_{t\in[0,T]}$ of the risk-free asset follows the following equation:

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1. \tag{3.4}$$

The dynamics $\{S_i(t)\}_{t\in[0,T]}$, $i = 1, \ldots, k$, of the other k risky assets evolve over time as

$$dS_i(t) = S_i(t)\Big[\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t)\Big], \quad S_i(0) = s_i > 0, \tag{3.5}$$

where $\mu_i(t) > r(t)$ and $\sigma_{ij}(t) > 0$ are the appreciation rate and volatility coefficients of the i-th risky asset at time t, respectively. We assume throughout that the following assumptions on the coefficients of the asset prices are satisfied: the interest rate $r(\cdot)$ is a deterministic and uniformly bounded function; the appreciation rate $\mu_i(\cdot)$ and the volatility coefficients $\sigma_{ij}(\cdot)$, for $i = 1, \ldots k$ and $j = 1, \ldots, n$, are $\mathbb{F}$-predictable and uniformly bounded processes on $[0, T]$, and that there exists a positive constant ε

such that $\sigma(t)\sigma(t)' \geq \varepsilon I_k$, for all $t \in [0, T]$, where $\sigma(t) = (\sigma_{ij}(t))_{k\times n}$ and I_k is the $(k \times k)$ identity matrix. It is worth mentioning that all of the model coefficients except the risk-free interest rate are $\mathbb{F}$-predictable processes, which may depend on the historical paths of the Brownian motion and the Markov chain. Such a path-dependent structure is more general than the Markovian regime-switching model considered in the literature, where the model parameters rely only on the current state of the Markov chain.

Remark 3.1. *We emphasize here the risk-free interest rate $r(\cdot)$ is assumed to be a deterministic function. Such an assumption is imposed for the sake of tractability. If $r(\cdot)$ were random, then it would not be taken into the conditional expectation in* (4.10)*. This makes solvability of the BSDE* (4.3) *very difficult. Actually, this key assumption has been applied by many authors including [7–9] and [20] to solve the mean-variance problem analytically.*

3.3. The mean-variance problem

Let $\pi(\cdot) := (\pi_1(\cdot), \ldots, \pi_k(\cdot))'$ be the vector of dollar amounts invested in the risky assets. An important restriction considered here is the prohibition of short-selling the risk assets, i.e., it must hold that $\pi_i(s) \geq 0$, for $i = 1, \ldots, k$ and $s \in [0, T]$. Let $X(\cdot)$ be the wealth process of the insurer corresponding to a strategy $u(\cdot) := (q(\cdot), \pi(\cdot)')'$, then for any initial wealth $X(0) = x_0 > 0$, the wealth process is governed by the following SDE:

$$\begin{aligned} dX(s) &= \frac{X(s) - \sum_{i=1}^{k} \pi_i(s)}{S_0(s)} dS_0(s) + \sum_{i=1}^{k} \frac{\pi_i(s)}{S_i(s)} dS_i(s) + dR^q(s) \\ &= \Big\{ r(s)X(s) + (p(s) - a(s))q(s) + \sum_{i=1}^{k} (\mu_i(s) - r(s))\pi_i(s) \Big\} ds \\ &\quad + b(s)q(s)dW_0(s) + \sum_{i=1}^{k} \sum_{j=1}^{n} \sigma_{ij}(s)\pi_i(s)dW_j(s) \\ &= \{ r(s)X(s) + u(s)'B(s) \} ds + u(s)'\Sigma(s)dW(s), \end{aligned} \tag{3.6}$$

where $B(s) := (p(s) - a(s), \mu_1(s) - r(s), \ldots, \mu_k(s) - r(s))'$ and

$$\Sigma(s) := \begin{pmatrix} b(s) & 0 \\ 0 & \sigma(s) \end{pmatrix}.$$

As time evolves, we need to consider the wealth process starting from time $t \in [0, T]$ and state $x(t) \in \mathbb{L}^2_{\mathcal{F}_t}(\Omega; \mathbb{R})$:

$$\begin{cases} dX(s) = \{ r(s)X(s) + u(s)'B(s) \} ds + u(s)'\Sigma(s)dW(s), \\ X(t) = x(t), \quad s \in [t, T]. \end{cases} \tag{3.7}$$

Let $\mathbb{R}_+ = [0, +\infty)$. For any strategy $u(\cdot) \in \mathbb{H}^2_{\mathbb{F}}(t, T; \mathbb{R}^{k+1}_+)$, the wealth equation (3.7) admits a unique solution $X^{t,x(t),u}(\cdot) \in \mathbb{L}^2_{\mathbb{F}}(\Omega; C(t, T; \mathbb{R}))$.

At any time t with state $X(t) = x(t)$, the insurer aims to minimize the mean-variance cost functional

$$J(t, x(t); u(\cdot)) := \frac{1}{2} \mathrm{Var}_t[X(T)] - \gamma x(t) \mathbb{E}_t[X(T)]$$

$$=\frac{1}{2}(\mathbb{E}_t[X^2(T)] - (\mathbb{E}_t[X(T)])^2) - \gamma x(t)\mathbb{E}_t[X(T)] \tag{3.8}$$

over $u(\cdot) \in \mathbb{H}^2_{\mathbb{F}}(t, T; \mathbb{R}^{k+1}_+)$. Here γ is a positive constant, $X(\cdot) = X^{t,x(t),u}(\cdot)$ and $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$. It is well known that the problem of minimizing (3.8) is time-inconsistent. In this paper, we look for the time-consistent equilibrium investment and risk control policies for the insurer. It is worthwhile to point out that the state-dependent risk-aversion $\gamma x(t)$ in (3.8) indicates that the insurer has a time-varying preference of risk according to his/her wealth levels. We will see later that $\gamma x(t)$ is always positive under the equilibrium strategy.

4. Open-loop equilibrium strategy

The problem of minimizing (3.8) formulated in the previous section is a time-inconsistent stochastic LQ control problem. However, this problem is not a conventional one as the strategy $u(\cdot)$ is constrained to take nonnegative values. In this section, we derive an open-loop equilibrium strategy for the mean-variance problem (3.8). The following definition is in a similar manner as that of [8].

Definition 4.1. *Let $u^*(\cdot) \in \mathbb{H}^2_{\mathbb{F}}(0, T; \mathbb{R}^{k+1}_+)$ be a given strategy and $X^*(\cdot)$ be the associated wealth process. The strategy $u^*(\cdot)$ is called an open-loop equilibrium if*

$$\liminf_{\varepsilon\downarrow 0} \frac{J(t, X^*(t); u^{t,\varepsilon,v}(\cdot)) - J(t, X^*(t); u^*(\cdot))}{\varepsilon} \geq 0, \tag{4.1}$$

where for any $t \in [0, T), \varepsilon > 0$ and $v(\cdot) \in \mathbb{H}^2_{\mathbb{F}}(t, T; \mathbb{R}^{k+1}_+)$,

$$u^{t,\varepsilon,v}(\cdot) = v(\cdot)\mathbf{1}_{[t,t+\varepsilon)}(\cdot) + u^*(\cdot)\mathbf{1}_{[t+\varepsilon,T]}(\cdot). \tag{4.2}$$

The next result provides a sufficient condition for the open-loop equilibrium strategy.

Theorem 4.2. *Let $u^*(\cdot)$ be a fixed strategy and $X^*(\cdot)$ be the associated wealth process. For each $t \in [0, T)$, define on the time interval $[t, T]$ the processes $(Y(\cdot; t), Z(\cdot; t), U(\cdot; t)) \in \mathbb{L}^2_{\mathbb{F}}(\Omega; D(t, T; \mathbb{R})) \times \mathbb{H}^2_{\mathbb{F}}(t, T; \mathbb{R}^{n+1}) \times \mathbb{J}^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$ as the unique solution to the following BSDE:*

$$\begin{cases} dY(s; t) = -r(s)Y(s; t)ds + Z(s; t)'dW(s) + U(s; t)'d\widetilde{\Phi}(s), \\ Y(T; t) = \frac{1}{2}X^*(T) - \frac{1}{2}\mathbb{E}_t[X^*(T)] - \frac{1}{2}\gamma X^*(t), \qquad s \in [t, T]. \end{cases} \tag{4.3}$$

Suppose $\Lambda(s; t) := B(s)Y(s; t) + \Sigma(s)Z(s; t)$ satisfies the following condition:

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathbb{E}_t\Big[\int_t^{t+\varepsilon} \langle\Lambda(s; t), v(s) - u^*(s)\rangle ds\Big] \geq 0, \ a.s., \ \forall t \in [0, T). \tag{4.4}$$

Then $u^*(\cdot)$ is an open-loop equilibrium strategy.

Proof. For each fixed $t \in [0, T)$ and $v(\cdot) \in \mathbb{H}^2_{\mathbb{F}}(t, T; \mathbb{R}^{k+1}_+)$, let $u^{t,\varepsilon,v}(\cdot)$ be defined by (4.2) and let $X^\varepsilon(\cdot)$ be the associated wealth process starting from $X^*(t)$. Then we have

$$\begin{aligned} &J(t, X^*(t); u^{t,\varepsilon,v}(\cdot)) - J(t, X^*(t); u^*(\cdot)) \\ &= \mathbb{E}_t\Big[(X^\varepsilon(T) - X^*(T))\Big(\frac{1}{2}(X^\varepsilon(T) + X^*(T)) - \frac{1}{2}\mathbb{E}_t[X^\varepsilon(T) + X^*(T)] - \gamma X^*(t)\Big)\Big]. \end{aligned} \tag{4.5}$$

Applying Itô's formula to $s \mapsto Y(s;t)(X^\varepsilon(s) - X^*(s))$ yields

$$
\begin{aligned}
& d[(X^\varepsilon(s) - X^*(s))Y(s;t)] \\
&\quad = (u^{t,\varepsilon,v}(s) - u^*(s))'[B(s)Y(s;t) + \Sigma(s)Z(s;t)]ds \\
&\qquad + [Y(s;t)(u^{t,\varepsilon,v}(s) - u^*(s))'\Sigma(s) + (X^\varepsilon(s) - X^*(s))Z(s;t)']dW(s) \\
&\qquad + (X^\varepsilon(s) - X^*(s))U(s;t)'d\widetilde{\Phi}(s).
\end{aligned} \tag{4.6}
$$

By the Burkholder-Davis-Gundy (B-D-G) inequality, it holds that

$$
\begin{aligned}
& \mathbb{E}\Big[\sup_{\tau\in[t,T]}\Big|\int_t^\tau [Y(s;t)(u^{t,\varepsilon,v}(s) - u^*(s))'\Sigma(s) + (X^\varepsilon(s) - X^*(s))Z(s;t)']dW(s)\Big|\Big] \\
& \leq C\mathbb{E}\Big[\Big|\int_t^T [Y(s;t)(u^{t,\varepsilon,v}(s) - u^*(s))'\Sigma(s) + (X^\varepsilon(s) - X^*(s))Z(s;t)']^2 ds\Big|^{\frac{1}{2}}\Big] \\
& \leq C\mathbb{E}\Big[\sup_{s\in[t,T]} Y(s;t)\Big(\int_t^T |u^{t,\varepsilon,v}(s) - u^*(s)|^2 ds\Big)^{\frac{1}{2}}\Big] \\
& \quad + C\mathbb{E}\Big[\sup_{s\in[t,T]} (X^\varepsilon(s) - X^*(s))\Big(\int_t^T |Z(s;t)|^2 ds\Big)^{\frac{1}{2}}\Big] \\
& \leq C\mathbb{E}\Big[\sup_{s\in[t,T]} |Y(s;t)|^2 + \int_t^T |u^{t,\varepsilon,v}(s) - u^*(s)|^2 ds\Big] \\
& \quad + C\mathbb{E}\Big[\sup_{s\in[t,T]} |X^\varepsilon(s) - X^*(s)|^2 + \int_t^T |Z(s;t)|^2 ds\Big] \\
& < +\infty,
\end{aligned}
$$

where C is a positive constant that may differ from line to line. Therefore, the $dW(s)$ term in (4.6) is a martingale. In addition, using Hölder and Young's inequalities, we have

$$
\begin{aligned}
& \mathbb{E}\Big[\sup_{\tau\in[t,T]}\Big|\int_t^\tau (u^{t,\varepsilon,v}(s) - u^*(s))'[B(s)Y(s;t) + \Sigma(s)Z(s;t)]ds\Big|\Big] \\
& \quad \leq C\mathbb{E}\Big[\sup_{s\in[t,T]} |Y(s;t)|^2 + \int_t^T |u^{t,\varepsilon,v}(s) - u^*(s)|^2 ds + \int_t^T |Z(s;t)|^2 ds\Big] \\
& \quad < +\infty,
\end{aligned}
$$

and

$$
\mathbb{E}\Big[\sup_{s\in[t,T]}\Big|Y(s;t)(X^\varepsilon(s) - X^*(s))\Big|\Big] \leq C\mathbb{E}\Big[\sup_{s\in[t,T]} |Y(s;t)|^2 + \sup_{s\in[t,T]} |X^\varepsilon(s) - X^*(s)|^2\Big] < +\infty.
$$

As a result, it can be shown from (4.6) that

$$
\mathbb{E}\Big[\sup_{\tau\in[t,T]}\Big|\int_t^\tau (X^\varepsilon(s) - X^*(s))U(s;t)'d\widetilde{\Phi}(s)\Big|\Big] < +\infty.
$$

This implies that the $d\widetilde{\Phi}(s)$ term in (4.6) is also a martingale. By taking the conditional expectation on both sides of (4.6), we obtain

$$
\mathbb{E}_t[(X^\varepsilon(T) - X^*(T))Y(T;t)]
$$

$$= \mathbb{E}_t\Big[(X^\varepsilon(T) - X^*(T))\Big(\frac{1}{2}X^*(T) - \frac{1}{2}\mathbb{E}_t[X^*(T)] - \frac{1}{2}\gamma X^*(t)\Big)\Big]$$
$$= \mathbb{E}_t\Big[\int_t^T (u^{t,\varepsilon,v}(s) - u^*(s))'[B(s)Y(s;t) + \Sigma(s)Z(s;t)]ds\Big]. \quad (4.7)$$

Combining (4.5) with (4.7), we have

$$\begin{aligned}
&J(t, X^*(t); u^{t,\varepsilon,v}(\cdot)) - J(t, X^*(t); u^*(\cdot))\\
&= \mathbb{E}_t\Big[(X^\varepsilon(T) - X^*(T))\Big(\frac{1}{2}(X^\varepsilon(T) + X^*(T)) - \frac{1}{2}\mathbb{E}_t[X^\varepsilon(T) + X^*(T)] - \gamma X^*(t)\Big)\\
&\qquad - 2(X^\varepsilon(T) - X^*(T))Y(T;t) + 2\int_t^T (u^{t,\varepsilon,v}(s) - u^*(s))'[B(s)Y(s;t) + \Sigma(s)Z(s;t)]ds\Big]\\
&= \mathbb{E}_t\Big[2\int_t^{t+\varepsilon} \langle \Lambda(s;t), v(s) - u^*(s)\rangle ds + \frac{1}{2}\{X^\varepsilon(T) - X^*(T) - \mathbb{E}_t[X^\varepsilon(T) - X^*(T)]\}^2\Big]\\
&\geq 2\mathbb{E}_t\Big[\int_t^{t+\varepsilon} \langle \Lambda(s;t), v(s) - u^*(s)\rangle ds\Big]. \qquad (4.8)
\end{aligned}$$

Therefore, it follows from condition (4.4) that

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0} \frac{J(t, X^*(t); u^{t,\varepsilon,v}(\cdot)) - J(t, X^*(t); u^*(\cdot))}{\varepsilon}\\
&\qquad \geq 2\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathbb{E}_t\Big[\int_t^{t+\varepsilon} \langle \Lambda(s;t), v(s) - u^*(s)\rangle ds\Big] \geq 0,
\end{aligned}$$

which implies that $u^*(\cdot)$ is an open-loop equilibrium strategy. □

Although (4.4) already provides a sufficient condition for an equilibrium strategy, it is not easily applicable as it involves a limit. Thanks to Proposition 3.4 in [14] and Theorem 3.6 in [8], it is straightforward to get the following sufficient condition of an equilibrium strategy.

Corollary 4.3. *Given a strategy* $u^*(\cdot) \in \mathbb{H}^2_{\mathbb{F}}(0, T; \mathbb{R}^{k+1}_+)$*, let* $X^*(\cdot)$ *be the associated wealth process and* $(Y(\cdot;t), Z(\cdot;t), U(\cdot;t)) \in \mathbb{L}^2_{\mathbb{F}}(\Omega; D(t, T; \mathbb{R})) \times \mathbb{H}^2_{\mathbb{F}}(t, T; \mathbb{R}^{n+1}) \times \mathbb{J}^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$ *be the unique solution to the BSDE* (4.3)*. If*

$$\langle \Lambda(t;t), v(t) - u^*(t)\rangle \geq 0, \ a.s., \ a.e.\ t \in [0, T]. \quad (4.9)$$

Then $u^*(\cdot)$ *is an open-loop equilibrium strategy.*

We are now in the position to construct a solution to the BSDE (4.3) such that the condition (4.9) holds. Let us first assume the following Ansatz:

$$Y(s;t) = M(s)X^*(s) - \mathbb{E}_t[M(s)X^*(s)] - N(s)X^*(t), \quad (4.10)$$

where (M, Γ, V) and (N, Δ, K) are solutions to the following BSDEs:

$$\begin{cases} dM(s) = -f(s, M(s), \Gamma(s), V(s))ds + \Gamma(s)'dW(s) + V(s)'d\widetilde{\Phi}(s), \quad M(T) = \frac{1}{2},\\ dN(s) = -g(s, N(s), \Delta(s), K(s))ds + \Delta(s)'dW(s) + K(s)'d\widetilde{\Phi}(s), \quad N(T) = \frac{1}{2}\gamma, \end{cases} \quad (4.11)$$

with f and g to be determined.

By Itô's formula, one can easily obtain

$$\begin{aligned} dM(s)X^*(s) =& \{[r(s)M(s) - f(s)]X^*(s) + [B(s)M(s) + \Sigma(s)\Gamma(s)]'u^*(s)\}ds \\ &+ [M(s)\Sigma(s)'u^*(s) + \Gamma(s)X^*(s)]'dW(s) + V(s)'X^*(s)d\widetilde{\Phi}(s), \end{aligned}$$

and then

$$d\mathbb{E}_t[M(s)X^*(s)] = \mathbb{E}_t\{[r(s)M(s) - f(s)]X^*(s) + [B(s)M(s) + \Sigma(s)\Gamma(s)]'u^*(s)\}ds.$$

Hence

$$\begin{aligned} dY(s;t) =& \Big\{[r(s)M(s) - f(s)]X^*(s) + [B(s)M(s) + \Sigma(s)\Gamma(s)]'u^*(s) \\ &- \mathbb{E}_t\{[r(s)M(s) - f(s)]X^*(s) + [B(s)M(s) + \Sigma(s)\Gamma(s)]'u^*(s)\} + gX^*(t)\Big\}ds \\ &+ [M(s)\Sigma(s)'u^*(s) + \Gamma(s)X^*(s) - \Delta(s)X^*(t)]'dW(s) + [V(s)X^*(s) - K(s)X^*(t)]'d\widetilde{\Phi}(s). \end{aligned} \tag{4.12}$$

Comparing the $dW(s)$ and $d\widetilde{\Phi}(s)$ terms with those of (4.3), we get

$$Z(s;t) = M(s)\Sigma(s)'u^*(s) + \Gamma(s)X^*(s) - \Delta(s)X^*(t), \tag{4.13}$$

$$U(s;t) = V(s)X^*(s) - K(s)X^*(t). \tag{4.14}$$

Substituting (4.13) into the condition (4.9) and noting $Y(s;s) = -N(s)X^*(s)$, we have

$$\langle -N(s)B(s)X^*(s) + M(s)\Sigma(s)\Sigma(s)'u^*(s) + \Sigma(s)[\Gamma(s) - \Delta(s)]X^*(s), v(s) - u^*(s)\rangle \geq 0.$$

Since we will construct a solution with $X^*(s) \geq 0$, it holds that

$$u^*(s) = \left\{\frac{1}{M(s)}(\Sigma(s)\Sigma(s)')^{-1}[B(s)N(s) - \Sigma(s)(\Gamma(s) - \Delta(s))]\right\}^+ X^*(s).$$

Denoting

$$\beta(s) := \left\{\frac{1}{M(s)}(\Sigma(s)\Sigma(s)')^{-1}[B(s)N(s) - \Sigma(s)(\Gamma(s) - \Delta(s))]\right\}^+,$$

and comparing the ds term in (4.12) with that in (4.3), we get

$$\begin{cases} f(s, M(s), \Gamma(s), V(s)) = 2r(s)M(s) + [M(s)B(s) + \Sigma(s)\Gamma(s)]'\beta(s), \\ g(s, N(s), \Delta(s), K(s)) = r(s)N(s). \end{cases} \tag{4.15}$$

It now suffices to solve the BSDEs (4.11). Its second equation can be easily solved since $r(\cdot)$ is a deterministic function. The solution is given by

$$N(s) = \frac{1}{2}\gamma e^{\int_s^T r(t)dt}, \quad \Delta(s) = K(s) \equiv 0. \tag{4.16}$$

Now, plugging the expression of $\beta(s)$ into the first equation in (4.11), we get the following BSDE governing (M, Γ, V) (suppressing s):

$$\begin{cases} dM = -\Big\{2rM + [MB + \Sigma\Gamma]'\Big[\frac{1}{M}(\Sigma\Sigma')^{-1}(BN - \Sigma\Gamma)\Big]^+\Big\}ds + \Gamma' dW(s) + V' d\widetilde{\Phi}(s), \\ M(T) = \frac{1}{2}. \end{cases} \tag{4.17}$$

Proposition 4.4. *The BSDE* (4.17) *admits a unique solution*

$$(M, \Gamma, V) \in \mathbb{L}^{\infty}_{\mathbb{F}}(0, T; \mathbb{R}) \times \mathbb{H}^2_{BMO(\mathbb{P})}(0, T; \mathbb{R}^{n+1}) \times \mathbb{J}^2_{BMO(\mathbb{P})}(0, T; \mathbb{R}^m)$$

such that $M \geq \delta$ for some positive constant δ, and in addition, V is uniformly bounded.

Proof. Let δ be a given positive constant. We consider the following auxiliary truncated BSDE:

$$\begin{cases} dM = -\Big\{2rM + [(M \vee \delta)B + \Sigma\Gamma]'\Big[\frac{1}{M\vee\delta}(\Sigma\Sigma')^{-1}(BN - \Sigma\Gamma)\Big]^+\Big\}ds \\ \qquad\qquad +\Gamma' dW(s) + V' d\widetilde{\Phi}(s), \\ M(T) = \frac{1}{2}. \end{cases} \tag{4.18}$$

This is in fact a regime-switching quadratic-exponential BSDE. By Lemma A.2 of [12], we know that (4.18) admits a unique solution $(M^\delta, \Gamma^\delta, V^\delta) \in \mathbb{L}^{\infty}_{\mathbb{F}}(0, T; \mathbb{R}) \times \mathbb{H}^2_{BMO(\mathbb{P})}(0, T; \mathbb{R}^{n+1}) \times \mathbb{J}^2_{BMO(\mathbb{P})}(0, T; \mathbb{R}^m)$, and in addition, V^δ is uniformly bounded. It should be noted that the unique solution $(M^\delta, \Gamma^\delta, V^\delta)$ is parameterized by δ.

Now we rewrite the BSDE (4.18) as

$$\begin{cases} dM^\delta = -\{2rM^\delta + B'(\Sigma\Sigma')^{-1}BN\}ds + (\Gamma^\delta)'[dW(s) - \theta(s)]dW(s) + (V^\delta)' d\widetilde{\Phi}(s), \\ M^\delta(T) = \frac{1}{2}, \end{cases} \tag{4.19}$$

where

$$\begin{aligned} \theta(s) :=& \Sigma'\Big[\frac{1}{M^\delta \vee \delta}(\Sigma\Sigma')^{-1}(BN - \Sigma\Gamma^\delta)\Big]^+ \\ &+ \frac{B'\Big\{\Big[(\Sigma\Sigma')^{-1}(BN - \Sigma\Gamma^\delta)\Big]^+ - (\Sigma\Sigma')^{-1}BN\Big\}}{|\Gamma^\delta|^2}\Gamma^\delta \mathbf{1}_{\{\Gamma^\delta \neq 0\}}. \end{aligned}$$

It is easy to check that $|\theta| \leq C(1 + |\Gamma^\delta|)$, which implies that $\theta \in \mathbb{H}^2_{BMO(\mathbb{P})}(0, T; \mathbb{R}^{n+1})$. Thus, the stochastic exponential of $\theta \cdot W$ denoted by $\mathcal{E}(\theta \cdot W)$ is a uniformly integrable $(\mathbb{P}, \mathbb{F})$-martingale. This allows us to define a new probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} := \mathcal{E}(\theta \cdot W)(T), \tag{4.20}$$

under which the process

$$W^{\mathbb{Q}}(t) := W(t) - \int_0^t \theta(s)ds$$

is an $n + 1$ dimensional standard Brownian motion and, at the same time, the probability law of the Markov chain remains unchanged. By standard theory of linear BSDEs, we can express M^δ by the following conditional expectation:

$$M^\delta(s) = \mathbb{E}^{\mathbb{Q}}_s\Big[\frac{1}{2}e^{\int_s^T 2r(t)dt} + \int_s^T e^{\int_s^v 2r(t)dt}B(v)'(\Sigma(v)\Sigma(v)')^{-1}B(v)N(v)dv\Big], \tag{4.21}$$

from which we deduce that there exists a positive constant η independent of δ and depending only on T and the bounds of the model parameters such that $M^\delta \geq \eta$. By taking $\delta = \eta$, we obtain that $(M^\delta, \Gamma^\delta, V^\delta) \in \mathbb{L}^{\infty}_{\mathbb{F}}(0, T; \mathbb{R}) \times \mathbb{H}^2_{BMO(\mathbb{P})}(0, T; \mathbb{R}^{n+1}) \times \mathbb{J}^2_{BMO(\mathbb{P})}(0, T; \mathbb{R}^m)$ is exactly the unique solution to the BSDE (4.17). □

With M, Γ, N, Δ obtained, we can construct a strategy

$$u^*(s) = \beta(s)X^*(s), \tag{4.22}$$

where

$$\beta(s) := \left\{ \frac{1}{M(s)} (\Sigma(s)\Sigma(s)')^{-1} [B(s)N(s) - \Sigma(s)\Gamma(s)] \right\}^+.$$

The following theorem confirms that the above is indeed an open-loop equilibrium strategy.

Theorem 4.5. *Let* (M, Γ, V) *and* (N, Δ, K) *be solutions to the BSDEs* (4.11). *Then* $u^*(\cdot)$ *given by* (4.22) *is an open-loop equilibrium strategy.*

Proof. Substituting the feedback strategy $u^*(\cdot)$ into the wealth equation (3.6), we get

$$\begin{cases} dX^*(s) = \{r(s) + \beta(s)'B(s)\}X^*(s)ds + \beta(s)'\Sigma(s)X^*(s)dW(s), \\ X^*(0) = x_0 > 0. \end{cases}$$

Then $X^*(s) > 0$ implying that $u^*(s) \geq 0$. Let us now prove that $u^*(\cdot)$ is square integrable. Applying Itô's formula to $s \mapsto M(s)X^*(s)^2$, it holds that (suppressing s)

$$\frac{d[M(X^*)^2]}{M(X^*)^2} = \left(\left[B + \Sigma \frac{\Gamma}{M} \right]' \beta + \beta'\Sigma\Sigma'\beta \right) ds + \left(2\Sigma'\beta + \frac{\Gamma}{M} \right)' dW(s) + \frac{V'}{M} d\widetilde{\Phi}(s).$$

In addition, by the definition for β, we have

$$\begin{aligned} \beta'\Sigma\Sigma'\beta &= \left\{ \left[\frac{1}{M} (\Sigma\Sigma')^{-1} (BN - \Sigma\Gamma) \right]^+ \right\}' \Sigma\Sigma' \left[\frac{1}{M} (\Sigma\Sigma')^{-1} (BN - \Sigma\Gamma) \right]^+ \\ &= \left[\frac{1}{M} (\Sigma\Sigma')^{-1} (BN - \Sigma\Gamma) \right]' \Sigma\Sigma' \left[\frac{1}{M} (\Sigma\Sigma')^{-1} (BN - \Sigma\Gamma) \right]^+ \\ &= \left[\frac{N}{M} B' - \frac{\Gamma'}{M} \Sigma' \right] \beta, \end{aligned}$$

from which we derive that

$$\frac{d[M(X^*)^2]}{M(X^*)^2} = \left(1 + \frac{N}{M} \right) B'\beta ds + \left(2\Sigma'\beta + \frac{\Gamma}{M} \right)' dW(s) + \frac{V'}{M} d\widetilde{\Phi}(s).$$

Therefore

$$M(T)X^*(T)^2 = M(0)x_0^2 e^{\int_0^T (1+M^{-1}N)B'\beta ds} \mathcal{E}\left(\left(2\Sigma'\beta + \frac{\Gamma}{M} \right) \cdot W \right)(T) \times \mathcal{E}\left(\frac{V'}{M} \cdot \widetilde{\Phi} \right)(T).$$

By the John-Nirenberg's inequality (see Theorem 2.2 in [10]), we obtain that there exists some $\varepsilon > 0$ such that

$$\mathbb{E}\left[e^{\varepsilon \int_0^T |\beta(s)|^2 ds} \right] < +\infty.$$

Since M, N and B are bounded, using the inequality $2ab \leq a^2 + b^2$, it follows that for any $p_1 > 1$,

$$\mathbb{E}\left[e^{p_1 \int_0^T (1+M^{-1}N)B'\beta ds} \right]$$

$$
\begin{aligned}
&\leq \mathbb{E}\Big[e^{\int_0^T \frac{p_1^2}{4\varepsilon}|(1+M^{-1}N)B'|^2 ds + \varepsilon\int_0^T |\beta(s)|^2 ds}\Big] \\
&\leq C\mathbb{E}\Big[e^{\varepsilon\int_0^T |\beta(s)|^2 ds}\Big] < +\infty.
\end{aligned}
\tag{4.23}
$$

Denoting

$$
\Psi_1 := \mathcal{E}\Big(\Big(2\Sigma'\beta + \frac{\Gamma}{M}\Big)\cdot W\Big)(T) \quad \text{and} \quad \Psi_2 := \mathcal{E}\Big(\frac{V'}{M}\cdot\widetilde{\Phi}\Big)(T).
$$

As $\left(2\Sigma'\beta + \frac{\Gamma}{M}\right)\cdot W$ is a *BMO*-martingale, the reverse Hölder inequality indicates that $\mathbb{E}[\Psi_1^{p_2}] < +\infty$, for some $p_2 > 1$.

Moreover, since $\frac{V}{M}$ is uniformly bounded, it holds that

$$
\mathbb{E}[\Psi_2^{p_3}] < +\infty, \quad \text{for any} \quad p_3 > 1. \tag{4.24}
$$

Using Young's inequality and taking $p > 1$ such that $1/p + 1/p_2 = 1$, we get

$$
\begin{aligned}
\mathbb{E}[|X^*(T)|^2] \leq& C\mathbb{E}\Big[e^{\int_0^T (1+M^{-1}N)B'\beta ds}\Psi_1\Psi_2\Big] \\
\leq& C\Big\{\mathbb{E}[\Psi_1^{p_2}] + \mathbb{E}\Big[e^{p\int_0^T (1+M^{-1}N)B'\beta ds}\Psi_2^p\Big]\Big\} \\
\leq& C\Big\{\mathbb{E}[\Psi_1^{p_2}] + \mathbb{E}[\Psi_2^{2p}] + \mathbb{E}\Big[e^{2p\int_0^T (1+M^{-1}N)B'\beta ds}\Big]\Big\} \\
<& +\infty.
\end{aligned}
\tag{4.25}
$$

From the wealth equation (3.6), it can be shown that the wealth-strategy pair $(X^*(\cdot), u^*(\cdot))$ satisfy the following equation:

$$
\begin{cases}
dy(s) = \{r(s)y(s) + u^*(s)'B(s)\}ds + u^*(s)'\Sigma(s)dW(s), \\
y(T) = X^*(T).
\end{cases}
\tag{4.26}
$$

Setting $z(s) = \Sigma(s)'u^*(s)$ and substituting it back into (4.26), it follows that

$$
(y(s), z(s)) = (X^*(s), \Sigma(s)'u^*(s))
$$

is the solution to the following BSDE:

$$
\begin{cases}
dy(s) = \{r(s)y(s) + B(s)'\big(\Sigma(s)\Sigma(s)'\big)^{-1}\Sigma(s)z(s)\}ds + z(s)'dW(s), \\
y(T) = X^*(T).
\end{cases}
\tag{4.27}
$$

In fact, (4.27) is a linear BSDE with a Lipschitz continuous and linear growth driver and a square integrable terminal value (see (4.25)). Therefore, the BSDE (4.27) admits a unique solution $(y(\cdot), z(\cdot)) \in \mathbb{L}^2_{\mathbb{F}}(\Omega; C(0,T;\mathbb{R})) \times \mathbb{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^{n+1})$. Consequently, following from the relationship between $u^*(\cdot)$ and $z(\cdot)$, we immediately obatin $u^*(\cdot) \in \mathbb{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^{k+1}_+)$.

Finally, applying Itô's formula to $s \mapsto M(s)X^*(s)$ and noticing the definition for $Z(\cdot;t)$ and $U(\cdot;t)$ (see (4.13)-(4.14)) and the fact that $\Delta = K = 0$ yield (suppressing s)

$$
d[MX^*] = -[2rM + (MB + \Sigma\Gamma)'\beta]X^*ds + \Gamma'X^*dW(s) + V'X^*d\widetilde{\Phi}(s)
$$

$$
\begin{aligned}
&+ [r + \beta' B]MX^* ds + \beta'\Sigma MX^* dW(s) + \beta'\Sigma\Gamma X^* ds \\
&= -rMX^* ds + Z(s;t)' dW(s) + U(s;t)' d\widetilde{\Phi}(s),
\end{aligned}
\tag{4.28}
$$

and then

$$
\begin{aligned}
dY(s;t) &= -rMX^* ds + Z(s;t) dW(s) + U(s;t) d\widetilde{\Phi}(s) + r\mathbb{E}_t[MX^*] ds + rNX^*(t) ds \\
&= -rY(s;t) ds + Z(s;t)' dW(s) + U(s;t)' d\widetilde{\Phi}(s).
\end{aligned}
\tag{4.29}
$$

Therefore, $(Y(\cdot;t), Z(\cdot;t), U(\cdot;t))$ defined by (4.10), (4.13) and (4.14) is a solution to the BSDE (4.3), and in addition, the condition (4.9) is easily checked. Then it follows from Corollary 4.3 that $u^*(\cdot)$ is an open-loop equilibrium strategy. □

Suppose that $\Gamma(\cdot)$ is partitioned such that $\Gamma(\cdot) = [\Gamma_1(\cdot), \Gamma_2(\cdot)']'$, where $\Gamma_1(\cdot)$ denotes the first entry of $\Gamma(\cdot)$, and $\Gamma_2(\cdot)$ denotes the remaining n entries. Recalling the closed-form expressions for $B(\cdot), \Sigma(\cdot)$ and $N(\cdot)$, we obtain finally the main result of this section.

Theorem 4.6. *The open-loop equilibrium investment and risk control policies for the mean-variance problem* (3.8) *are respectively given by*

$$
q^*(s) = \frac{1}{M(s) b^2(s)} \Big[\frac{1}{2} \gamma e^{\int_s^T r(t)dt} (p(s) - a(s)) - b(s)\Gamma_1(s) \Big]^+ X^*(s), \tag{4.30}
$$

$$
\pi^*(s) = \frac{1}{M(s)} (\sigma(s)\sigma(s))^{-1} \Big[\frac{1}{2} \gamma e^{\int_s^T r(t)dt} B_2(s) - \sigma(s)\Gamma_2(s) \Big]^+ X^*(s), \tag{4.31}
$$

where $B_2(s) = (\mu_1(s) - r(s), \ldots, \mu_k(s) - r(s))'$.

5. An example of Markovian regime-switching model

In this section, we consider an example of the Markovian regime-switching model. Let

$$
a(\cdot) := \hat{a}(\cdot, \alpha(\cdot)),\ b(\cdot) := \hat{b}(\cdot, \alpha(\cdot)),\ p(\cdot) := \hat{p}(\cdot, \alpha(\cdot)),\ \mu_i(\cdot) := \hat{\mu}_i(\cdot, \alpha(\cdot)),\ \sigma_{ij}(\cdot) := \hat{\sigma}_{ij}(\cdot, \alpha(\cdot)),
$$

where $\hat{a}(\cdot,\cdot), \hat{b}(\cdot,\cdot), \hat{p}(\cdot,\cdot), \hat{\mu}_i(\cdot,\cdot), \hat{\sigma}_{ij}(\cdot,\cdot)$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, are deterministic bounded functions. Under such circumstance, the model of study turns into the classical Markovian regime-switching case which is widely studied in the literature. It is obvious in this setting that $\Gamma(t) \equiv 0$ for all $t \in [0, T]$. Then the BSDE (4.17) becomes

$$
\begin{cases}
dM(s) = -\Big\{ 2r(s)M(s) + \frac{1}{2}\gamma e^{\int_s^T r(t)dt} \hat{B}(s,\alpha(s))' \big(\hat{\Sigma}(s,\alpha(s))\hat{\Sigma}(s,\alpha(s))'\big)^{-1} \hat{B}(s,\alpha(s)) \Big\} ds \\
\qquad\qquad + V(s)' d\widetilde{\Phi}(s), \\
M(T) = \frac{1}{2},
\end{cases}
\tag{5.1}
$$

where $\hat{B}(s,\alpha(s)) := (\hat{p}(s,\alpha(s)) - \hat{a}(s,\alpha(s)), \hat{\mu}_1(s,\alpha(s)) - r(s), \ldots, \hat{\mu}_k(s,\alpha(s)) - r(s))'$ and

$$
\hat{\Sigma}(s,\alpha(s)) := \begin{pmatrix} \hat{b}(s,\alpha(s)) & 0 \\ 0 & \hat{\sigma}(s,\alpha(s)) \end{pmatrix}.
$$

To find a solution $(M(\cdot), V(\cdot))$ to (5.1), we try a process $M(\cdot)$ of the following form

$$M(s) = \phi(s, \alpha(s)), \tag{5.2}$$

where $\phi(\cdot, \cdot)$ is a deterministic and differentiable function which is to be determined. It is clear that ϕ must satisfy the terminal condition $\phi(T, e_i) = 1/2$, for $i = 1, \ldots, m$. Applying Itô's formula to the right-hand side of (5.2) yields

$$\begin{aligned} d\phi(s, \alpha(s)) =& \Big\{\dot{\phi}(s, \alpha(s)) + \sum_{j=1}^{m} \Big[\phi(s, e_j) - \phi(s, \alpha(s))\Big]\lambda_j(s)\Big\}ds \\ &+ \sum_{j=1}^{m} \Big[\phi(s, e_j) - \phi(s, \alpha(s-))\Big]d\widetilde{\Phi}_j(s). \end{aligned} \tag{5.3}$$

Comparing the coefficients with those of (5.1), we get the following system of ODEs governing ϕ:

$$\begin{cases} \dot{\phi}(s, e_i) + 2r(s)\phi(s, e_i) + \frac{1}{2}\gamma e^{\int_s^T r(t)dt}\hat{B}(s, e_i)'(\hat{\Sigma}(s, e_i)\hat{\Sigma}(s, e_i)')^{-1}\hat{B}(s, e_i) \\ \qquad + \sum\limits_{j=1}^{m} q_{ij}\phi(s, e_j) = 0, \\ \phi(T, e_i) = \frac{1}{2}, \qquad i = 1, \ldots, m. \end{cases} \tag{5.4}$$

As (5.4) is linear with uniformly bounded coefficients, the existence and uniqueness of a solution is evident. Consequently, the open-loop equilibrium strategy (4.22) can be rewritten as

$$u^*(s) = \mathcal{U}(s, \alpha(s), X^*(s)),$$

where the function $\mathcal{U}(s, e_i, x)$ is given by

$$\mathcal{U}(s, e_i, x) = \frac{1}{2\phi(s, e_i)}\gamma e^{\int_s^T r(t)dt}(\hat{\Sigma}(s, e_i)\hat{\Sigma}(s, e_i)')^{-1}\hat{B}(s, e_i)x. \tag{5.5}$$

Theorem 5.1. *Let $\phi(\cdot, e_i)$, for $i = 1, \ldots, m$, be given by* (5.4)*. Then the open-loop equilibrium investment and risk control policies for the mean-variance problem* (3.8) *under the Markovian regime-switching model are respectively given by*

$$q^*(s) = q^*(s, \alpha(s), X^*(s)) = \frac{\gamma e^{\int_s^T r(t)dt}(\hat{p}(s, \alpha(s)) - \hat{a}(s, \alpha(s)))}{2\phi(s, \alpha(s))\hat{b}^2(s, \alpha(s))}X^*(s), \tag{5.6}$$

$$\pi^*(s) = \pi^*(s, \alpha(s), X^*(s)) = \frac{\gamma e^{\int_s^T r(t)dt}}{2\phi(s, \alpha(s))}(\hat{\sigma}(s, \alpha(s))\hat{\sigma}(s, \alpha(s)))^{-1}\hat{B}_2(s, \alpha(s))X^*(s), \tag{5.7}$$

where $\hat{B}_2(s, \alpha(s)) := (\hat{\mu}_1(s, \alpha(s)) - r(s), \ldots, \hat{\mu}_k(s, \alpha(s)) - r(s))'$.

Remark 5.2. It should be noted that in the Markovian regime-switching framework, the equilibrium strategy $u^*(\cdot)$ at time s is a feedback of $(\alpha(s), X^*(s))$. In the non-Markovian setting, however, since the parameters are $\mathbb{F}$-predictable processes, the past-dependence of the Brwonian motion W and the Markov chain α has been shown in $\beta(\cdot)$ (see (4.22)).

Example 5.3. Let us consider the Markovian regime-switching model with $m = 2$ and $k = n = 1$. The other parameters are set as follows: $T = 1, r(s) = 0.04, \gamma = 3, \hat{p}(s, e_1) = 3.5, \hat{p}(s, e_2) = 1.8, \hat{a}(s, e_1) = 3, \hat{a}(s, e_2) = 1.5, \hat{b}(s, e_1) = 1.5, \hat{b}(s, e_2) = 1.2, \hat{\mu}(s, e_1) = 0.2, \hat{\mu}(s, e_2) = 0.5, \hat{\sigma}(s, e_1) = 1, \hat{\sigma}(s, e_2) = 1.5, q_{11} = -0.5, q_{22} = -0.5$. We solve the system of ODEs (5.4) numerically by using MatLab, and easily get the ratios of investment and risk control policies to wealth, i.e., $q^*(s, e_i, x)/x$ and $\pi^*(s, e_i, x)/x$, which are called the liability ratio and the investment ratio, respectively. These ratios are depicted in Figure 1 under the two regimes.

Figure 2 plots the liability ratio and the investment ratio with different risk aversion parameter γ in Regime 1. This figure shows when γ becomes larger, the insurer tends to increase the number of insurance policies and, at the same time, invest more in the risky assets. This result is consistent with that in [2] (without the insurance risk).

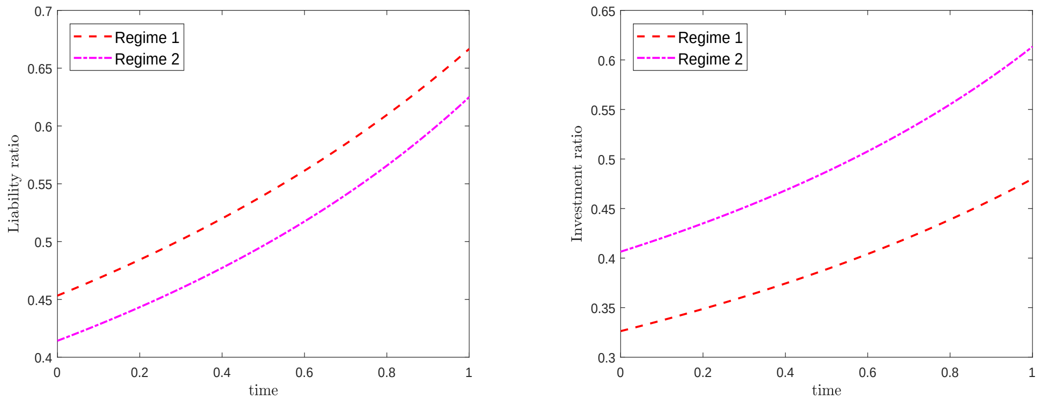

Figure 1. The liability ratio and the investment ratio in the two regimes.

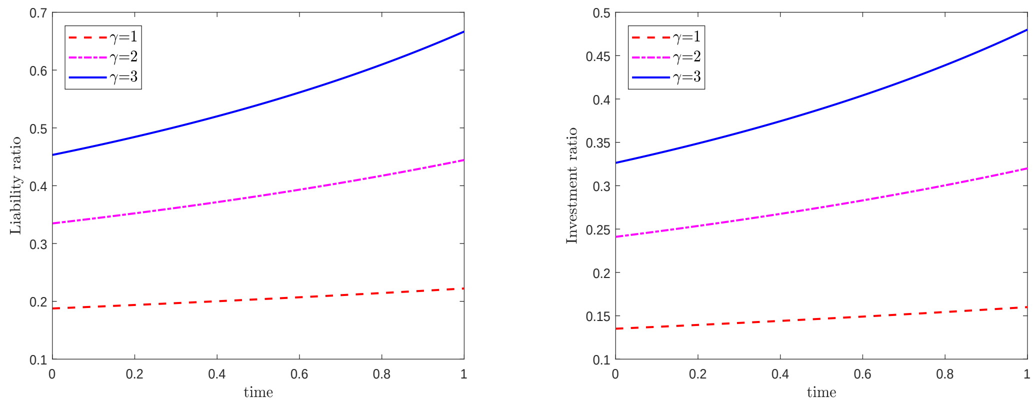

Figure 2. The liability ratio and the investment ratio with different γ in Regime 1.

6. Conclusion

This paper studies a time-inconsistent mean-variance problem under a non-Markovian regime-switching model where short-selling is not allowed. The open-loop equilibrium investment and risk control polices are derived explicitly based on the FBSDE technique and BMO-martingale theory. Due to some technical difficulties, we only consider the problem with a deterministic risk-free interest rate. One of the potential research topics in the future is to extend the results to the case with fully random coefficients. As jumps in the price processes of the underlying risky assets always describe extra-ordinary market news or sudden events, another good research direction is to convert the present study to a non-Markovian regime-switching, jump-diffusion model.

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Conflict of interest

The authors declare that they have no competing interests.

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