Research article

Generalized \((\alpha, \beta, \gamma)\)-derivations on Lie \(C^*\)-algebras

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Abstract: The Hyers-Ulam stability of \((\alpha, \beta, \gamma)\)-derivations on Lie \(C^*\)-algebras is discussed by following functional inequality

\[ f(ax + by) + f(ax - by) = 2f(ax) + bf(y) + bf(-y), \]

where \(a, b\) are nonzero fixed complex numbers.

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1. Introduction and preliminaries

The derivation theory of Lie algebras play a key role in Lie theory. In particular, Physically motivated relations between two Lie algebras have been extensively discussed [27]. The problems for the structures and characteristics of \((\alpha, \beta, \gamma)\)-derivations of Lie algebras have been extensively investigated by a range of scholars, as for this, many scholars have made useful researches (see [22, 28, 37]). The authors set up the structure and properties of \((\alpha, \beta, \gamma)\)-derivations of Lie algebras.

In this work, the definition of a Lie \(C^*\)-algebra come from [29, 30, 34]). In [28], the definition of \((\alpha, \beta, \gamma)\)-derivation can be found.

1940, the stability problem of group homomorphisms was raised by Ulam [38]. In 1941, Hyers [20] answers this question with a qualified yes to the question of Ulam for additive groups in Banach spaces. Hyers’ theorem was generalized by Aoki [2], Rassias [35] and Găvruta [17] for linear mappings. In recent years, a lot of experts and scholars have studied in this area and made many achievements
Gilányi [18] and [36] considered the functional inequality
\[ \|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\| \] (1.1)
then \( f \) satisfies the Jordan-von Neumann functional equation
\[ 2f(x) + 2f(y) = f(x + y) + f(x - y), \]
respectively. The Hyers-Ulam stability of the above functional inequality is discussed by Fechner [16] and Gilányi [19]. Park [31, 32] gave the definition of additive \( \rho \)-functional inequalities and discussed the Hyers-Ulam stability of the additive \( \rho \)-functional inequalities in different spaces.

To obtain a Jordan and von Neumann type characterization theorem for the quasi-inner-product spaces, Drygas [11] considered the functional equation
\[ f(x + y) + f(x - y) = 2f(x) + f(y), \]
which solution is called a Drygas mapping. The general solution of the above functional equation was given by Ebanks, Kannappan and Sahoo [13] as
\[ f(x) = Q(x) + A(x), \]
here \( A \) is an additive mapping and \( Q \) is a quadratic mapping.

In this work, we consider the stability of \((\alpha, \beta, \gamma)\)-derivations on Lie \( C^* \)-algebras by the general Drygas functional equation
\[ f(ax + by) + f(ax - by) = f(2ax) + bf(y) + bf(-y), \] (1.3)
the coefficients \( a, b \) is complex number, the proof of stability of the (1.3) is difference in [13]. The additive mapping \( A \) and quadratic mapping \( Q \) is constructed by the function relations, this method is called “directed method”. In the (1.3), \( a, b \) action will cause difficulties for the stability of functional inequalities. We can overcome the influence of \( a, b \), the stability of \((\alpha, \beta, \gamma)\)-derivations using the fixed method. The beautiful examples about \((\alpha, \beta, \gamma)\)-derivations can be found in [41].

The Hyers-Ulam stability analysis on \( C^* \)-algebras about functional equations have been discussed by fixed point theorem (see [5, 8, 14, 15, 21]).

Next, the concept of the “generalized complete metric space” is introduced following Luxemburg [26].

**Definition 1.1.** Let \( X \) be an abstract (nonempty) set, the elements of which are denoted by \( x, y, \cdots \) and assume that on the Cartesian product \( X \times X \) a distance function \( d(x, y)(0 \leq d(x, y) \leq \infty) \) is defined, satisfying the following conditions

1. \( d(x, y) = 0 \) if and only if \( x = y \),
2. \( d(x, y) = d(y, x) \) (symmetry),
3. \( d(x, y) \leq d(x, z) + d(z, y) \) (triangle inequality),
4. every \( d \)-Cauchy sequence in \( X \) is \( d \)-convergent, i.e. \( \lim_{n \to \infty} d(x_n, x_m) = 0 \) for a sequence \( x_n \in X(n = 1, 2, \cdots) \) implies the existence of an element \( x \in X \) with \( \lim_{n \to \infty} d(x, x_n) = 0 \), (\( x \) is unique).
By the concept, every two points in $X$ may be have the infinite distance. The space is called a generalized complete metric space.

We recall fixed point theorem that plays an key role to prove the stability of derivation.

**Theorem 1.2.** \([4, 10]\) Let $(X, d)$ be a complete generalized metric space and $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for any $x \in X$, either

\[ d(J^n x, J^{n+1} x) = \infty \]

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty$, for all $n \geq n_0$;
2. the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X | d(J^n x, y) < \infty\}$;
4. $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Now, using some thoughts from \([4, 10, 15]\) we discuss the stability for $(\alpha, \beta, \gamma)$-derivations and Lie $C^*$-algebra homomorphisms on Lie $C^*$-algebras related to (1.3) via the above fixed point theorem.

### 2. The stability of $(\alpha, \beta, \gamma)$-derivations

Now, suppose that $s$ is complex fixed point and $A$ is a Lie $C^*$-algebra with norm $\| \cdot \|$. The following lemma is necessary to prove our main theorems.

**Lemma 2.1.** \([30]\) Suppose $X$ and $Y$ are linear spaces, $f : X \rightarrow Y$ is an additive map satisfying $f(\mu x) = \mu f(x)$, $\forall x \in X$ and $\mu \in T^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$. Then $f$ is $\mathbb{C}$-linear.

**Lemma 2.2.** Assume $f : \mathcal{A} \rightarrow \mathcal{A}$ is a map satisfying

\[ ||f(ax + by) + f(ax - by) - f(2ax) - bf(y) - bf(-y)|| \leq ||s(f(ax - by) + f(ax + by) - f(2ax))|| \]  

(2.1)

$\forall x, y \in \mathcal{A}$, $|s| \leq |1 - 2b| \leq 1$. Then $f$ is additive.

**Proof.** If $x = y = 0$ in (2.1), then $f(0) = 0$. If $x = \frac{b}{a} y$ in (2.1) with $b \neq 0$, one obtain $f(-y) = -f(y)$.

Next, we discuss that $f$ is additive. Since $f(-y) = -f(y)$ in (2.1),

\[ f(ax + by) + f(ax - by) - f(2ax) = 0 \]

for $\forall x, y \in \mathcal{A}$. So $f$ is additive. \(\square\)

**Theorem 2.3.** If there are a mapping $\phi : \mathcal{A}^2 \rightarrow [0, \infty)$

\[ \frac{1}{2} \phi(2x, 2y) \leq L\phi(x, y), \quad \forall x, y \in \mathcal{A}; \]  

(2.2)

and a mapping $\psi : \mathcal{A}^2 \rightarrow [0, \infty)$ with a constant $0 < L < 1$

\[ \psi \left( \frac{x}{2}, \frac{y}{2} \right) \leq L^2 \frac{1}{2^2} \psi(x, y), \quad \forall x, y \in \mathcal{A}. \]  

(2.3)
Let $f : \mathcal{A} \to \mathcal{A}$ satisfy
\[
\|f(ax + by) + f(ax - by) - \mu f(2ax) - bf(y) - bf(-y)\| \\
\leq \|s(f(ax - by) + f(ax + by) - \mu f(2ax))\| + \phi(x, y),
\] (2.4)
\[
\|\alpha f[x, y] - \beta[f(x), y] - \gamma[x, f(y)]\| \leq \psi(x, y),
\] (2.5)
\forall x, y \in \mathcal{A}, \mu \in T^1, some \alpha, \beta, \gamma, a, b and |s| \leq |1 - 2b| \leq 1. Then we can find a unique $(\alpha, \beta, \gamma)$-derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfies (1.3) and
\[
\|f(x) - \delta(x)\| \leq \frac{1}{2(1 - |s|)(1 - L)} \phi\left(\frac{x}{a}, 0\right), \quad \forall x \in \mathcal{A}.
\] (2.6)

**Proof.** Suppose $\Omega$ is a set of all mappings from $\mathcal{A}$ into $\mathcal{A}$, on $\Omega$, a generalized metric is introduced,
\[
d(g, h) = \inf \left\{ C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C \phi\left(\frac{x}{a}, 0\right), \forall x \in \mathcal{A}\right\}.
\]

Then $(\Omega, d)$ becomes a generalized complete metric space. One define a map $T : \Omega \to \Omega$ by
\[
Tg(x) = \frac{1}{2}g(2x), \forall g \in \Omega, x \in \mathcal{A}.
\]

Let $g, h \in \Omega$ with $d(g, h) \leq C$, here $C \in (0, \infty)$ is an arbitrary constant. Then we obtain $\|g(x) - h(x)\| \leq C \phi\left(\frac{x}{a}, 0\right)$,
\[
\|Tg(x) - Th(x)\| \leq C \phi(2x, 0) \leq LC\phi(x, 0), \forall x \in \mathcal{A},
\]
i.e. $d(Tg - Th) \leq Ld(g, h), \forall g, h \in \Omega$. Therefore, $T$ is a strictly contractive self-mapping on $\Omega$ associated with the Lipschitz constant $L$.

If $x = y = 0$ in (2.4), $f(0) = 0$.

If $y = 0$ and $\mu = 1$ in (2.4), then
\[
\|2f(ax) - f(2ax)\| \leq |s|\|2f(ax) - f(2ax)\| + \phi(x, 0), \quad \forall x \in \mathcal{A}.
\]

Thus
\[
\left\| \frac{f(2x)}{2} - f(x) \right\| \leq \frac{1}{1 - |s|} \frac{1}{2} \phi\left(\frac{x}{a}, 0\right)
\]
for $\forall x \in \mathcal{A}$. Then we have $d(Tf, f) \leq \frac{1}{2(1 - |s|)}$. By Theorem 1.2, there is a unique fixed point of $T$, map $\delta$, in the set $\Omega_1 = \{g \in \Omega : d(f, g) < \infty\}$,
\[
\delta(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x), \forall x \in \mathcal{A},
\] (2.7)
since $\lim_{n \to \infty} d(T^n f, \delta) = 0$. Again by Theorem 1.2,
\[
d(f, \delta) \leq \frac{1}{1 - L} d(Tf, f) \leq \frac{1}{2(1 - |s|)(1 - L)}, \forall x \in \mathcal{A}.
\]
Then (2.6) holds.

By (2.4) and (2.7) and the property of $\phi$,

$$\|\delta(\alpha x + by) + \delta(\alpha x - by) - \mu \delta(2ax) - b\delta(y) - b\delta(-y)\|$$

$$= \lim_{n \to \infty} \frac{1}{2^n} \|f(2^n\alpha x + 2^ny) + f(2^n\alpha x - 2^ny) - \mu f(2a2^n x)$$

$$- bf(2^ny) - bf(-2^ny)\|$$

$$\leq \lim_{n \to \infty} \frac{1}{2^n} \|s(f(\alpha x^2 + b2^ny) + f(\alpha 2^ny - b2^ny) - \mu f(2a2^n x))\|$$

$$+ \lim_{n \to \infty} \frac{1}{2^n} \phi(2^n x, 0)$$

$$\leq \|s(\delta(\alpha x + by) + \delta(\alpha x - by) - \mu \delta(2ax))\| + \lim_{n \to \infty} L^n \phi(x, 0).$$

That is, $\delta$ is additive by Lemma 2.2. Next, letting $y = 0$, we get $2\delta(\alpha x) = \mu \delta(2ax)$ and so the map $\delta$ is $\mathbb{C}$-linear. Therefore, by the property of $\psi$, (2.5) and (2.7), then

$$\|\alpha \delta[x, y] - \beta[\delta(x), y] - \gamma[x, \delta(y)]\|$$

$$= \lim_{n \to \infty} 4^n\|\alpha f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - \beta[f\left(x/2^n\right), y/2^n] - \gamma[x/2^n, f(y/2^n)]\|$$

$$\leq \lim_{n \to \infty} 4^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$$

$$\leq \lim_{n \to \infty} L^{2^n} \psi(x, y) = 0$$

for $\forall x, y \in \mathcal{A}$, some $\alpha, \beta$ and $\gamma \in \mathbb{C}$. Thus

$$\alpha \delta[x, y] = \beta[\delta(x), y] + \gamma[x, \delta(y)], \forall x, y \in \mathcal{A},$$

for some $\alpha, \beta$ and $\gamma \in \mathbb{C}$. Hence $\delta$ is an unique derivation satisfying (2.6). \hfill \square

**Corollary 2.4.** If $r, k$ and $\theta$ belong to real numbers, $0 < r < 1, 0 < k < 2$ and $\theta \geq 0$. Let the map $f : \mathcal{A} \to \mathcal{A}$ satisfy

$$\|f(\alpha x + by) + f(\alpha x - by) - \mu f(2ax) - b f(y) - b f(-y)\|$$

$$\leq \|s(f(\alpha x - by) + f(\alpha x + by) - \mu f(2ax))\| + \theta(||x||^r + ||y||^r),$$

$$\|\alpha f[x, y] - \beta[f(x), y] - \gamma[x, f(y)]\| \leq \theta(||x||^k + ||y||^k)$$

for $\forall x, y \in \mathcal{A}, \mu \in T^1$ and $|s| \leq |1 - 2b| \leq 1$. Then we can find a unique $(\alpha, \beta, \gamma)$-derivation $\delta : \mathcal{A} \to \mathcal{A}$,

$$\|f(x) - \delta(x)\| \leq \frac{1}{(1 - |s|\alpha r (2 - 2^r))} ||x||^r$$

for $\forall x \in \mathcal{A}$.

**Proof.** Let $\phi(x, y) = \theta(||x||^r + ||y||^r), \psi(x, y) = \theta(||x||^k + ||y||^k)$ and $L = 2^{r-1}$ in Theorem 2.3, the desired result is obtained. \hfill \square
Theorem 2.5. If there exists a map $\psi : \mathcal{A}^2 \to [0, \infty)$ satisfying (2.3). Let a map $f : \mathcal{A} \to \mathcal{A}$ satisfy

$$\|f(\alpha x + y) + f(\alpha x - y) - \mu f(2ax) - bf(y) - bf(-y)\|$$

$$\leq \|s(f(\alpha x - by) + f(\alpha x + by) - \mu f(2ax))\|,$$

(2.8)

$$\|\alpha f[x, y] - \beta[f(x), y] - \gamma[x, f(y)]\| \leq \psi(x, y)$$

(2.9)

for $\forall x, y \in \mathcal{A}, \mu \in T^1$ and $|s| \leq |1 - 2b| \leq 1$. Thus the map $f : \mathcal{A} \to \mathcal{A}$ is a $(\alpha, \beta, \gamma)$-derivation.

Proof. Let $\mu = 1$ in (2.8), the map $f$ is additive by Lemma 2.2. Let $y = 0$ in (2.8), we get

$$\|2f(\alpha x) - \mu f(2ax)\| \leq 0$$

for $\forall x \in \mathcal{A}, \mu \in T^1$. So $f(\alpha x) = \mu f(x), \forall x \in \mathcal{A}$ and $\mu \in T^1$. The map $f$ is $C$-linear by Lemma 2.1. On account of $f$ is additive, by (2.9),

$$\|\alpha f([x, y]) - \beta[f(x), y] - \gamma[x, f(y)]\|$$

$$= \lim_{n \to \infty} 4^n\|\alpha f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - \beta\left[f\left(\frac{x}{2^n}\right), \frac{y}{2^n}\right] - \gamma\left[\frac{x}{2^n}, f\left(\frac{y}{2^n}\right)\right]\|

\leq \lim_{n \to \infty} L^{2n}\psi(x, y) = 0$$

for $\forall x, y \in \mathcal{A}$. Thus

$$\alpha f([x, y]) = \beta[f(x), y] + \gamma[x, f(y)], \forall x, y \in \mathcal{A}.$$

\(\Box\)

Corollary 2.6. If $k$ and $\theta$ belong to real numbers with $0 < k < 2$ and $\theta \geq 0$. Assume a map $f : \mathcal{A} \to \mathcal{A}$ satisfies

$$\|f(\alpha x + by) + f(\alpha x - by) - \mu f(2ax) - bf(y) - bf(-y)\|$$

$$\leq \|s(f(\alpha x - by) + f(\alpha x + by) - \mu f(2ax))\|,$$

$$\|\alpha f[x, y] - \beta[f(x), y] - \gamma[x, f(y)]\| \leq \theta(||x||^k + ||y||^k)$$

for $\forall x, y \in \mathcal{A}, \mu \in T^1$ and $|s| \leq |1 - 2b| \leq 1$. Then the map $f$ is a $(\alpha, \beta, \gamma)$-derivation.

Lemma 2.7. If $f : \mathcal{A} \to \mathcal{A}$ is a map satisfying

$$\|f(ax + by) + f(ax - by) - f(2ax) - bf(y) - bf(-y)\|$$

$$\geq \|s(f(ax - by) + f(ax + by) - f(2ax))\|$$

for $\forall x, y \in \mathcal{A}, |s| \geq |1 - 2b| \geq 1$. Then $f$ is additive.

Proof. Using the same technique with the Lemma 2.2, we can show that the Lemma 2.7. \(\Box\)
Theorem 2.8. Assume the map $\phi : \mathcal{A}^2 \to [0, \infty)$ satisfies (2.2) and a map $\psi : \mathcal{A}^2 \to [0, \infty)$ satisfies (2.3). Let the map $f : \mathcal{A} \to \mathcal{A}$ satisfy
\[
\|f(ax + by) + f(ax - by) - \mu f(2ax) - bf(y) - bf(-y)\| \\
\geq \|s(f(ax - by) + f(ax + by) - \mu f(2ax))\| - \phi(x, y),
\]
\[
\|\alpha f[x, y] - \beta f(x, y) - \gamma[x, f(y)]\| \leq \psi(x, y)
\]
for $\forall x, y \in \mathcal{A}, \mu \in T^1$, some $\alpha, \beta, \gamma, a, b$, and $|s| \geq |1 - 2b| \geq 1$. Then we can find a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying (1.3), and
\[
\|f(x) - \delta(x)\| \leq \frac{1}{2(1 - |s|)(1 - L)} \phi\left(\frac{x}{a}, 0\right)
\]
for $\forall x \in \mathcal{A}$.

Proof. In a similar vein of Theorem 2.3, the theorem can be proved. □

Corollary 2.9. Suppose $r, k, \theta \in \mathbb{R}$ and $0 < r < 1$, $0 < k < 2$, $\theta \geq 0$, let the map $f : \mathcal{A} \to \mathcal{A}$ satisfy
\[
\|f(ax + by) + f(ax - by) - \mu f(2ax) - bf(y) - bf(-y)\| \\
\geq \|s(f(ax - by) + f(ax + by) - \mu f(2ax))\| - \theta(||x||^r + ||y||^r),
\]
\[
\|\alpha f[x, y] - \beta f(x, y) - \gamma[x, f(y)]\| \leq \theta(||x||^k + ||y||^k)
\]
for $\forall x, y \in \mathcal{A}, \mu \in T^1$, some $\alpha, \beta, \gamma, a, b$, and $|s| \geq |1 - 2b| \geq 1$. Then there is only one $(\alpha, \beta, \gamma)$-derivation $\delta : \mathcal{A} \to \mathcal{A}$ satisfying
\[
\|f(x) - \delta(x)\| \leq \frac{1}{(1 - |s|)|a|^r(2 - 2^r)||x||^r}
\]
for $\forall x \in \mathcal{A}$.

Proof. In Theorem 2.8, let $\phi(x, y) = \theta(||x||^r + ||y||^r), \psi(x, y) = \theta(||x||^k + ||y||^k)$, $\forall x, y \in \mathcal{A}$ and $L = 2^{r-1}$, then the Corollary is proved. □

Theorem 2.10. If the map $\psi : \mathcal{A}^2 \to [0, \infty)$ satisfies (2.3). The map $f : \mathcal{A} \to \mathcal{A}$ satisfies
\[
\|f(ax + by) + f(ax - by) - \mu f(2ax) - bf(y) - bf(-y)\| \\
\geq \|s(f(ax - by) + f(ax + by) - \mu f(2ax))\|,
\]
\[
\|\alpha f[x, y] - \beta f(x, y) - \gamma[x, f(y)]\| \leq \psi(x, y)
\]
for $\forall x, y \in \mathcal{A}, \mu \in T^1$, some $\alpha, \beta, \gamma, a, b$, and $|s| \geq |1 - 2b| \geq 1$. Then the map $f : \mathcal{A} \to \mathcal{A}$ is a $(\alpha, \beta, \gamma)$-derivation.

Corollary 2.11. If $k, \theta \in \mathbb{R}$, $0 < k < 2$, $\theta \geq 0$, assume the map $f : \mathcal{A} \to \mathcal{A}$ satisfies
\[
\|f(ax + by) + f(ax - by) - \mu f(2ax) - bf(y) - bf(-y)\| \\
\geq \|s(f(ax - by) + f(ax + by) - \mu f(2ax))\|,
\]
\[
\|\alpha f[x, y] - \beta f(x, y) - \gamma[x, f(y)]\| \leq \theta(||x||^k + ||y||^k)
\]
for $\forall x, y \in \mathcal{A}, \mu \in T^1$, $|s| \geq |1 - 2b| \geq 1$. Then the map $f : \mathcal{A} \to \mathcal{A}$ is a $(\alpha, \beta, \gamma)$-derivation.
3. Conclusions

In this work, the general Drygas functional equation is introduced, the Hyers-Ulam stability of $(\alpha, \beta, \gamma)$-derivations on Lie $C^*$-algebras is discussed by general Drygas functional inequality with the participation of coefficient $a$ and $b$.

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Conflict of interest

The authors of this paper declare that they have no conflict of interest.

References


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