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Research article

New solitary wave solutions for the conformable Klein-Gordon equation with quantic nonlinearity

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Abstract: We present new exact solutions in the form of solitary waves for the conformable Klein-Gordon equation with quintic nonlinearity. We use functional variable method which converts a conformable PDE to a second-order ordinary differential equation through a traveling wave transformation. We obtain periodic wave and solitary wave solutions including particularly kink-profile and bell-profile type solutions. The present method is a direct and concise technique which has the potential to be applicable to many other conformable PDEs arising in physics and engineering.

Keywords: functional variable technique; conformable derivative; conformable Klein-Gordon with quantic nonlinearity; exact wave solutions
Mathematics Subject Classification: 35A09, 35E05

1. Introduction

Nonlinear conformable evolution equations (NLCEEs) became significantly useful tools in the modeling of many problems in sciences and technology. Exact wave solutions of these models are very important and active research area. NLCEEs are getting the attention of researchers and becoming phenomenal subject in the contemporary science. Many systems in mathematical physics and fluid dynamics are modeled via fractional differential equations. Exact wave solutions of these models are quite active and important research area in science. For the numerical and exact solutions of NLCEEs, there are some efficient techniques in the literature such as method of $(G'/G)-$expansion, extended sinh-Gordon equation expansion, Kudryashov, exp-function, exponential rational function, modified Khater, functional variable, improved Bernoulli sub-equation function, sub-equation, tanh, Jacobi elliptic function expansion, auxiliary equation, extended direct algebraic, etc., see [1–27]. The functional variable (FV) method was introduced in [28] and was further developed in the studies [29–33]. FV method treats nonlinear PDEs with linear techniques and constructs interesting type of soliton solutions (kink, black, white, pattern, etc). The conformable fractional derivatives don't have a physical meaning as the Caputo or Riemann-Liouville derivatives. This situation is a general open problem for fractional calculus. Despite this many physical applications of conformable fractional derivative appear in the literature. Dazhi Zhao and Maokang Luo generalized the conformable fractional derivative and give the physical interpretation of generalized conformable derivative. In addition, with the help of this fractional derivative and some important formulas, one can convert conformable fractional partial differential equations into integer-order differential equations by travelling wave transformation [39].

The aim of the present paper is present new exact solutions to conformable Klein-Gordon (KG) equation with quintic nonlinearity by employing FV method. Nonlinear conformable Klein-Gordon equation has the form (for $\alpha = 1$, see [34])

$$D_t^{2\alpha} u - k^2 u_{xx} + \gamma u - \lambda u^n + \sigma u^{2n-1} = 0, \tag{1.1}$$

in which u represents wave profile, and $k, \gamma, \lambda, \sigma \neq 0$ are real valued constants. KG equation arises in theoretical physics, particularly in the area of relativistic quantum mechanics and it is used in modeling of dislocations in crystals.

For $n = 3$, Eq (1.1) is known as conformable Klein-Gordon equation with quintic nonlinearity [24]

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - k^2 \frac{\partial^2 u}{\partial x^2} + \gamma u - \lambda u^3 + \sigma u^5 = 0, \quad \sigma \neq 0. \tag{1.2}$$

In particular, if $\sigma = 0$, then Eq (1.2) reduces to some other PDEs including the ones in [35, 36].

(i) Conformable Klein-Gordon equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - \frac{\partial^2 u}{\partial x^2} + \kappa u + \beta u^3 = 0. \tag{1.3}$$

(ii) Conformable Landau-Ginzburg-Higgs equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - p \frac{\partial^2 u}{\partial x^2} - m^2 u + g^2 u^3 = 0. \tag{1.4}$$

(iii) Conformable Φ-four equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - \frac{\partial^2 u}{\partial x^2} + u - u^3 = 0. \tag{1.5}$$

(iv) Conformable Duffing equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + bu + cu^3 = 0. \tag{1.6}$$

(v) Conformable Sine-Gordon equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - \frac{\partial^2 u}{\partial x^2} + u - \frac{1}{6}u^3 = 0. \tag{1.7}$$

Next, we overview method of functional variable.

2. Method of functional variable

Consider the NLCEE:

$$F(u, D_t^{\alpha} u, u_x, D_t^{2\alpha} u, u_{xx}, \ldots) = 0, \qquad t \geq 0, \quad 0 < \alpha \leq 1, \tag{2.1}$$

in which F is a polynomial function in terms of unknown function u, and $D_t^{\alpha} u$ is defined as [37]

$$D_t^{\alpha} u(x, t) = \lim_{\varepsilon \to 0} \frac{u(x, t + \varepsilon t^{1-\alpha}) - u(x, t)}{\varepsilon}, \tag{2.2}$$

where $0 < t$, $\alpha \in (0, 1]$.

Now, let us define the wave variable [38]

$$u(x, t) = U(\xi), \quad \xi = x - \omega \frac{t^{\alpha}}{\alpha}, \tag{2.3}$$

in which ω is a parameter which will be determined later. Hence, we can write that

$$D_t^{\alpha} u = -\omega U'(\xi), \quad u_x = U'(\xi), \quad D_t^{2\alpha} u = \omega^2 U''(\xi), \quad \ldots.$$

By writing Eq (2.3) in Eq (2.1), we get ordinary differential equations:

$$G(U(\xi), U'(\xi), U''(\xi), U'''(\xi), \ldots) = 0. \tag{2.4}$$

Now, define a transformation:

$$U_{\xi} = F(U), \tag{2.5}$$

from which, we obtain

$$\begin{aligned} U_{\xi\xi} &= \frac{1}{2}(F^2)', \\ U_{\xi\xi\xi} &= \frac{1}{2}(F^2)'' \sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \frac{1}{2}[(F^2)''' F^2 + (F^2)''(F^2)'], \end{aligned} \tag{2.6}$$

$$\vdots$$

in which ”′” stands for $\frac{d}{dU}$.

Using Eq (2.6) in Eq (2.3), ordinary differential Eq (2.3) can be reduced to:

$$G(U, F, F', F'', F''', \ldots) = 0. \tag{2.7}$$

Now, let us consider the equation

$$(U(\xi)_\xi)^2 = aU^2(\xi) + bU^{2+n}(\xi) + cU^{2+2n}(\xi),\ 0 < n, \tag{2.8}$$

in which a, b, c are parameters.

Next, we present a set of exact wave solutions of (2.8), see e.g., [39]:

Case 1. If $a > 0$, then (2.8) admits hyperbolic function solution:

$$U_1(\xi) = \left[\frac{-ab\sec h^2(\frac{n\sqrt{a}}{2}\xi)}{b^2 - ac(1 - \tanh(\frac{n\sqrt{a}}{2}\xi))^2}\right]^{\frac{1}{n}}. \tag{2.9}$$

Case 2. If $a,\ c > 0$, then (2.8) admits the following hyperbolic function solution

$$U_2(\xi) = \left[\frac{a\csc h^2(\frac{n\sqrt{a}}{2}\xi)}{b + 2\sqrt{ac}\coth(\frac{n\sqrt{a}}{2}\xi)}\right]^{\frac{1}{n}}, \tag{2.10}$$

$$U_3(\xi) = \left[\frac{4a\left(\cosh(n\sqrt{a}\xi) + \sinh(n\sqrt{a}\xi)\right)}{4ac - \left(b + \cosh(n\sqrt{a}\xi) + \sinh(n\sqrt{a}\xi)\right)^2}\right]^{\frac{1}{n}}, \tag{2.11}$$

$$U_4(\xi) = \left[\frac{8a^2\sec h(n\sqrt{a}\xi)}{b^2 + 4a(a-c) - 4ab\sec h(n\sqrt{a}\xi) + (b^2 - 4a(a+c))\tanh(n\sqrt{a}\xi)}\right]^{\frac{1}{n}}, \tag{2.12}$$

$$U_5(\xi) = \left[\frac{a\csc h(\frac{n\sqrt{a}}{2}\xi)}{b\sinh(\frac{n\sqrt{a}}{2}\xi) + 2\sqrt{ac}\cosh(\frac{n\sqrt{a}}{2}\xi)}\right]^{\frac{1}{n}}, \tag{2.13}$$

$$U_6(\xi) = \left[\frac{a\sec h(\frac{n\sqrt{a}}{2}\xi)}{2\sqrt{ac}\sinh(\frac{n\sqrt{a}}{2}\xi) - b\cosh(\frac{n\sqrt{a}}{2}\xi)}\right]^{\frac{1}{n}}. \tag{2.14}$$

Case 3. If $a > 0$ and $b^2 - 4ac > 0$, then (2.8) admits the following hyperbolic function solution

$$U_7(\xi) = \left[\frac{2a\sec h(n\sqrt{a}\xi)}{-b\sec h(n\sqrt{a}\xi) \pm \sqrt{b^2 - 4ac}}\right]^{\frac{1}{n}}. \tag{2.15}$$

Case 4. If $a > 0$ and $b^2 - 4ac < 0$, then (2.8) admits the following hyperbolic function solution

$$U_8(\xi) = \left[\frac{2a\csc h(n\sqrt{a}\xi)}{\pm\sqrt{4ac - b^2} - b\csc h(n\sqrt{a}\xi)}\right]^{\frac{1}{n}}. \tag{2.16}$$

Case 5. If $a > 0$ and $b^2 - 4ac = 0$, then (2.8) admits the following hyperbolic function solution

$$U_9(\xi) = \left[-\frac{a}{c}\left(1 \pm \tanh(\frac{n}{2}\sqrt{a}\xi)\right)\right]^{\frac{1}{n}}, \tag{2.17}$$

$$U_{10}(\xi) = \left[-\frac{a}{c}\left(1 \pm \coth(\frac{n}{2}\sqrt{a}\xi)\right)\right]^{\frac{1}{n}}. \tag{2.18}$$

Case 6. If $a < 0$ and $c > 0$, then (2.8) admits the following triangular function solution

$$U_{11}(\xi) = \left[\frac{2a}{-b \pm \sqrt{b^2 - 4ac}\sin(n\sqrt{-a}\xi)}\right]^{\frac{1}{n}}, \tag{2.19}$$

$$U_{12}(\xi) = \left[\frac{2a}{-b \pm \sqrt{b^2 - 4ac}\cos(n\sqrt{-a}\xi)}\right]^{\frac{1}{n}}, \tag{2.20}$$

$$U_{13}(\xi) = \left[\frac{a\sec^2(\frac{n\sqrt{-a}}{2}\xi)}{-b + 2\sqrt{-ac}\tan(\frac{n\sqrt{-a}}{2}\xi)}\right]^{\frac{1}{n}}, \tag{2.21}$$

$$U_{14}(\xi) = \left[\frac{a\csc^2(\frac{n\sqrt{-a}}{2}\xi)}{-b + 2\sqrt{-ac}\cot(\frac{n\sqrt{-a}}{2}\xi)}\right]^{\frac{1}{n}}, \tag{2.22}$$

$$U_{15}(\xi) = \left[\frac{-a\left(1 + (\tan(n\sqrt{-a}\xi) \pm \sec(n\sqrt{-a}\xi))^2\right)}{b - 2\sqrt{-ac}\tan(n\sqrt{-a}\xi) \pm \sec(n\sqrt{-a}\xi)}\right]^{\frac{1}{n}}, \tag{2.23}$$

$$U_{16}(\xi) = \left[\frac{-a\csc(\frac{n\sqrt{-a}}{2}\xi)}{b\sin(\frac{n\sqrt{-a}}{2}\xi) + 2\sqrt{-ac}\cos(\frac{n\sqrt{-a}}{2}\xi)}\right]^{\frac{1}{n}}, \tag{2.24}$$

$$U_{17}(\xi) = \left[\frac{a\sec(\frac{n\sqrt{-a}}{2}\xi)}{2\sqrt{-ac}\sin(\frac{n\sqrt{-a}}{2}\xi) - b\cos(\frac{n\sqrt{-a}}{2}\xi)}\right]^{\frac{1}{n}}. \tag{2.25}$$

Case 7. If $a > 0$and $b = 0$, then (2.8) admits the following hyperbolic function solution

$$U_{18}(\xi) = \left[\pm\sqrt{\frac{a}{c}}\csc h(n\sqrt{a}\xi)\right]^{\frac{1}{n}}, \qquad (c > 0), \tag{2.26}$$

$$U_{19}(\xi) = \left[\pm\sqrt{-\frac{a}{c}}\sec h(n\sqrt{a}\xi)\right]^{\frac{1}{n}}, \qquad (c < 0). \tag{2.27}$$

Case 8. If $a < 0$and $b = 0$, then (2.8) admits the following triangular function solution

$$U_{20}(\xi) = \left[\pm\sqrt{-\frac{a}{c}}\csc(n\sqrt{a}\xi)\right]^{\frac{1}{n}}, \qquad (c > 0), \tag{2.28}$$

$$U_{21}(\xi) = \left[\pm\sqrt{-\frac{a}{c}}\sec(n\sqrt{-a}\xi)\right]^{\frac{1}{n}}, \qquad (c < 0). \tag{2.29}$$

Case 9. If $a > 0$ and $c = 0$, then (2.8) admits the following hyperbolic function solution

$$U_{22}(\xi) = \left[-\frac{a}{b}\csc h^2(\frac{n\sqrt{a}}{2}\xi)\right]^{\frac{1}{n}}, \tag{2.30}$$

$$U_{23}(\xi) = \left[\frac{a}{b}\sec h^2(\frac{n\sqrt{a}}{2}\xi)\right]^{\frac{1}{n}}. \tag{2.31}$$

Case 10. If $a < 0$ and $c = 0$, then (2.8) admits the following triangular function solution

$$U_{24}(\xi) = \left[\frac{a}{b}\csc^2(\frac{n\sqrt{-a}}{2}\xi)\right]^{\frac{1}{n}}, \tag{2.32}$$

$$U_{25}(\xi) = \left[\frac{a}{b}\sec^2(\frac{n\sqrt{-a}}{2}\xi)\right]^{\frac{1}{n}}. \tag{2.33}$$

3. Conformable Klein-Gordon with quintic nonlinearity

Using transformation of traveling wave; $u(x,t) = U(\xi)$, $\xi = x - \omega\frac{t^\alpha}{\alpha}$, Eq (1.1) is written as:

$$(w^2 - k^2)U_{\xi\xi} + \gamma U - \lambda U^3 + \sigma U^5 = 0, \tag{3.1}$$

or

$$U_{\xi\xi} = \frac{1}{w^2 - k^2}\left[-\gamma U + \lambda U^3 - \sigma U^5\right]. \tag{3.2}$$

Writing Eq (2.5) in Eq (3.2), we get:

$$\frac{1}{2}(F^2)' = \frac{1}{w^2 - k^2}\left[-\gamma U + \lambda U^3 - \sigma U^5\right], \tag{3.3}$$

where the prime denotes differentiation for ξ. From the integrating of Eq (3.3), we obtain:

$$F(U)^2 = \frac{1}{w^2 - k^2}\left[-\gamma U^2 + \frac{2\lambda}{4}U^4 - \frac{\sigma}{3}U^6\right]. \tag{3.4}$$

Using the traveling wave transformation (2.5), we have

$$(U_\xi)^2 = aU^2 + bU^4 + cU^6, \tag{3.5}$$

where

$$a = -\frac{\gamma}{w^2 - k^2}, b = \frac{\lambda}{2(w^2 - k^2)}, c = -\frac{\sigma}{3(w^2 - k^2)}.$$

By using the relations (16–40), we obtain exact solutions of conformable KG equation with quintic nonlinearity (1.2).

Case 1. If $\frac{\gamma}{w^2-k^2} < 0$, then (1.2) admits the following hyperbolic function solution

$$u_1(x,t) = \left[\frac{\frac{\gamma\lambda}{2} \sec h^2(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{\frac{\lambda^2}{2} - \frac{\gamma\sigma}{3}(1-\tanh(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})))^2} \right]^{\frac{1}{2}}. \tag{3.6}$$

Case 2. If $\frac{\gamma}{w^2-k^2} < 0$, $\frac{\sigma}{3(w^2-k^2)} < 0$, then (1.2) admits the following hyperbolic function solution

$$u_2(x,t) = \left[\frac{-\gamma \csc h^2(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{\frac{\lambda}{2} + 2\sqrt{\frac{\gamma\sigma}{3}} \coth(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))} \right]^{\frac{1}{2}}, \tag{3.7}$$

$$U_3(x,t) = \left[\frac{-4\gamma\left(\cosh(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) + \sinh(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right)}{\frac{4\gamma\sigma}{3(w^2-k^2)} - \left(\frac{\lambda}{2} + \cosh(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) + \sinh(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right)^2} \right]^{\frac{1}{2}}, \tag{3.8}$$

$$u_4(x,t) = \left[\frac{8\gamma^2 \sec h(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{\frac{\lambda^2}{4} + 4\gamma(\gamma-\frac{\sigma}{3}) + 2\gamma\lambda \sec h(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) + (\frac{\lambda^2}{4} - 4\gamma(\gamma-\frac{\sigma}{3}))\tanh(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))} \right]^{\frac{1}{2}}, \tag{3.9}$$

$$u_5(x,t) = \left[\frac{-\gamma \csc h(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{\frac{\lambda}{2}\sinh(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) + 2\sqrt{\frac{\gamma\sigma}{3}} \cosh(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))} \right]^{\frac{1}{2}}, \tag{3.10}$$

$$u_6(x,t) = \left[\frac{-\gamma \sec h(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{2\sqrt{\frac{\gamma\sigma}{3}}\sinh(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) - \frac{\lambda}{2} \cosh(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))} \right]^{\frac{1}{2}}. \tag{3.11}$$

Case 3. If $\frac{\gamma}{w^2-k^2} < 0$ and $\lambda^2 > \frac{16}{3}\gamma\sigma$, then (1.2) admits the following hyperbolic function solution

$$u_7(x,t) = \left[\frac{-2\gamma \sec h(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{-\frac{\lambda}{2}\sec h(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) \pm \sqrt{3\lambda^2 - 16\gamma\sigma}} \right]^{\frac{1}{2}}. \tag{3.12}$$

Case 4. If $\frac{\gamma}{w^2-k^2} < 0$ and $\lambda^2 < \frac{16}{3}\gamma\sigma$, then (1.2) admits the following hyperbolic function solution

$$u_8(x,t) = \left[\frac{-2\gamma \csc h(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{\pm\sqrt{16\gamma\sigma - 3\lambda^2} - \frac{\lambda}{2}\csc h(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))} \right]^{\frac{1}{2}}. \tag{3.13}$$

Case 5. If $\frac{\gamma}{w^2-k^2} < 0$ and $\lambda = \pm 4\sqrt{\frac{\gamma\sigma}{3}}$, then (1.2) admits the following hyperbolic function solution

$$u_9(x,t) = \left[-\frac{3\gamma}{\sigma}\left(1 \pm \tanh(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right)\right]^{\frac{1}{2}}, \tag{3.14}$$

$$u_{10}(x,t) = \left[-\frac{3\gamma}{\sigma}\left(1 \pm \coth(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right)\right]^{\frac{1}{2}}. \tag{3.15}$$

Case 6. If $\frac{\gamma}{w^2-k^2} > 0$ and $\frac{\sigma}{3(w^2-k^2)} < 0$, then (1.2) admits the following triangular function solution

$$u_{11}(x,t) = \left[\frac{-2\gamma}{-\frac{\lambda}{2} \pm \sqrt{3\lambda^2 - 16\gamma\sigma}\sin(2\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}\right]^{\frac{1}{2}}, \tag{3.16}$$

$$u_{12}(x,t) = \left[\frac{-2\gamma}{-\frac{\lambda}{2} \pm \sqrt{3\lambda^2 - 16\gamma\sigma}\cos(2\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}\right]^{\frac{1}{2}}, \tag{3.17}$$

$$u_{13}(x,t) = \left[\frac{-\gamma\sec^2(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{-\frac{\lambda}{2} + 2\sqrt{-\frac{\gamma\sigma}{3}}\tan(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}\right]^{\frac{1}{2}}, \tag{3.18}$$

$$u_{14}(x,t) = \left[\frac{-\gamma\csc^2(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{-\frac{\lambda}{2} + 2\sqrt{-\frac{\gamma\sigma}{3}}\cot(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}\right]^{\frac{1}{2}}, \tag{3.19}$$

$$u_{15}(x,t) = \left[\frac{\gamma\left(1 + (\tan(2\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) \pm \sec(2\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})))^2\right)}{\frac{\lambda}{2} - 2\sqrt{-\frac{\gamma\sigma}{3}}\tan(2\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) \pm \sec(2\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}\right]^{\frac{1}{2}}, \tag{3.20}$$

$$u_{16}(x,t) = \left[\frac{\gamma\csc(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{\frac{\lambda}{2}\sin(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) + 2\sqrt{-\frac{\gamma\sigma}{3}c}\cos(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}\right]^{\frac{1}{2}}, \tag{3.21}$$

$$u_{17}(x,t) = \left[\frac{-\gamma\sec(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}{2\sqrt{-\frac{\gamma\sigma}{3}}\sin(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha})) - \frac{\lambda}{2}\cos(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))}\right]^{\frac{1}{2}}. \tag{3.22}$$

Case 7. If $\frac{\gamma}{w^2-k^2} < 0$ and $\lambda = 0$, then (1.2) admits the following hyperbolic function solution

$$u_{18}(x,t) = \left[\pm\sqrt{\frac{3\gamma}{\sigma}}\csc h(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right]^{\frac{1}{2}}, \qquad (\frac{\sigma}{3(w^2-k^2)} < 0), \tag{3.23}$$

$$u_{19}(x,t) = \left[\pm\sqrt{-\frac{3\gamma}{\sigma}}\sec h(2\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right]^{\frac{1}{2}}, \qquad (\frac{\sigma}{3(w^2-k^2)} > 0). \tag{3.24}$$

Case 8. If $\frac{\gamma}{w^2-k^2} > 0$ and $\lambda = 0$, then (1.2) admits the following triangular function solution

$$u_{20}(x,t) = \left[\pm\sqrt{-\frac{3\gamma}{\sigma}}\csc(2\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right]^{\frac{1}{2}}, \qquad (\frac{\sigma}{3(w^2-k^2)} < 0), \tag{3.25}$$

$$u_{21}(x,t) = \left[\pm\sqrt{-\frac{3\gamma}{\sigma}}\sec(2\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right]^{\frac{1}{2}}, \qquad (\frac{\sigma}{3(w^2-k^2)} > 0). \tag{3.26}$$

Case 9. If $\frac{\gamma}{w^2-k^2} < 0$ and $\sigma = 0$, then (1.2) admits the following hyperbolic function solution

$$u_{22}(x,t) = \left[\frac{2\gamma}{\lambda}\csc h^2(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right]^{\frac{1}{2}}, \tag{3.27}$$

$$u_{23}(x,t) = \left[-\frac{2\gamma}{\lambda}\sec h^2(\sqrt{-\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right]^{\frac{1}{2}}. \tag{3.28}$$

Case 10. If $\frac{\gamma}{w^2-k^2} > 0$ and $\sigma = 0$, then ((1.2)) admits the following triangular function solution

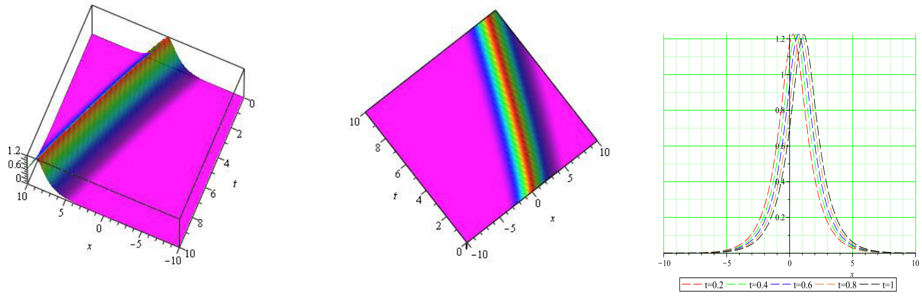
$$u_{24}(x,t) = \left[-\frac{2\gamma}{\lambda}\csc^2(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right]^{\frac{1}{2}}, \tag{3.29}$$

$$u_{25}(x,t) = \left[-\frac{2\gamma}{\lambda}\sec^2(\sqrt{\frac{\gamma}{w^2-k^2}}(x-\omega\frac{t^\alpha}{\alpha}))\right]^{\frac{1}{2}}. \tag{3.30}$$

4. Graphical representations

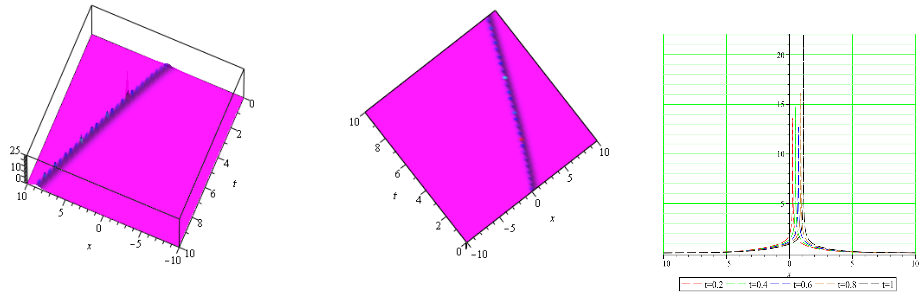
In this part, some graphical representations of exact wave solutions of conformable KG equation are presented in three different forms. 3D plots of exact solutions $|u_3|$, $|u_3|$, $|u_3|$ are displayed in Figures 1(a), 2(a), 3(a), respectively. Figures 1(b), 2(b), and 3(b) demonstrate the shape of contour plot of exact wave solutions $|u_3|$, $|u_3|$ and $|u_3|$. 2D line plot of exact wave solutions $|u_3|$, $|u_3|$ and $|u_3|$ are presented in Figures 1(c), 2(c), and 3(c) with $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$, $t = 1$.

Solitary wave solutions (3.6)–(3.15), (3.23), (3.24), (3.26) and (3.27) represent bell-profile and kink-profile solitary wave solutions, and solutions (3.16)–(3.22), (3.25) and (3.28) are triangular periodic wave solutions. These solutions may be useful to explain some physical phenomena in dynamical systems that are described by the system of conformable fractional equations for Klein-Gordon with quantic nonlinearity.



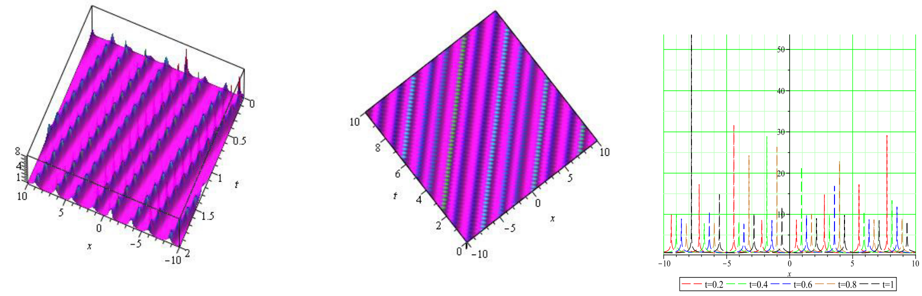
(a) **(b)** **(c)**

Figure 1. 3D-plot of the modulus (left), the contour plot (middle) and 2D-polar plot (right) parts of the exact wave solution of $|u_1|$ when $\sigma = 0.5,\ \omega = 1, \gamma = 1, \lambda = 1, k = 1.5$,and $\alpha = 0.9$.



(a) **(b)** **(c)**

Figure 2. 3D-plot of the modulus (left), the contour plot (middle) and 2D-polar plot (right) parts of the exact wave solution of $|u_2|$ when $\sigma = 3,\ \omega = 1, \gamma = 0.75, \lambda = 1.5, k = 2$,and $\alpha = 0.9$.



(a) **(b)** **(c)**

Figure 3. 3D-plot of the modulus (left), the contour plot (middle) and 2D-polar plot (right) parts of the exact wave solution of $|u_{11}|$ when $\sigma = -1,\ \omega = 2, \gamma = 1.5, \lambda = 2, k = 0.5$,and $\alpha = 1$.

5. Conclusions and outlook

We presented new exact solutions of conformable Klein-Gordon equation with quantic nonlinearity by using method of functional variable. Solutions were expressed in terms of solitary waves such as kink-profile and bell-profile. Moreover, we obtain exact periodic solutions of the KG equation. Computational results show that FV method is a highly efficient technique in the solutions of conformable PDEs. In a future research work, we will investigate the applicability of these results to some fractional-stochastic differential equations.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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