AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 6856–6873.
DOI: 10.3934/math.2020440
Received: 11 June 2020
Accepted: 25 August 2020
Published: 04 September 2020

Research article

New inequalities via n-polynomial harmonically exponential type convex functions

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Abstract: In this work we introduced a new class of functions called n-polynomial harmonically exponential type convex and study some of their algebraic properties. Several new inequalities via n-polynomial harmonically exponential type convexity are established. Some special cases for suitable choices of parameters are given in details.

Keywords: Hermite-Hadamard inequality; Hölder's inequality; harmonically convexity; n-polynomial harmonically convexity
Mathematics Subject Classification: 26A51, 26A33, 26D07, 26D10, 26D15

1. Introduction

Theory of convexity have a lot of applications in pure and applied mathematics and played an important and fundamental role in the developments of various branches of engineering, financial mathematics, economics and optimization. In recent years, the concept of convex functions and its variant forms have been extended and generalized using innovative techniques to study complicated problems. It is well known that convexity is closely related to inequality theory.

A significant class of convex functions, called harmonic convex was introduced by Anderson et al. [2] and İşcan [15] independently. Noor [20, 21] have shown that the optimality conditions of the differentiable harmonic convex functions on the harmonic convex set can be expressed by a class of variational inequality, which he called the harmonic variational inequality. Recently Mahir Kadakal and İşcan [24] introduced a generalized form of convexity namely n-polynomial convex function. Noor [4], keeping his work in the field of convex analysis, introduced a new generalized class of convex function called n-polynomial harmonic convex function. Due to widespread views and applications, many mathematicians put an effort, hardworking to collaborate on different ideas and concepts in the field of convex analysis. Many generalizations, variants and extensions for the convexity have attracted the attention of many researchers, see [6, 10, 18, 24].

After studying literature about convex analysis, motivated and inspired by the ongoing activities and research in this dynamic field, we found out that there exists a special class of function known as exponential convex function and nowadays a lot of peoples are working in this field. Antczak [3] and Dragomir [8] introduced the class of exponential type convexity. After Antczak and Dragomir, Awan et al. [5] studied and investigated a new class of exponentially convex functions. Recently Mahir Kadakal and İşcan introduced a new definition of exponential type convexity in [16]. The fruitful importance of exponential type convexity is used to manipulate for statistical learning, stochastic optimization and sequential prediction (see [1, 22, 24] and the references therein). Interested readers are referred to [9, 11].

The aim of this paper is to introduce a new class of functions called n-polynomial harmonically exponential type convex and study some of their algebraic properties. Several new inequalities via n-polynomial harmonically exponential type convexity are establish. The interesting techniques and the fruitful ideas of this paper may stimulate further research in this dynamic field. Before we start, we need the following necessary known definitions.

2. Preliminaries

Let $\psi : I \to \Re$ be a real valued function. A function ψ is said to be convex, if

$$\psi\left(t\mu_1 + (1-t)\mu_2\right) \le t\psi\left(\mu_1\right) + (1-t)\psi\left(\mu_2\right) \tag{2.1}$$

holds for all $\mu_1, \mu_2 \in I$ and $t \in [0,1]$. Any paper on Hermite inequalities seems to be incomplete without mentioning the well-known Hermite-Hadamard inequality which states: if $\psi : I \to \Re$ is a convex function for all $\mu_1, \mu_2 \in I$, then

$$\psi\left(\frac{\mu_1+\mu_2}{2}\right) \le \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\psi(x)dx \le \frac{\psi(\mu_1)+\psi(\mu_2)}{2}. \tag{2.2}$$

Since the researchers have shown keen interest in above inequality, as a result various generalizations and improvements have been appeared in the literature. Interested readers can refer to [2–19, 23–25].

Definition 2.1. [15] A function $\psi : H \subseteq (0,+\infty) \to \Re$ is said to be harmonically convex, if

$$\psi\left(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\right) \le t\psi(\mu_1) + (1-t)\psi(\mu_2) \tag{2.3}$$

holds for all $\mu_1, \mu_2 \in H$ and $t \in [0,1]$.

For the harmonically convex function, İşcan [15] provided the Hermite-Hadamard type inequality.

Theorem 2.2. *[15] Let $\psi : H \subseteq (0, +\infty) \to \mathfrak{R}$ be a harmonically convex function. If $\psi \in L[\mu_1, \mu_2]$ for all $\mu_1, \mu_2 \in H$ with $\mu_1 < \mu_2$, then*

$$\psi\Big(\frac{2\mu_1\mu_2}{\mu_1 + \mu_2}\Big) \leq \frac{\mu_1\mu_2}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \frac{\psi(x)}{x^2} dx \leq \frac{\psi(\mu_1) + \psi(\mu_2)}{2}. \tag{2.4}$$

Definition 2.3. [24] A nonnegative function $\psi : I \to \mathfrak{R}$ is called n–polynomial convex, if

$$\psi\left(t\mu_1 + (1-t)\mu_2\right) \leq \frac{1}{n}\sum_{i=1}^{n}\left[1 - (1-t)^i\right]\psi\left(\mu_1\right) + \frac{1}{n}\sum_{i=1}^{n}\left[1 - t^i\right]\psi\left(\mu_2\right) \tag{2.5}$$

holds for every $\mu_1, \mu_2 \in I$, $n \in \mathbb{N}$ and $t \in [0, 1]$.

Definition 2.4. [4] A nonnegative function $\psi : I \to [0, +\infty)$ is called n–polynomial harmonically convex, if

$$\psi\Big(\frac{\mu_1\mu_2}{t\mu_2 + (1-t)\mu_1}\Big) \leq \frac{1}{n}\sum_{i=1}^{n}\left[1 - (1-t)^i\right]\psi\left(\mu_1\right) + \frac{1}{n}\sum_{i=1}^{n}\left[1 - t^i\right]\psi\left(\mu_2\right) \tag{2.6}$$

holds for every $\mu_1, \mu_2 \in I$, $n \in \mathbb{N}$ and $t \in [0, 1]$.

Remark 1. If we put $n = 1$ in definition 2.4, then we get definition 2.1.

Remark 2. Every nonnegative harmonic convex function is also an n-polynomial harmonic convex function. Indeed for all $t \in [0, 1]$, it follows from the following inequalities $t \leq \frac{1}{n}\sum_{i=1}^{n}[1 - (1-t)^i]$ *and* $1 - t \leq \frac{1}{n}\sum_{i=1}^{n}[1 - t^i]$.

Motivated by the above results and references, in Section 3 we give the idea and explore some of the algebraic properties of n-polynomial harmonically exponential type convex function. In Section 4, we derive the new version of Hermite-Hadamard inequality by using the new introduced definition. In Section 5, several new special cases are discussed in details for suitable choices of parameters. In the last Section, a brief conclusion is provided as well.

3. Algebraic properties of n-polynomial harmonically exponential type convex function

We are going to introduce a new definition called n–polynomial harmonically exponential type convex function.

Definition 3.1. A nonnegative real valued function $\psi : H \subseteq (0, +\infty) \to [0, +\infty)$ is called n–polynomial harmonically exponential type convex, if

$$\psi\Big(\frac{\mu_1\mu_2}{t\mu_2 + (1-t)\mu_1}\Big) \leq \frac{1}{n}\sum_{i=1}^{n}\left(e^t - 1\right)^i\psi\left(\mu_1\right) + \frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t} - 1\right)^i\psi\left(\mu_2\right) \tag{3.1}$$

holds for every $\mu_1, \mu_2 \in H$, $n \in \mathbb{N}$ and $t \in [0, 1]$.

Remark 3. Taking $n = 1$ in definition 3.1, we obtain the following new definition about harmonically exponential type convex function

$$\psi\Big(\frac{\mu_1\mu_2}{t\mu_2 + (1-t)\mu_1}\Big) \leq \left(e^t - 1\right)\psi\left(\mu_1\right) + \left(e^{1-t} - 1\right)\psi\left(\mu_2\right). \tag{3.2}$$

Example 1. *$\psi(x) = ln(x)$ is harmonically exponential type convex, since it is harmonically convex function for positive values of x.*

Example 2. *$\psi(x) = \sqrt{x}$ is harmonically exponential type convex, since it is harmonically convex function for nonnegative values of x.*

Example 3. *$\psi(x) = 1/x^2$ is harmonically exponential type convex, since it is harmonically convex function for positive values of x.*

Remark 4. If we take $n = 2$ in definition 3.1, we obtain the following new definition about 2-polynomial harmonically exponential type convex function

$$\psi\left(\frac{\mu_1\mu_2}{t\mu_2 + (1-t)\mu_1}\right) \leq \left(\frac{e^{2t} - e^t}{2}\right)\psi(\mu_1) + \left(\frac{e^{2(1-t)} - e^{1-t}}{2}\right)\psi(\mu_2). \tag{3.3}$$

This is a clear advantage of the proposed new definition with respect to other known functions on the topic mentioned above.

Remark 5. It's easy to show that, if the function ψ is n–polynomial harmonically convex then ψ is n–polynomial harmonically exponential type convex. Indeed for all $t \in [0, 1]$, the following inequalities hold:

$$e^t \geq t \quad \text{and} \quad e^{1-t} \geq 1 - t.$$

This mean that, the new class of n-polynomial harmonically exponential type convex function is very larger with respect the known class of functions like n-polynomial convex and n-polynomial harmonically convex. This is an advantage of the proposed new definition 3.1.

Example 4. *$\psi(x) = x^2 e^{x^2}$ is non-decreasing convex function on $(0, 1)$, so it is harmonic convex function (see [7]). By using Remark 2, it is n-polynomial harmonic convex function and by using Remark 5, it is n-polynomial harmonic exponential type convex function.*

Example 5. *$\psi(x) = e^x$ is increasing convex function, so it is harmonic convex function (see [7]). So according to Remark 2 and Remark 5, it is n-polynomial harmonic exponential type convex function.*

Example 6. *$\psi(x) = \sin(-x)$ is non-decreasing convex function on $(0, \frac{\pi}{2})$, so it is harmonic convex function $\forall x \in (0, \frac{\pi}{2})$ (see [7]). So by using Remark 2 and Remark 5, it is n-polynomial harmonic exponential type convex function.*

Example 7. *$\psi(x) = x$ is non-decreasing convex function on $(0, \infty)$, so it is harmonic convex function $\forall x \in (0, \infty)$ (see [7]). So by the following Remark 2 and Remark 5, we get $\psi(x)$ is n-polynomial harmonic exponential type convex function.*

Example 8. *$\psi(x) = \ln x$ is harmonic convex function on $(0, \infty)$ (see [7]). So by the following Remark 2 and Remark 5, we get $\psi(x)$ is n-polynomial harmonic exponential type convex function.*

Now, we will study some of their algebraic properties.

Theorem 3.2. *Let $\psi, \psi_1, \psi_2 : H \subseteq (0, +\infty) \to [0, +\infty)$. If ψ, ψ_1 and ψ_2 are three n–polynomial harmonically exponential type convex functions, then*

(1) $\psi_1 + \psi_2$ is n–polynomial harmonically exponential type convex function;

(2) For nonnegative real number c, $c\psi$ is n–polynomial harmonically exponential type convex function.

Proof. (1) Let ψ_1 and ψ_2 be n-polynomial harmonically exponential type convex function, then

$$\begin{aligned}
&(\psi_1+\psi_2)\left(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\right)\\
&=\psi_1\left(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\right)+\psi_2\left(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\right)\\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left(e^t-1\right)^i\psi_1(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^i\psi_1(\mu_2)\\
&+\frac{1}{n}\sum_{i=1}^{n}\left(e^t-1\right)^i\psi_2(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^i\psi_2(\mu_2)\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(e^t-1\right)^i\left[\psi_1(\mu_1)+\psi_2(\mu_1)\right]+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^i\left[\psi_1(\mu_2)+\psi_2(\mu_2)\right]\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(e^t-1\right)^i(\psi_1+\psi_2)(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^i(\psi_1+\psi_2)(\mu_2).
\end{aligned}$$

(2) Let ψ be n–polynomial harmonically exponential type convex function, then

$$\begin{aligned}
&(c\psi)\left(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\right)\\
&\leq c\left[\frac{1}{n}\sum_{i=1}^{n}\left(e^t-1\right)^i\psi(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^i\psi(\mu_2)\right]\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(e^t-1\right)^i c\psi(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^i c\psi(\mu_2)\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(e^t-1\right)^i(c\psi)(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^i(c\psi)(\mu_2).
\end{aligned}$$

□

Theorem 3.3. *Let $\psi_1 : H \to [0,+\infty)$ be harmonically convex function and $\psi_2 : [0,+\infty) \to [0,+\infty)$ is non-decreasing and n-polynomial exponential type convex function. Then the function $\psi_2\circ\psi_1 : H \to [0,+\infty)$ is n-polynomial harmonically exponential type convex.*

Proof. For all $\mu_1,\mu_2 \in H$, and $t \in [0,1]$, we have

$$\begin{aligned}
&(\psi_2\circ\psi_1)\left(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\right)\\
&=\psi_2\left(\psi_1\left(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\right)\right)\\
&\leq \psi_2\left(t\psi_1(\mu_1)+(1-t)\psi_1(\mu_2)\right)\\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left(e^t-1\right)^i\psi_2(\psi_1(\mu_1))+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^i\psi_2(\psi_1(\mu_2))
\end{aligned}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(e^{t}-1\right)^{i}\left(\psi_2\circ\psi_1\right)(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^{i}\left(\psi_2\circ\psi_1\right)(\mu_2).$$

□

Theorem 3.4. *Let* $0<\mu_1<\mu_2$, $\psi_j:[\mu_1,\mu_2]\to[0,+\infty)$ *be a family of n–polynomial harmonically exponential type convex functions and* $\psi(u)=\sup_j\psi_j(u)$. *Then* ψ *is an n–polynomial harmonically exponential type convex function and* $U=\{u\in[\mu_1,\mu_2]:\psi(u)<+\infty\}$ *is an interval.*

Proof. Let $\mu_1,\mu_2\in U$ and $t\in[0,1]$, then

$$\begin{aligned}
&\psi\left(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\right)\\
&=\sup_j\psi_j\left(\frac{\mu_1\mu_2}{(t\mu_2+(1-t)\mu_1)}\right)\\
&\le\frac{1}{n}\sum_{i=1}^{n}\left(e^{t}-1\right)^{i}\sup_j\psi_j(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^{i}\sup_j\psi_j(\mu_2)\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(e^{t}-1\right)^{i}\psi(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^{i}\psi(\mu_2)<+\infty,
\end{aligned}$$

which completes the proof. □

Theorem 3.5. *If the function* $\psi:[\mu_1,\mu_2]\subseteq(0,+\infty)\to[0,+\infty)$ *is n–polynomial harmonically exponential type convex, then* ψ *is bounded on* $[\mu_1,\mu_2]$.

Proof. Let $L=\max\{\psi(\mu_1),\psi(\mu_2)\}$ and $x\in[\mu_1,\mu_2]$ be an arbitrary point. Then there exists $t\in[0,1]$ such that $x=\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}$. Thus, since $e^t\le e$ and $e^{1-t}\le e$, we have

$$\begin{aligned}
\psi(x)&=\psi\left(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\right)\\
&\le\frac{1}{n}\sum_{i=1}^{n}\left(e^{t}-1\right)^{i}\psi(\mu_1)+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^{i}\psi(\mu_2)\\
&\le\frac{1}{n}\sum_{i=1}^{n}\left(e^{t}-1\right)^{i}L+\frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^{i}L\\
&\le\frac{2L}{n}\sum_{i=1}^{n}(e-1)^{i}=M.
\end{aligned}$$

We have shown that ψ is bounded above from real number M. Interested reader can also prove the fact that ψ is bounded below using the same idea as in Theorem 2.4 in [16]. □

Remark 6. Interested readers can find many other nice properties of this new class of functions. We omit here the details.

4. Hermite-Hadamard type inequality via n-polynomial harmonically exponential type convex functions

The purpose of this section is to derive a new version of Hermite-Hadamard type via n-polynomial harmonically exponential type convexity.

Theorem 4.1. *Let $\psi : [\mu_1, \mu_2] \to [0, +\infty)$ be an n-polynomial harmonically exponential type convex function. If $\psi \in L[\mu_1, \mu_2]$, then*

$$\frac{n}{2\sum_{i=1}^{n}\left(\sqrt{e}-1\right)^{i}}\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big) \le \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx \tag{4.1}$$

$$\le \Big[\frac{\psi(\mu_1)+\psi(\mu_2)}{n}\Big]\sum_{i=1}^{n}[e-2]^{i}.$$

Proof. Since ψ is n-polynomial harmonically exponential type convex function, we have

$$\psi\Big(\frac{xy}{ty+(1-t)x}\Big) \le \frac{1}{n}\sum_{i=1}^{n}(e^{t}-1)^{i}\psi(x) + \frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^{i}\psi(y),$$

which lead to

$$\psi\Big(\frac{2xy}{x+y}\Big) \le \frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{e}-1\right)^{i}\psi(x) + \frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{e}-1\right)^{i}\psi(y).$$

Using the change of variables, we get

$$\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big) \le \frac{1}{n}\sum_{i=1}^{n}\Big[\left(\sqrt{e}-1\right)^{i}\Big]\Big[\psi\Big(\frac{\mu_1\mu_2}{(t\mu_2+(1-t)\mu_1)}\Big)+\psi\Big(\frac{\mu_1\mu_2}{(t\mu_1+(1-t)\mu_2)}\Big)\Big].$$

Integrating with respect to t on $[0,1]$ the above inequality, we obtain

$$\frac{n}{2\sum_{i=1}^{n}\left(\sqrt{e}-1\right)^{i}}\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big) \le \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx,$$

which completes the left side inequality. For the right side inequality, changing the variable of integration as $x = \frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}$ and using Definition 3.1 for the function ψ, we have

$$\begin{aligned}
&\frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx \\
&= \int_0^1 \psi\Big(\frac{\mu_1\mu_2}{t\mu_2+(1-t)\mu_1}\Big)dt \\
&\le \int_0^1\Big[\frac{1}{n}\sum_{i=1}^{n}(e^{t}-1)^{i}\psi(\mu_1) + \frac{1}{n}\sum_{i=1}^{n}\left(e^{1-t}-1\right)^{i}\psi(\mu_2)\Big]dt \\
&= \frac{\psi(\mu_1)}{n}\sum_{i=1}^{n}\int_0^1 (e^{t}-1)^{i}dt + \frac{\psi(\mu_2)}{n}\sum_{i=1}^{n}\int_0^1\left(e^{1-t}-1\right)^{i}dt
\end{aligned}$$

$$= \left[\frac{\psi(\mu_1) + \psi(\mu_2)}{n}\right] \sum_{i=1}^{n} [e-2]^i,$$

which completes the proof. □

Corollary 1. *Choosing $n = 1$ in Theorem 4.1, then*

$$\frac{1}{2\left(\sqrt{e}-1\right)}\psi\left(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\right) \le \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx \le (e-2)[\psi(\mu_1)+\psi(\mu_2)].$$

5. Other results

In order to obtain some new results using n–polynomial harmonically exponential type convex function, we need the following lemma:

Lemma 5.1. *[4] Let $\psi : [\mu_1, \mu_2] \subseteq (0, +\infty) \to \Re$ be a differentiable function and $\rho, \sigma \in [0, 1]$. If $\psi' \in L[\mu_1, \mu_2]$, then the following identity holds:*

$$\begin{aligned} &\frac{\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{2} + \frac{2-\rho-\sigma}{2}\psi\left(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\right) - \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx \\ &= \frac{\mu_1\mu_2(\mu_2-\mu_1)}{4}\int_0^1\Bigg[\frac{4(1-\rho-t)}{((1-t)\mu_2+(1+t)\mu_1)^2}\psi'\left(\frac{2\mu_1\mu_2}{(1-t)\mu_2+(1+t)\mu_1}\right) \\ &+ \frac{4(\sigma-t)}{(t\mu_1+(2-t)\mu_2)^2}\psi'\left(\frac{2\mu_1\mu_2}{t\mu_1+(2-t)\mu_2}\right)\Bigg]dt. \end{aligned} \tag{5.1}$$

For simplicity, we denote

$$A_{\mu_1,\mu_2} = (1-t)\mu_2 + (1+t)\mu_1 \quad \text{and} \quad B_{\mu_1,\mu_2} = t\mu_1 + (2-t)\mu_2. \tag{5.2}$$

The following special functions will be used in sequel:

$$\Gamma(\mu) = \int_0^{+\infty} e^{-t}t^{\mu-1}dt, \quad \mu > 0;$$

$$\beta(\mu_1,\mu_2) = \int_0^1 t^{\mu_1-1}(1-t)^{\mu_2-1}dt, \quad \mu_1,\mu_2 > 0;$$

$$\beta(\mu_1,\mu_2) = \frac{\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\mu_1+\mu_2)}, \quad \mu_1,\mu_2 > 0;$$

$${}_2F_1(\mu_1,\mu_2;\mu_3;\mu) = \frac{1}{\beta(\mu_2,\mu_3-\mu_2)}\int_0^1 t^{\mu_2-1}(1-t)^{\mu_3-\mu_2-1}(1-\mu t)^{-\mu_1}dt,$$

where $\mu_3 > \mu_2 > 0$ and $|\mu| < 1$.

Theorem 5.2. *Let $\psi : [\mu_1, \mu_2] \subseteq (0, +\infty) \to \Re$ be a differentiable function such that $\psi' \in L[\mu_1, \mu_2]$ and $\rho, \sigma \in [0, 1]$. If the function $|\psi'|^q$ is an n–polynomial harmonically exponential type convex, then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\left|\frac{\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{2} + \frac{2-\rho-\sigma}{2}\psi\left(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\right) - \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right|$$

$$\leq \mu_1\mu_2(\mu_2-\mu_1) \times\left[\varphi_1^{\frac{1}{p}}\left(C_1|\psi'(\mu_1)|^q+C_2|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}+\varphi_2^{\frac{1}{p}}\left(C_3|\psi'(\mu_1)|^q+C_4|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\right], \tag{5.3}$$

where

$$\varphi_1=\int_0^1|1-\rho-t|^p dt=\frac{(1-\rho)^{p+1}+\rho^{p+1}}{p+1},$$

$$\varphi_2=\int_0^1|\sigma-t|^p dt=\frac{(1-\sigma)^{p+1}+\sigma^{p+1}}{p+1},$$

$$C_1=\frac{1}{2n}\sum_{i=1}^n\int_0^1\frac{1}{A^{2q}_{\mu_1,\mu_2}}(e^{1-t}-1)^i dt,\quad C_2=\frac{1}{2n}\sum_{i=1}^n\int_0^1\frac{1}{A^{2q}_{\mu_1,\mu_2}}(e^{1+t}-1)^i dt,$$

$$C_3=\frac{1}{2n}\sum_{i=1}^n\int_0^1\frac{1}{B^{2q}_{\mu_1,\mu_2}}(e^{2-t}-1)^i dt,\quad C_4=\frac{1}{2n}\sum_{i=1}^n\int_0^1\frac{1}{B^{2q}_{\mu_1,\mu_2}}(e^{t}-1)^i dt,$$

and A_{μ_1,μ_2}, B_{μ_1,μ_2} are defined from (5.2).

Proof. From Lemma 5.1, Hölder's inequality, n–polynomial harmonically exponential type convexity of $|\psi'|^q$ and properties of modulus, we have

$$\begin{aligned}
&\left|\frac{\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{2}+\frac{2-\rho-\sigma}{2}\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big)-\frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right|\\
&\leq\frac{\mu_1\mu_2(\mu_2-\mu_1)}{4}\Big[\int_0^1\left|\frac{4(1-\rho-t)}{((1-t)\mu_2+(1+t)\mu_1)^2}\right|\left|\psi'\Big(\frac{2\mu_1\mu_2}{(1-t)\mu_2+(1+t)\mu_1}\Big)\right|dt\\
&+\int_0^1\left|\frac{4(\sigma-t)}{(t\mu_1+(2-t)\mu_2)^2}\right|\left|\psi'\Big(\frac{2\mu_1\mu_2}{t\mu_1+(2-t)\mu_2}\Big)\right|dt\Big]\\
&\leq\mu_1\mu_2(\mu_2-\mu_1)\Big\{\Big(\int_0^1|1-\rho-t|^p dt\Big)^{\frac{1}{p}}\\
&\times\Big[\int_0^1\frac{1}{A^{2q}_{\mu_1,\mu_2}}\Big(\frac{1}{2n}\sum_{i=1}^n\left(e^{1-t}-1\right)^i|\psi'(\mu_1)|^q+\frac{1}{2n}\sum_{i=1}^n\left(e^{1+t}-1\right)^i|\psi'(\mu_2)|^q\Big)dt\Big]^{\frac{1}{q}}\\
&+\Big(\int_0^1|\sigma-t|^p dt\Big)^{\frac{1}{p}}\\
&\times\Big[\int_0^1\frac{1}{B^{2q}_{\mu_1,\mu_2}}\Big(\frac{1}{2n}\sum_{i=1}^n\left(e^{2-t}-1\right)^i|\psi'(\mu_1)|^q+\frac{1}{2n}\sum_{i=1}^n(e^{t}-1)^i|\psi'(\mu_2)|^q\Big)dt\Big]^{\frac{1}{q}}\Big\}\\
&=\frac{\mu_1\mu_2(\mu_2-\mu_1)}{4}\\
&\times\left[\varphi_1^{\frac{1}{p}}\left(C_1|\psi'(\mu_1)|^q+C_2|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}+\varphi_2^{\frac{1}{p}}\left(C_3|\psi'(\mu_1)|^q+C_4|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which completes the proof. □

Corollary 2. *Taking $n=1$ in Theorem 5.2, then*

$$\left|\frac{\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{2}+\frac{2-\rho-\sigma}{2}\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big)-\frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right|$$

$$\leq \mu_1\mu_2(\mu_2-\mu_1)$$
$$\times\left[\varphi_1^{\frac{1}{p}}\left(D_1|\psi'(\mu_1)|^q + D_2|\psi'(\mu_2)|^q\right)^{\frac{1}{q}} + \varphi_2^{\frac{1}{p}}\left(D_3|\psi'(\mu_1)|^q + D_4|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\right],$$

where

$$D_1 = \frac{1}{2}\int_0^1 \frac{1}{A_{\mu_1,\mu_2}^{2q}}(e^{1-t}-1)dt, \quad D_2 = \frac{1}{2}\int_0^1 \frac{1}{A_{\mu_1,\mu_2}^{2q}}(e^{1+t}-1)dt,$$
$$D_3 = \frac{1}{2}\int_0^1 \frac{1}{B_{\mu_1,\mu_2}^{2q}}(e^{2-t}-1)dt, \quad D_4 = \frac{1}{2}\int_0^1 \frac{1}{B_{\mu_1,\mu_2}^{2q}}(e^{t}-1)dt.$$

Corollary 3. *Taking $\rho = \sigma$ in Theorem 5.2, then*

$$\left|\rho\frac{\psi(\mu_1)+\psi(\mu_2)}{2} + (1-\rho)\psi\left(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\right) - \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right|$$
$$\leq \mu_1\mu_2(\mu_2-\mu_1)\varphi^{\frac{1}{p}}$$
$$\times\left[\left(C_1|\psi'(\mu_1)|^q + C_2|\psi'(\mu_2)|^q\right)^{\frac{1}{q}} + \left(C_3|\psi'(\mu_1)|^q + C_4|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\right],$$

where $\varphi_1 = \varphi_2 = \varphi$.

Corollary 4. *Choosing $\rho = \sigma = 0$ in Theorem 5.2, then*

$$\left|\psi\left(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\right) - \frac{2\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right| \leq \frac{\mu_1\mu_2(\mu_2-\mu_1)}{\sqrt[p]{p+1}}$$
$$\times\left[\left(C_1|\psi'(\mu_1)|^q + C_2|\psi'(\mu_2)|^q\right)^{\frac{1}{q}} + \left(C_3|\psi'(\mu_1)|^q + C_4|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 5. *Choosing $\rho = \sigma = \frac{1}{2}$ in Theorem 5.2, then*

$$\left|\frac{\psi(\mu_1)+\psi(\mu_2)}{2} + \psi\left(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\right) - \frac{2\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right|$$
$$\leq \mu_1\mu_2(\mu_2-\mu_1)\sqrt[p]{\frac{4}{p+1}}$$
$$\times\left[\left(C_1|\psi'(\mu_1)|^q + C_2|\psi'(\mu_2)|^q\right)^{\frac{1}{q}} + \left(C_3|\psi'(\mu_1)|^q + C_4|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 6. *Taking $\rho = \sigma = \frac{1}{3}$ in Theorem 5.2, then*

$$\left|\frac{\psi(\mu_1)+\psi(\mu_2)}{2} + 2\psi\left(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\right) - \frac{3\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right|$$
$$\leq 3\mu_1\mu_2(\mu_2-\mu_1)\sqrt[p]{4\left(\frac{\left(\frac{2}{3}\right)^{p+1} + \left(\frac{1}{3}\right)^{p+1}}{p+1}\right)}$$
$$\times\left[\left(C_1|\psi'(\mu_1)|^q + C_2|\psi'(\mu_2)|^q\right)^{\frac{1}{q}} + \left(C_3|\psi'(\mu_1)|^q + C_4|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 7. *Taking $\rho = \sigma = 1$ in Theorem 5.2, then*

$$\left|\frac{\psi(\mu_1)+\psi(\mu_2)}{2} - \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right| \leq \frac{\mu_1\mu_2(\mu_2-\mu_1)}{\sqrt[p]{p+1}}$$
$$\times\left[\left(C_1|\psi'(\mu_1)|^q + C_2|\psi'(\mu_2)|^q\right)^{\frac{1}{q}} + \left(C_3|\psi'(\mu_1)|^q + C_4|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\right].$$

Theorem 5.3. *Let $\psi : [\mu_1,\mu_2] \subseteq (0,+\infty) \to \Re$ be a differentiable function such that $\psi' \in L[\mu_1,\mu_2]$ and $\rho,\sigma \in [0,1]$. If the function $|\psi'|^q$ is an n-polynomial harmonically exponential type convex, then for $p,q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, we have*

$$\begin{aligned}&\left|\frac{\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{2} + \frac{2-\rho-\sigma}{2}\psi\Big(\frac{\mu_1\mu_2}{\mu_1+\mu_2}\Big) - \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right| \\ &\leq \frac{\mu_1\mu_2(\mu_2-\mu_1)}{4} \\ &\times\Bigg[\frac{4}{(\mu_1+\mu_2)^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\Big)\Big)^{\frac{1}{p}}\left(C_5|\psi'(\mu_1)|^q + C_6|\psi'(\mu_2)|^q\right)^{\frac{1}{q}} \\ &+ \frac{1}{\mu_1^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{2\mu_1}\Big)\Big)^{\frac{1}{p}}\left(C_7|\psi'(\mu_1)|^q + C_8|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\Bigg],\end{aligned} \tag{5.4}$$

where

$$C_5 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1|1-\rho-\sigma|^q(e^{1-t}-1)^i dt,$$

$$C_6 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1|1-\rho-\sigma|^q(e^{1+t}-1)^i dt,$$

$$C_7 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1|\sigma-t|^q(e^{2-t}-1)^i dt, \quad C_8 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1|\sigma-t|^q(e^{t}-1)^i dt.$$

Proof. Applying Lemma 5.1, Hölder's inequality, n-polynomial harmonically exponential type convexity of $|\psi'|^q$ and properties of modulus, we have

$$\begin{aligned}&\left|\frac{\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{2} + \frac{2-\rho-\sigma}{2}\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big) - \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right| \\ &\leq \frac{\mu_1\mu_2(\mu_2-\mu_1)}{4}\Big[\int_0^1\left|\frac{4(1-\rho-t)}{((1-t)\mu_2+(1+t)\mu_1)^2}\right|\left|\psi'\Big(\frac{2\mu_1\mu_2}{(1-t)\mu_2+(1+t)\mu_1}\Big)\right|dt \\ &+ \int_0^1\left|\frac{4(\sigma-t)}{(t\mu_1+(2-t)\mu_2)^2}\right|\left|\psi'\Big(\frac{2\mu_1\mu_2}{t\mu_1+(2-t)\mu_2}\Big)\right|dt\Big] \\ &\leq \frac{\mu_1\mu_2(\mu_2-\mu_1)}{4}\Bigg\{4\Big(\int_0^1\frac{1}{A_{\mu_1,\mu_2}^{2p}}dt\Big)^{\frac{1}{p}} \\ &\times\Big[\int_0^1|1-\rho-\sigma|^q\Big(\frac{1}{2n}\sum_{i=1}^{n}\Big(e^{1-t}-1\Big)^i|\psi'(\mu_1)|^q + \frac{1}{2n}\sum_{i=1}^{n}\Big(e^{1+t}-1\Big)^i|\psi'(\mu_2)|^q\Big)dt\Big]^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&+4\Big(\int_0^1 \frac{1}{B_{\mu_1,\mu_2}^{2p}}dt\Big)^{\frac{1}{p}}\\
&\times\Big[\int_0^1 |\sigma-t|^q\Big(\frac{1}{2n}\sum_{i=1}^{n}\Big(e^{2-t}-1\Big)^i|\psi'(\mu_1)|^q+\frac{1}{2n}\sum_{i=1}^{n}(e^t-1)^i|\psi'(\mu_2)|^q\Big)dt\Big]^{\frac{1}{q}}\Big\}\\
&=\frac{\mu_1\mu_2(\mu_2-\mu_1)}{4}\\
&\times\Big[\frac{4}{(\mu_1+\mu_2)^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\Big)\Big)^{\frac{1}{p}}(C_5|\psi'(\mu_1)|^q+C_6|\psi'(\mu_2)|^q)^{\frac{1}{q}}\\
&+\frac{1}{\mu_1^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{2\mu_1}\Big)\Big)^{\frac{1}{p}}(C_7|\psi'(\mu_1)|^q+C_8|\psi'(\mu_2)|^q)^{\frac{1}{q}}\Big],
\end{aligned}
$$

which completes the proof. □

Corollary 8. *Taking $n=1$ in Theorem 5.3, then*

$$
\begin{aligned}
&\Big|\frac{\rho\psi(\mu_1)+\sigma\psi(\mu_2)}{2}+\frac{2-\rho-\sigma}{2}\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big)-\frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\Big|\\
&\le\frac{\mu_1\mu_2(\mu_2-\mu_1)}{4}\\
&\times\Big[\frac{4}{(\mu_1+\mu_2)^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\Big)\Big)^{\frac{1}{p}}(D_5|\psi'(\mu_1)|^q+D_6|\psi'(\mu_2)|^q)^{\frac{1}{q}}\\
&+\frac{1}{\mu_1^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{2\mu_1}\Big)\Big)^{\frac{1}{p}}(D_7|\psi'(\mu_1)|^q+D_8|\psi'(\mu_2)|^q)^{\frac{1}{q}}\Big],
\end{aligned}
$$

where

$$D_5=\frac{1}{2}\int_0^1|1-\rho-\sigma|^q(e^{1-t}-1)dt,$$

$$D_6=\frac{1}{2}\int_0^1|1-\rho-\sigma|^q(e^{1+t}-1)dt,$$

$$D_7=\frac{1}{2}\int_0^1|\sigma-t|^q(e^{2-t}-1)dt,\quad D_8=\frac{1}{2}\int_0^1|\sigma-t|^q(e^{t}-1)dt.$$

Corollary 9. *Taking $\rho=\sigma$ in Theorem 5.3, then*

$$
\begin{aligned}
&\Big|\rho\frac{\psi(\mu_1)+\psi(\mu_2)}{2}+(1-\rho)\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big)-\frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\Big|\\
&\le\frac{\mu_1\mu_2(\mu_2-\mu_1)}{4}\\
&\times\Big[\frac{4}{(\mu_1+\mu_2)^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\Big)\Big)^{\frac{1}{p}}(E_1|\psi'(\mu_1)|^q+E_2|\psi'(\mu_2)|^q)^{\frac{1}{q}}\\
&+\frac{1}{\mu_1^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{2\mu_1}\Big)\Big)^{\frac{1}{p}}(E_3|\psi'(\mu_1)|^q+E_4|\psi'(\mu_2)|^q)^{\frac{1}{q}}\Big],
\end{aligned}
$$

where

$$E_1 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1 |1-2\rho|^q (e^{1-t}-1)^i dt,$$

$$E_2 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1 |1-2\rho|^q (e^{1+t}-1)^i dt,$$

$$E_3 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1 |\rho-t|^q (e^{2-t}-1)^i dt,$$

$$E_4 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1 |\rho-t|^q (e^{t}-1)^i dt.$$

Corollary 10. *Taking $\rho = \sigma = 0$ in Theorem 5.3, then*

$$\begin{aligned}
&\left|\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big) - \frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right| \le \frac{\mu_1\mu_2(\mu_2-\mu_1)}{4} \\
&\times\Big[\frac{4}{(\mu_1+\mu_2)^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\Big)\Big)^{\frac{1}{p}}\left(E_5|\psi'(\mu_1)|^q + E_6|\psi'(\mu_2)|^q\right)^{\frac{1}{q}} \\
&+ \frac{1}{\mu_1^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{2\mu_1}\Big)\Big)^{\frac{1}{p}}\left(E_7|\psi'(\mu_1)|^q + E_8|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\Big],
\end{aligned}$$

where

$$E_5 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1 (e^{1-t}-1)^i dt, \quad E_6 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1 (e^{1+t}-1)^i dt,$$

$$E_7 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1 t^q(e^{2-t}-1)^i dt, \quad E_8 = \frac{1}{2n}\sum_{i=1}^{n}\int_0^1 t^q(e^{t}-1)^i dt.$$

Corollary 11. *Choosing $\rho = \sigma = \frac{1}{2}$ in Theorem 5.3, then*

$$\begin{aligned}
&\left|\frac{\psi(\mu_1)+\psi(\mu_2)}{2} + \psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big) - \frac{2\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right| \le \frac{\mu_1\mu_2(\mu_2-\mu_1)}{2} \\
&\times\Big[\frac{1}{\mu_1^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{2\mu_1}\Big)\Big)^{\frac{1}{p}}\left(E_9|\psi'(\mu_1)|^q + E_{10}|\psi'(\mu_2)|^q\right)^{\frac{1}{q}}\Big],
\end{aligned}$$

where

$$E_9 = \frac{1}{2^{q+1}n}\sum_{i=1}^{n}\int_0^1 |1-2t|^q (e^{2-t}-1)^i dt,$$

$$E_{10} = \frac{1}{2^{q+1}n}\sum_{i=1}^{n}\int_0^1 |1-2t|^q (e^{t}-1)^i dt.$$

Corollary 12. *Choosing $\rho = \sigma = \frac{1}{3}$ in Theorem 5.3, then*

$$\left|\frac{\psi(\mu_1)+\psi(\mu_2)}{2} + 2\psi\Big(\frac{2\mu_1\mu_2}{\mu_1+\mu_2}\Big) - \frac{3\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right| \le \frac{3\mu_1\mu_2(\mu_2-\mu_1)}{4}$$

$$\times\Big[\frac{4}{(\mu_1+\mu_2)^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\Big)\Big)^{\frac{1}{p}}(G_1|\psi'(\mu_1)|^q+G_2|\psi'(\mu_2)|^q)^{\frac{1}{q}}$$
$$+\frac{1}{\mu_1^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{2\mu_1}\Big)\Big)^{\frac{1}{p}}(G_3|\psi'(\mu_1)|^q+G_4|\psi'(\mu_2)|^q)^{\frac{1}{q}}\Big],$$

where

$$G_1=\frac{1}{3^q2n}\sum_{i=1}^{n}\int_0^1(e^{1-t}-1)^idt,\quad G_2=\frac{1}{3^q2n}\sum_{i=1}^{n}\int_0^1(e^{1+t}-1)^idt,$$
$$G_3=\frac{1}{3^q2n}\sum_{i=1}^{n}\int_0^1|1-3t|^q(e^{2-t}-1)^idt,$$
$$G_4=\frac{1}{3^q2n}\sum_{i=1}^{n}\int_0^1|1-3t|^q(e^{t}-1)^idt.$$

Corollary 13. *Taking $\rho=\sigma=1$ in Theorem 5.3, then*

$$\left|\frac{\psi(\mu_1)+\psi(\mu_2)}{2}-\frac{\mu_1\mu_2}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\frac{\psi(x)}{x^2}dx\right|\le\frac{\mu_1\mu_2(\mu_2-\mu_1)}{4}$$
$$\times\Big[\frac{4}{(\mu_1+\mu_2)^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\Big)\Big)^{\frac{1}{p}}(G_5|\psi'(\mu_1)|^q+G_6|\psi'(\mu_2)|^q)^{\frac{1}{q}}$$
$$+\frac{1}{\mu_1^2}\Big({}_2F_1\Big(2p,1;2;\frac{\mu_1-\mu_2}{2\mu_1}\Big)\Big)^{\frac{1}{p}}(G_7|\psi'(\mu_1)|^q+G_8|\psi'(\mu_2)|^q)^{\frac{1}{q}}\Big],$$

where

$$G_5=\frac{1}{2n}\sum_{i=1}^{n}\int_0^1(e^{1-t}-1)^idt,\quad G_6=\frac{1}{2n}\sum_{i=1}^{n}\int_0^1(e^{1+t}-1)^idt,$$
$$G_7=\frac{1}{2n}\sum_{i=1}^{n}\int_0^1|1-t|^q(e^{2-t}-1)^idt,$$
$$G_8=\frac{1}{2n}\sum_{i=1}^{n}\int_0^1|1-t|^q(e^{t}-1)^idt.$$

6. Applications

In this section, we recall the following special means for two positive real numbers μ_1,μ_2 where $\mu_1<\mu_2$:

(1) The arithmetic mean

$$A=A(\mu_1,\mu_2)=\frac{\mu_1+\mu_2}{2}.$$

(2) The geometric mean

$$G=G(\mu_1,\mu_2)=\sqrt{\mu_1\mu_2}.$$

(3) The harmonic mean

$$H=H(\mu_1,\mu_2)=\frac{2\mu_1\mu_2}{\mu_1+\mu_2}.$$

(4) The Logarithmic mean

$$L = L(\mu_1, \mu_2) = \frac{\mu_2 - \mu_1}{\ln \mu_2 - \ln \mu_1}.$$

(5) The p-Logarithmic mean

$$L_p = L_p(\mu_1, \mu_2) = \Big(\frac{\mu_2^{p+1} - \mu_1^{p+1}}{(p+1)(\mu_2 - \mu_1)}\Big)^{\frac{1}{p}}, \quad p \in \Re - \{-1, 0\}.$$

(6) The identric mean

$$I = I(\mu_1, \mu_2) = \frac{1}{e}\Big(\frac{\mu_2^{\mu_2}}{\mu_1^{\mu_1}}\Big)^{\frac{1}{\mu_2 - \mu_1}}.$$

These means have a lot of applications in areas and different type of numerical approximations. However, the following simple relationship are known in the literature.

$$H(\mu_1, \mu_2) \leq G(\mu_1, \mu_2) \leq L(\mu_1, \mu_2) \leq I(\mu_1, \mu_2) \leq A(\mu_1, \mu_2).$$

Proposition 1. *Let* $0 < \mu_1 < \mu_2$. *Then we get the following inequality*

$$\frac{n}{2\sum_{i=1}^n \big(\sqrt{e} - 1\big)^i} H(\mu_1, \mu_2) \leq \frac{G^2(\mu_1, \mu_2)}{L(\mu_1, \mu_2)} \leq A(\mu_1, \mu_2)\frac{2}{n}\sum_{i=1}^n [e-2]^i. \tag{6.1}$$

Proof. Taking $\psi(x) = x$ for $x > 0$ in Theorem 4.1, then inequality (6.1) is easily captured. □

Proposition 2. *Let* $0 < \mu_1 < \mu_2$. *Then we get the following inequality*

$$\frac{n}{2\sum_{i=1}^n \big(\sqrt{e} - 1\big)^i} H^2(\mu_1, \mu_2) \leq G^2(\mu_1, \mu_2) \leq A(\mu_1^2, \mu_2^2)\frac{2}{n}\sum_{i=1}^n [e-2]^i. \tag{6.2}$$

Proof. Taking $\psi(x) = x^2$ for $x > 0$ in Theorem 4.1, then inequality (6.2) is easily captured. □

Proposition 3. *Let* $0 < \mu_1 < \mu_2$. *Then we get the following inequality*

$$\frac{n}{2\sum_{i=1}^n \big(\sqrt{e} - 1\big)^i} H(\mu_1, \mu_2) \leq I(\mu_1, \mu_2) \leq G(\mu_1, \mu_2)\frac{2}{n}\sum_{i=1}^n [e-2]^i. \tag{6.3}$$

Proof. Taking $\psi(x) = \ln x$ for $x > 0$ in Theorem 4.1, then inequality (6.3) is easily captured. □

Proposition 4. *Let* $0 < \mu_1 < \mu_2$. *Then we get the following inequality*

$$\begin{aligned} &\frac{n}{2\sum_{i=1}^n \big(\sqrt{e} - 1\big)^i} H^2(\mu_1, \mu_2)\ln H(\mu_1, \mu_2) \\ &\qquad \leq G^2(\mu_1, \mu_2)\ln I(\mu_1, \mu_2) \leq A(\mu_1^2 \ln \mu_1, \mu_2^2 \ln \mu_2)\frac{2}{n}\sum_{i=1}^n [e-2]^i. \end{aligned} \tag{6.4}$$

Proof. Taking $\psi(x) = x^2 \ln x$ for $x > 0$ in Theorem 4.1, then inequality (6.4) is easily captured. □

Proposition 5. *Let* $0 < \mu_1 < \mu_2$. *Then we get the following inequality*

$$\frac{n}{2\sum_{i=1}^n \big(\sqrt{e} - 1\big)^i} H^{p+2}(\mu_1, \mu_2) \leq G^2(\mu_1, \mu_2)L_p^p(\mu_1, \mu_2) \leq A(\mu_1^{p+2}, \mu_2^{p+2})\frac{2}{n}\sum_{i=1}^n [e-2]^i. \tag{6.5}$$

Proof. Taking $\psi(x) = x^{p+2}$ for $x > 0$ in Theorem 4.1, then inequality (6.5) is easily captured. □

7. Conclusions

We have introduced and investigated some algebraic properties of a new class of functions namely n-polynomial harmonically exponential type convex. We showed that this class of functions had some nice properties, which other convex functions had as well. We proved that our new class of n-polynomial harmonically exponential type convex function is very larger with respect to the known class of functions, like n-polynomial convex and n-polynomial harmonically convex. New version of Hermite-Hadamard type inequality and an integral identity for the differentiable functions are obtained. For different choices of two parameters ρ and σ some special cases from our results are given. In recent years, many researchers put the effort into the theory of inequalities to bring a new dimension to mathematical analysis and applied mathematics with different features in the literature. Due to widespread views and applications, the theory of inequalities has become an attractive, interesting and absorbing field for the researchers. It is high time to find the applications of these inequalities along with efficient numerical methods. We believe that our new class of functions will have very deep research in this fascinating field of inequalities and also in pure and applied sciences. The interesting techniques and fruitful ideas of this paper can be extended on the co-ordinates along with fractional calculus. In future our goal is that we will continue our research work in this direction furthermore.

Acknowledgement

The research of the author Saad Ihsan Butt has been fully supported by H.E.C. Pakistan under NRPU project 7906.

Conflict of interest

The authors declare no conflict of interest.

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