Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 6800–6816.
DOI:10.3934/math.2020437
Received: 07 June 2020
Accepted: 24 August 2020
Published: 02 September 2020

Research article

Geometric properties of harmonic functions associated with the symmetric conjecture points and exponential function

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Abstract: In this paper, some classes of univalent harmonic functions are introduced by subordination, where the analytic parts of which are exponential starlike (or convex) functions with respect to the symmetric conjecture points. According to the relationships of the analytic part and the co-analytic part, the geometric properties, such as coefficient estimates, distortion theorems, integral expressions, estimates and growth conditions and covering theorem, of the classes are obtained.

Keywords: harmonic function; exponential function; subordination; symmetric conjecture point; coefficient estimates
Mathematics Subject Classification: 30C45, 30C65

1. Introduction

Let $\mathcal{A}$ be the class of functions h of the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

where h is analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We denote $\mathcal{S}, \mathcal{S}^*$ and $\mathcal{K}$ the subclasses of $\mathcal{A}$ consisting of univalent, starlike and convex functions respectively ([1,2]) and denote $\mathcal{P} = \{p : p(0) = 1, \mathrm{Re}\, p(z) > 0, z \in \mathbb{U}\}$.

An analytic function $s : \mathbb{U} = \{z : |z| < 1\} \to \mathbb{C}$ is subordinate to an analytic function $t : \mathbb{U} \to \mathbb{C}$, if there is a function v satisfying $v(0) = 0$ and $|v(z)| < 1$ $(z \in \mathbb{U})$, such that $s(z) = t(v(z))(z \in \mathbb{U})$. Note that $s(z) \prec t(z)$. Especially, if t is univalent in $\mathbb{U}$, then the following conclusion is true (see [1]):

$$s(z) \prec t(z) \Longleftrightarrow s(0) = t(0) \text{ and } s(\mathbb{U}) \subset t(\mathbb{U}).$$

In 1933, a classical Fekete-Szegö problem for $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ was introduced by Fekete and Szegö [3] as follows,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \mu \leq 0, \\ 1 + 2\exp(\frac{-2\mu}{1-\mu}), & 0 \leq \mu \leq 1, \\ 4\mu - 3, & \mu \geq 1. \end{cases}$$

The result is sharp.

Using the subordination, the classes $\mathcal{S}^*(\phi)$ and $\mathcal{K}(\phi)$ of starlike and convex functions were defined by Ma and Minda [4] in 1994. The function $h(z) \in \mathcal{S}^*(\phi)$ iff $\frac{zh'(z)}{h(z)} \prec \phi(z)$ and $\frac{zh''(z)}{h'(z)} \prec \phi(z)$, where $h \in \mathcal{A}$ and $\phi \in \mathcal{P}$. Moreover, Fekete-Szegö problems of the classes were obtained by Ma and Minda [4]. The problem of Fekete-Szegö has always been a hot topic in geometry function theory. Many authors studied and obtained many results (see [5–7]).

Let $\phi(z) = \frac{1+Az}{1+Bz}$ and $-1 \leq B < A \leq 1$. The classes $\mathcal{S}^*(\phi)$ and $\mathcal{K}(\phi)$ reduce to $\mathcal{S}^*(\frac{1+Az}{1+Bz})$ and $\mathcal{K}(\frac{1+Az}{1+Bz})$, which are the classes of Janowski starlike and convex functions respectively (see [8]).

Without loss of generality, both $S^*(\frac{1+z}{1-z}) = S^*$ and $K(\frac{1+z}{1-z}) = K$ represent the well-known classes of starlike and convex function respectively.

In 2015, Mediratta et al. [9] introduced the family of exponential starlike functions $S^*(e^z)$, that is

$$S^*(e^z) = \left\{ h \in \mathcal{A} : \frac{zh'(z)}{h(z)} \prec e^z, \quad z \in \mathbb{U} \right\}$$

or, equivalently

$$S^*(e^z) = \left\{ h \in \mathcal{A} : \left| \log \frac{zh'(z)}{h(z)} \right| < 1, \quad z \in \mathbb{U} \right\}.$$

According to the properties of the exponential function e^z and the subordination relationship, the class $S^*(e^z)$ maps the unit disc $\mathbb{U}$ onto a region, which is symmetric with respect to the real axis and 1.

In 1959, the class $\mathcal{S}_s^*$ of starlike functions with respect to symmetric points was introduced by Sakaguchi [10]. The function $h \in \mathcal{S}_s^*$ if and only if

$$\mathrm{Re} \frac{zh'(z)}{h(z) - h(-z)} > 0.$$

In 1987, the classes $\mathcal{S}_c^*$ and $\mathcal{S}_{cs}^*$ of starlike functions with respect to conjugate points and symmetric conjugate points were introduced by El-Ashwa and Thomas [11] as follows,

$$h \in \mathcal{S}_c^* \Longleftrightarrow \mathrm{Re} \frac{zh'(z)}{h(z) + \overline{h}(\overline{z})} > 0 \quad \text{and} \quad h \in \mathcal{S}_{sc}^* \Longleftrightarrow \mathrm{Re} \frac{zh'(z)}{h(z) - \overline{h}(-\overline{z})} > 0.$$

For analytic functions $h(z)$ and $g(z) (z \in \mathbb{U})$. Let S_H define the class of harmonic mappings with the following form (see [12, 13])

$$f(z) = h(z) + \overline{g(z)}, \quad z \in \mathbb{U}, \tag{1.2}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \ \ |b_1| = \alpha \in [0, 1). \tag{1.3}$$

In particular, h is called the analytic part and g is called the co-analytic part of f.

It is well known that the function $f = h + \overline{g}$ is locally univalent and sense preserving in $\mathbb{U}$ if and only if $|h'(z)| > |g'(z)|$ $(z \in \mathbb{U})$([14]).

According to the above conclusion, the coefficient estimations, distortion theorems, integral expressions, Jacobi estimates and growth condition in geometric properties of covering theorem of the co-analytic part can be obtained by using the analytic part of harmonic functions. In the recent years, various subclasses of S_H were researched by many authors as follows.

In 2007, the subclass of S_H with $h \in \mathcal{K}$ was studied by Klimek and Michalski [15].

In 2014, the subclass of S_H with $h \in \mathcal{S}$ was studied by Hotta and Michalski [16].

In 2015, the subclasses of S_H with $h \in S^*(\frac{1+(1-2\beta)z}{1-z})$ and $h \in K(\frac{1+(1-2\beta)z}{1-z})$ were studied by Zhu and Huang [17].

In this paper, by using subordination relationship, we studied the subclasses of S_H with $\frac{h(z)-\overline{h}(-\overline{z})}{2} \in S^*(e^z)$ and $\frac{h(z)-\overline{h}(-\overline{z})}{2} \in K(e^z)$.

Definition 1. *Suppose $f = h + \overline{g} \in S_H$ of the form (1.3). Let $HS^{*,\alpha}_{sc}(e)$ denote the class of harmonic univalent exponential starlike functions with symmetric conjecture point consisting of f with $h \in S^*_{sc}(e)$. That is, the function $f = h + \overline{g} \in HS^{*,\alpha}_{sc}(e)$ if and only if*

$$\frac{2zh'(z)}{h(z) - \overline{h}(-\overline{z})} \prec e^z.$$

Also, let $HK^{\alpha}_{sc}(e)$ denote the class of harmonic univalent convex exponential functions with symmetric conjecture point consisting of f with $h \in K_{sc}(e)$, that is, $f = h + \overline{g} \in HK^{\alpha}_{sc}(e)$ if and only if

$$\frac{2(zh'(z))'}{(h(z) - \overline{h}(-\overline{z}))'} \prec e^z.$$

We know that $h(z) \in K_{sc}(e) \Longleftrightarrow zh'(z) \in S^*_{sc}(e)$.

2. Preliminary preparation

In order to obtain our results, we need Lemmas as follows.

Lemma 1. *([18]). Let $\omega(z) = c_0 + c_1 z + \ldots + c_n z^n + \ldots$ be analytic satisfying $|\omega(z)| \le 1$ in $\mathbb{U}$. Then*

$$|c_n| \le 1 - |c_0|^2, n = 1, 2, \ldots, \tag{2.1}$$

and

$$|c_2 - \gamma c_1^2| \le \max\{1, |\gamma|\}. \tag{2.2}$$

Lemma 2. *Let*

$$\frac{2zh'(z)}{h(z) - \overline{h}(-\overline{z})} = p(z),$$

we have

$$h(z) = \int_0^z p(\eta) \exp \int_0^\eta \frac{p(t) + \overline{p}(-\overline{t}) - 2}{2t} dt d\eta.$$

Lemma 3. *If* $h(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in S^*_{sc}(e)$*, then*

$$|a_{2n}| \leq \frac{(2n-1)!!}{(2n)!!} \quad \textit{and} \quad |a_{2n+1}| \leq \frac{(2n-1)!!}{(2n)!!}. \tag{2.3}$$

The estimate is sharp for h given by

$$h(z) = \frac{1+z}{\sqrt{1-z^2}} - 1.$$

If $h(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in K^*_{sc}(e)$*, then*

$$|a_{2n}| \leq \frac{(2n-1)!!}{2n(2n)!!} \quad \textit{and} \quad |a_{2n+1}| \leq \frac{(2n-1)!!}{(2n+1)(2n)!!}. \tag{2.4}$$

The estimate is sharp for h given by

$$h(z) = \arcsin z - \log(1 + \sqrt{1-z^2}) + \log 2.$$

Proof. Let $h(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in S^*_{sc}(e)$. According to Definition 1, we have

$$\frac{2zh'(z)}{h(z) - \overline{h}(-\overline{z})} \prec e^z.$$

There exists a function $p(z) = 1 + \sum\limits_{k=1}^{\infty} p_k z^k$ satisfying

$$\frac{2zh'(z)}{h(z) - \overline{h}(-\overline{z})} = p(z), \tag{2.5}$$

that is,

$$1 + \sum_{k=1}^{\infty} p_k z^k \prec e^z = 1 + z + \frac{z^2}{2!} + \cdots.$$

Using the results of Rogosinski [19], we have $|p_k| \leq 1$ for $k \geq 1$.

By means of comparing the coefficients of the both sides of (2.5), we get

$$2na_{2n} = p_{2n-1} + a_3 p_{2n-3} + \cdots + a_{2n-1} p_1,$$

and

$$2na_{2n+1} = p_{2n} + a_3 p_{2n-2} + \cdots + a_{2n-1} p_2.$$

Let $\phi(n) = 1 + |a_3| + \cdots + |a_{2n-1}|$. It is easy to verify that

$$|a_{2n}| \leq \frac{1}{2n}\phi(n) \tag{2.6}$$

and

$$|a_{2n+1}| \leq \frac{1}{2n}\phi(n). \tag{2.7}$$

From (2.7), we have

$$\phi(n+1) \leq \frac{(2n+1)!!}{(2n)!!}. \tag{2.8}$$

According to (2.6), (2.7) and (2.8), we can obtain (2.3).

If $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K_{sc}(e)$, then $zh'(z) \in S^*_{sc}(e)$. Using the results in (2.3), we can obtain (2.4) easily. □

Lemma 4. *Let* $\mu \in \mathbb{C}$.

(1) If $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_{sc}(e)$, *then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{2}\max\{1, \frac{1}{2}|\mu - 1|\}. \tag{2.9}$$

(2) If $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K_{sc}(e)$, *then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{6}\max\{1, \frac{1}{8}|3\mu - 4|\}. \tag{2.10}$$

The estimates are sharp.

Proof. Let $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*_{sc}(e)$. According to the subordination relationship and Definition 1, we get

$$\frac{2zh'(z)}{h(z) - \overline{h}(-\overline{z})} = e^{\nu(z)},$$

that is,

$$\log \frac{2zh'(z)}{h(z) - \overline{h}(-\overline{z})} = \nu(z), \tag{2.11}$$

where $\nu(z) = c_1 z + c_2 z^2 + \cdots$ is an analytic function with $\nu(0) = 0$ and $|\nu(z)| < 1$ $(z \in \mathbb{U})$.

By means of comparing the coefficients of two sides of (2.11), we get

$$a_2 = \frac{1}{2}c_1, \qquad a_3 = \frac{1}{2}c_2 + \frac{1}{4}c_1^2 \qquad \text{and} \qquad a_4 = \frac{1}{4}c_3 + \frac{3}{8}c_2 c_1 + \frac{5}{48}c_1^3.$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{1}{2}\{c_2 - \frac{1}{2}(\mu - 1)c_1^2\}.$$

Using the fact that (2.2) in Lemma 1, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{1}{2}\max\{1, \frac{1}{2}|\mu - 1|\}.$$

The bound is sharp for h given as follows,

$$h(z) = \int_0^z \exp(\xi + \int_0^\xi \frac{e^t + e^{-t} - 2}{2t} dt) d\xi \quad \text{or} \quad h(z) = \int_0^z \exp(\xi^2 + \int_0^\xi \frac{e^{t^2} - 1}{t} dt) d\xi.$$

Taking $h(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K_{sc}(e) \Longleftrightarrow zh'(z) = z + \sum_{n=2}^{\infty} n a_n z^n \in S^*_{sc}(e)$ into consideration, it is easy to obtain (2.10). The bound is sharp for h given as follows,

$$h(z) = \int_0^z \frac{1}{\eta} \int_0^\eta \exp(\xi + \int_0^\xi \frac{e^t + e^{-t} - 2}{2t} dt) d\xi d\eta \quad \text{or} \quad h(z) = \int_0^z \frac{1}{\eta} \int_0^\eta \exp(\xi^2 + \int_0^\xi \frac{e^{t^2} - 1}{t} dt) d\xi d\eta.$$

□

Lemma 5. *Suppose $h(z) \in \mathcal{A}$ and $|z| = r \in [0, 1)$.*

(1) Let $h(z) \in S^*(e)$. Then

$$\exp(-r + \int_0^r \frac{e^{-\eta} - 1}{\eta} d\eta) < |h'(z)| < \exp(r + \int_0^r \frac{e^{\eta} - 1}{\eta} d\eta) \tag{2.12}$$

and

$$r \exp \int_0^r \frac{e^{-\eta} - 1}{\eta} d\eta < |h(z)| < r \exp \int_0^r \frac{e^{\eta} - 1}{\eta} d\eta. \tag{2.13}$$

(2) Let $h(z) \in K(e)$. Then

$$\exp \int_0^r \frac{e^{-\eta} - 1}{\eta} d\eta < |h'(z)| < \exp \int_0^r \frac{e^{\eta} - 1}{\eta} d\eta \tag{2.14}$$

and

$$r \exp \int_0^r \frac{e^{-\eta} - 1}{2\eta} d\eta < |h(z)| < r \exp \int_0^r \frac{e^{\eta} - 1}{2\eta} d\eta. \tag{2.15}$$

Proof. Let $h(z) \in S^*(e)$ and $|z| = r \in [0, 1)$. According to the subordination relationship and Definition 1, there exists an analytic function $\nu(z) = c_1 z + c_2 z^2 + \cdots$ satisfying $\nu(0) = 0$ and $|\nu(z)| < |z|$, such that

$$\frac{zh'(z)}{h(z)} = e^{\nu(z)}.$$

Thus, it is concluded that

$$e^{-r} < |\frac{zh'(z)}{h(z)}| < e^r, \tag{2.16}$$

that is,

$$e^{-r} < \operatorname{Re} \frac{zh'(z)}{h(z)} < e^r. \tag{2.17}$$

Since

$$\operatorname{Re} \frac{zh'(z)}{h(z)} = r \frac{\partial}{\partial r} \log |h(z)|. \tag{2.18}$$

By (2.17) and (2.18), we get

$$r \exp \int_0^r \frac{e^{-\eta} - 1}{\eta} d\eta < |h(z)| < r \exp \int_0^r \frac{e^{\eta} - 1}{\eta} d\eta. \tag{2.19}$$

From (2.16) and (2.19), we have

$$r\exp(-r+\int_0^r \frac{e^{-\eta}-1}{\eta}d\eta) < |zh'(z)| < r\exp(r+\int_0^r \frac{e^{\eta}-1}{\eta}d\eta).$$

Similar to the previous proof. We let $h(z) \in K(e)$ and $|z| = r \in [0, 1)$, then

$$e^{-r} < |1+\frac{zh''(z)}{h'(z)}| < e^{r}.$$

After simple calculation, we have

$$e^{-r}-1 < \mathrm{Re}\frac{zh''(z)}{h'(z)} < e^{r}-1. \tag{2.20}$$

By (2.18) and (2.20), we get

$$\exp\int_0^r \frac{e^{-\eta}-1}{\eta}d\eta < |h'(z)| < \exp\int_0^r \frac{e^{\eta}-1}{\eta}d\eta.$$

Using the conclusion in [19], for $h(z) \in K(e)$, we have $\frac{2zh'(z)}{h(z)}-1 \prec e^{z}$. According to the subordination relationship, we have

$$e^{-r} < |\frac{2zh'(z)}{h(z)}-1| < e^{r},$$

After simple calculation, we have

$$\frac{e^{-r}-1}{2} < \mathrm{Re}\frac{zh'(z)}{h(z)}-1 < \frac{e^{r}-1}{2}. \tag{2.21}$$

By (2.18) and (2.21), we get

$$r\exp\int_0^r \frac{e^{-\eta}-1}{2\eta}d\eta < |h(z)| < r\exp\int_0^r \frac{e^{\eta}-1}{2\eta}d\eta.$$

Therefore, we complete the proof of Lemma 5. □

Lemma 6. *([20]). If $h(z) \in \mathcal{S}^*_{sc}(e)$, then $\frac{h(z)-\overline{h}(-\overline{z})}{2} \in \mathcal{S}^*(e)$.*

Lemma 7. *If $h(z) \in \mathcal{K}_{sc}(e)$, then $\frac{h(z)-\overline{h}(-\overline{z})}{2} \in \mathcal{K}(e)$.*

Lemma 8. *Suppose $h(z) \in \mathcal{A}$ and $|z| = r \in [0.1)$.*

(1) Let $h \in \mathcal{S}^*_{sc}(e)$. Then

$$\phi_1(r) < |h'(z)| < \phi_2(r),$$

where

$$\phi_1(r) = \exp(-r+\int_0^r \frac{e^{-\eta}-1}{\eta}d\eta), \quad \phi_2(r) = \exp(r+\int_0^r \frac{e^{\eta}-1}{\eta}d\eta). \tag{2.22}$$

(2) Let $h \in \mathcal{K}_{sc}(e)$. Then

$$\psi_1(r) < |h'(z)| < \psi_2(r),$$

where

$$\psi_1(r) = \frac{1}{r}\int_0^r \exp(-t+\int_0^t \frac{e^{-\eta}-1}{\eta}d\eta)dt, \quad \psi_2(r) = \frac{1}{r}\int_0^r \exp(t+\int_0^t \frac{e^{\eta}-1}{\eta}d\eta)dt. \tag{2.23}$$

Proof. Suppose $h(z) \in \mathcal{S}^*_{sc}(e)$. It is quite similar to the proof of Lemma 5, we have

$$e^{-r}\left|\frac{h(z)-\overline{h}(-\overline{z})}{2}\right| \leq |zh'(z)| \leq e^{r}\left|\frac{h(z)-\overline{h}(-\overline{z})}{2}\right|. \tag{2.24}$$

According to Lemma 6 and (2.13) of Lemma 5, we have

$$r\exp\int_0^r \frac{e^{-\eta}-1}{\eta}d\eta < \left|\frac{h(z)-\overline{h}(-\overline{z})}{2}\right| < r\exp\int_0^r \frac{e^{\eta}-1}{\eta}d\eta. \tag{2.25}$$

By (2.24) and (2.25), we can obtain (2.22).

If $h(z) \in \mathcal{K}_{sc}(e)$, then

$$e^{-r}\left|\frac{(h(z)-\overline{h}(-\overline{z}))'}{2}\right| \leq |(zh'(z))'| \leq e^{r}\left|\frac{(h(z)-\overline{h}(-\overline{z}))'}{2}\right|. \tag{2.26}$$

According to Lemmas 7 and (2.14) of Lemma 5, we have

$$\exp\int_0^r \frac{e^{-\eta}-1}{\eta}d\eta < \left|\frac{(h(z)-\overline{h}(-\overline{z}))'}{2}\right| < \exp\int_0^r \frac{e^{\eta}-1}{\eta}d\eta. \tag{2.27}$$

By (2.26) and (2.27), we get

$$\exp\left(-r+\int_0^r \frac{e^{-\eta}-1}{\eta}d\eta\right) \leq |(zh'(z))'| \leq \exp\left(r+\int_0^r \frac{e^{\eta}-1}{\eta}d\eta\right). \tag{2.28}$$

By (2.28), integrating along a radial line $\xi = te^{i\theta}$, we obtained immediately,

$$|zh'(z)| \leq \int_0^r \exp\left(t+\int_0^t \frac{e^{\eta}-1}{\eta}d\eta\right)dt$$

The verification for the remainder of (2.23) is given as follows. Let $H(z) := zh'(z)$, which is univalent. Suppose that $\xi_1 \in \Gamma = H(\{z : |z| = r\})$ is the nearest point to the origin. By means of rotation, we suppose that $\xi_1 > 0$ and $z_1 = H^{-1}(\xi_1)$. Let $\gamma = \{\xi : 0 \leq \xi \leq \xi_1\}$ and $L = H^{-1}(\gamma)$. If $\varsigma = H^{-1}(\xi)$, then $d\xi = H'(\varsigma)d\varsigma$. Hence

$$\xi_1 = \int_0^{\xi_1} d\xi = \int_0^{z_1} H'(\varsigma)d\varsigma \geq \int_0^r |H'(te^{i\theta})|dt \geq \int_0^r \exp\left(-t+\int_0^t \frac{e^{-\eta}-1}{\eta}d\eta\right)dt.$$

Thus the proof of Lemma 8 is completed. □

3. Main results

Next, the integral expressions for functions of the classes defined in Definition 1 are obtained.

Theorem 1. *Let ω and ν be analytic in $\mathbb{U}$ with $|\omega(0)| = \alpha, \nu(0) = 0, |\omega(z)| < 1$ and $|\nu(z)| < 1$. If $f = h + \overline{g} \in HS^{*,\alpha}_{sc}(e)$. Then*

$$f(z) = \int_0^z \varphi(\xi)d\xi + \overline{\int_0^z \omega(\xi)\varphi(\xi)d\xi}, \tag{3.1}$$

where

$$\varphi(\xi) = e^{\nu(\xi)}\exp\int_0^{\xi} \frac{e^{\nu(t)}+e^{\overline{\nu}(-\overline{t})}-2}{2t}dt. \tag{3.2}$$

Proof. Let $f = h + \overline{g} \in HS^{*,\alpha}_{sc}(e)$. Using Definition 1 and the subordination relationship, there exist analytic functions ω and ν satisfying $\omega(0) = b_1, \nu(0) = 0, |\omega(z)| < 1$ and $|\nu(z)| < 1$ $(z \in \mathbb{U})$, such that

$$\frac{2zh'(z)}{h(z) - \overline{h}(-\overline{z})} = e^{\nu(z)}, \tag{3.3}$$

and

$$g'(z) = \omega(z)h'(z). \tag{3.4}$$

If we substitute z by $-\overline{z}$ in (3.4), we obtain

$$\frac{-2\overline{z}h'(-\overline{z})}{h(-\overline{z}) - \overline{h}(z)} = e^{\nu(-\overline{z})}. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\frac{2z(h(z) - \overline{h}(-\overline{z}))'}{h(z) - \overline{h}(-\overline{z})} = e^{\nu(z)} + e^{\overline{\nu}(-\overline{z})}. \tag{3.6}$$

A routine computation for the equality (3.6) gives rise to the following equation,

$$\frac{\overline{h}(z) - h(-\overline{z})}{2} = z \exp \int_0^z \frac{e^{\nu(t)} + e^{\overline{\nu}(-\overline{t})} - 2}{2t} dt. \tag{3.7}$$

Plugging (3.7) back into (3.3), we have

$$h'(z) = e^{\nu(z)} \exp \int_0^z \frac{e^{\nu(t)} + e^{\overline{\nu}(-\overline{t})} - 2}{2t} dt. \tag{3.8}$$

If the equality (3.8) is integrated from the both sides of it, then

$$h(z) = \int_0^z e^{\nu(\xi)} \exp \int_0^\xi \frac{e^{\nu(t)} + e^{\overline{\nu}(-\overline{t})} - 2}{2t} dt d\xi.$$

Inserting (3.8) into (3.4), it is easy to show that

$$g(z) = \int_0^z \omega(\xi) e^{\nu(\xi)} \exp \int_0^\xi \frac{e^{\nu(t)} + e^{\overline{\nu}(-\overline{t})} - 2}{2t} dt d\xi.$$

Thus, the proof of Theorem 1 is completed. □

Taking Theorem 1 and $h \in K_{sc}(e) \iff zh'(z) \in S^*_{sc}(e)$ into consideration, we get the following result.

Theorem 2. *Let ω and ν be analytic in $\mathbb{U}$ satisfying $|\omega(0)| = \alpha, \nu(0) = 0, |\omega(z)| < 1$ and $|\nu(z)| < 1$. If $f \in HK^{\alpha}_{sc}(e)$, then*

$$f(z) = \int_0^z \frac{1}{\eta} \int_0^\eta \varphi(\xi) d\xi d\eta + \overline{\int_0^z \frac{\omega(\eta)}{\eta} \int_0^\eta \varphi(\xi) d\xi d\eta}.$$

where $\varphi(\xi)$ is defined by (3.2).

In the following, the coefficient estimates of the class $HS^{*,\alpha}_{sc}(e)$ will be obtained.

Theorem 3. *Let h and g be given by (1.3). If* $f = h + \overline{g} \in HS^{*,\alpha}_{sc}(e)$*, then*

$$|b_{2n}| \leq \begin{cases} \frac{1-\alpha^2}{2} + \frac{\alpha}{2}, & n = 1, \\ \frac{(1-\alpha^2)}{2n}\left(1 + \sum\limits_{k=2}^{n}(4k-3)\frac{(2k-3)!!}{(2k-2)!!}\right) + \alpha\frac{(2n-1)!!}{(2n)!!}, & n \geq 2, \end{cases} \tag{3.9}$$

and

$$|b_{2n+1}| \leq \begin{cases} \frac{2(1-\alpha^2)}{3} + \frac{\alpha}{2}, & n = 1, \\ \frac{(1-\alpha^2)}{2n+1}\left(1 + \sum\limits_{k=2}^{n}(4k-3)\frac{(2k-3)!!}{(2k-2)!!} + \frac{(2n-1)!!}{(2n-2)!!}\right) + \alpha\frac{(2n-1)!!}{(2n)!!}, & n \geq 2. \end{cases} \tag{3.10}$$

The estimates are sharp and the extremal function is

$$f_0^{\alpha}(z) = \frac{1+z}{\sqrt{1-z^2}} - 1 + \overline{\int_0^z \frac{(\alpha + (1-\alpha^2-\alpha)t)}{(1-t)^2\sqrt{(1-t^2)}}dt}.$$

Specially, if $f \in HS^{*,0}_{sc}(e)$*, then*

$$|b_{2n}| \leq \begin{cases} \frac{1}{2}, & n = 1, \\ \frac{1}{2n}\left(1 + \sum\limits_{k=2}^{n}(4k-3)\frac{(2k-3)!!}{(2k-2)!!}\right), & n \geq 2, \end{cases}$$

and

$$|b_{2n+1}| \leq \begin{cases} \frac{2}{3}, & n = 1, \\ \frac{1}{2n+1}\left(1 + \sum\limits_{k=2}^{n}(4k-3)\frac{(2k-3)!!}{(2k-2)!!} + \frac{(2n-1)!!}{(2n-2)!!}\right), & n \geq 2. \end{cases}$$

The estimates are sharp and the extremal function is

$$f_1^0(z) = \frac{1+z}{\sqrt{1-z^2}} - 1 + \overline{\frac{2z^3+3z^2-1}{3(1-z^2)^{\frac{3}{2}}} + \frac{1}{3}}.$$

Proof. Let h and g be given by (1.3). Using the fact that $g' = \omega h'$ satisfying $\omega(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ analytic in $\mathbb{U}$, we obtain

$$2nb_{2n} = \sum_{p=1}^{2n} pa_p c_{2n-p} \quad (a_1 = 1, n \geq 1) \tag{3.11}$$

and

$$(2n+1)b_{2n+1} = \sum_{p=1}^{2n+1} pa_p c_{2n+1-p} \quad (a_1 = 1, n \geq 1). \tag{3.12}$$

It is easy to show that

$$2n|b_{2n}| \leq \sum_{p=1}^{2n} p|a_p||c_{2n-p}|$$

and

$$(2n+1)|b_{2n+1}| \leq \sum_{p=1}^{2n+1} p|a_p||c_{2n+1-p}|.$$

Since $g' = \omega h'$, it follows that $c_0 = b_1$. By (2.1), it is obvious that $|c_k| \leq 1-\alpha^2$ for $k \in \mathbb{N}$. Therefore,

$$|b_{2n}| \leq \begin{cases} \frac{1-\alpha^2}{2} + |a_2|\alpha, & n=1, \\ \frac{(1-\alpha^2)}{2n}(1+\sum\limits_{k=2}^{2n-1} k|a_k|) + \alpha|a_{2n}|, & n \geq 2, \end{cases} \tag{3.13}$$

and

$$|b_{2n+1}| \leq \begin{cases} \frac{1-\alpha^2}{3}(1+2|a_2|) + |a_3|\alpha, & n=1, \\ \frac{(1-\alpha^2)}{2n+1}(1+\sum\limits_{k=2}^{2n} k|a_k|) + \alpha|a_{2n+1}|, & n \geq 2. \end{cases} \tag{3.14}$$

According to Lemma 3, (3.13) and (3.14), after the simple calculation, (3.9) and (3.10) can be obtained easily. We also obtain the extreme function. □

By using the same methods in Theorem 3, the following results are obtained.

Theorem 4. *Let h and g of the form (1.3). If $f = h + \overline{g} \in HK_{sc}^{\alpha}(e)$, then*

$$|b_{2n}| \leq \begin{cases} \frac{1-\alpha^2}{4} + \frac{\alpha}{4}, & n=1, \\ \frac{(1-\alpha^2)}{(2n)^2}\left(1+\sum\limits_{k=2}^{n}(4k-3)\frac{(2k-3)!!}{(2k-2)!!}\right) + \alpha\frac{(2n-1)!!}{2n(2n)!!}, & n \geq 2, \end{cases}$$

and

$$|b_{2n+1}| \leq \begin{cases} \frac{2(1-\alpha^2)}{9} + \frac{\alpha}{6}, & n=1, \\ \frac{(1-\alpha^2)}{(2n+1)^2}\left(1+\sum\limits_{k=2}^{n}(4k-3)\frac{(2k-3)!!}{(2k-2)!!} + \frac{(2n-1)!!}{(2n-2)!!}\right) + \alpha\frac{(2n-1)!!}{(2n+1)(2n)!!}, & n \geq 2. \end{cases}$$

For functions of the classes defined in the paper, Fekete-Szegö inequality of which are listed below.

Theorem 5. *Let $f = h + \overline{g}$ with h and g given by (1.3) and $\mu \in \mathbb{C}$.*
(1) If $f \in HS_{sc}^{*,\alpha}(e)$, then

$$|b_3 - \mu b_2^2| \leq \frac{(1-\alpha^2)}{3}\left\{1 + \frac{3|\mu|(1-\alpha^2)}{4} + \frac{|2-3\mu b_1|}{2}\right\} + \frac{\alpha}{2}\max\left\{1, \frac{|\mu b_1 - 1|}{2}\right\},$$

$$|b_{2n} - b_{2n-1}| \leq \begin{cases} \frac{1}{2}(1-\alpha^2) + \frac{3\alpha}{2}, & n=1, \\ (1-\alpha^2)\left((\frac{1}{2n}+\frac{1}{2n-1})(1+\sum\limits_{k=2}^{n}(4k-3)\frac{(2k-3)!!}{(2k-2)!!}) - \frac{(2n-3)!!}{(2n-1)(2n-4)!!}\right) & \\ \quad +\alpha\left(\frac{(2n-1)!!}{(2n)!!} + \frac{(2n-3)!!}{(2n-2)!!}\right), & n \geq 2, \end{cases}$$

and

$$|b_{2n+1} - b_{2n}| \leq (1-\alpha^2)\left((\frac{1}{2n+1}+\frac{1}{2n})(1+\sum_{k=2}^{n}(4k-3)\frac{(2k-3)!!}{(2k-2)!!}) + \frac{(2n-1)!!}{(2n+1)(2n-2)!!}\right) + 2\alpha\frac{(2n-1)!!}{(2n)!!}, \quad n \geq 1.$$

(2) If $f \in HK_{sc}^{\alpha}(e)$, then

$$|b_3 - \mu b_2^2| \leq \frac{(1-\alpha^2)}{3}\left\{1 + \frac{3|\mu|(1-\alpha^2)}{4} + \frac{|2-3\mu b_1|}{4}\right\} + \frac{\alpha}{6}\max\left\{1, \frac{|4-3\mu b_1|}{8}\right\},$$

$$|b_{2n}-b_{2n-1}| \leq \begin{cases} \frac{1}{2}(1-\alpha^2)+\frac{5\alpha}{4}, & n=1,\\ (1-\alpha^2)\left((\frac{1}{2n}+\frac{1}{2n-1})(1+2\sum_{k=2}^{n}\frac{(2k-3)!!}{(2k-2)!!})-\frac{(2n-3)!!}{(2n-1)(2n-2)!!}\right) & \\ \quad +\alpha\left(\frac{(2n-1)!!}{(2n)(2n)!!}+\frac{(2n-3)!!}{(2n-1)(2n-2)!!}\right), & n\geq 2, \end{cases}$$

and

$$|b_{2n+1}-b_{2n}| \leq \ (1-\alpha^2)\left((\frac{1}{2n+1}+\frac{1}{2n})(1+2\sum_{k=2}^{n}\frac{(2k-3)!!}{(2k-2)!!})+\frac{(2n-1)!!}{(2n+1)(2n)!!}\right) +\alpha(\frac{1}{2n+1}+\frac{1}{2n})\frac{(2n-1)!!}{(2n)!!}, \quad n\geq 1.$$

Proof. From the relation (3.11) and (3.12), we have

$$2b_2 = c_1 + 2a_2c_0, \ \ 3b_3 = c_2 + 2a_2c_1 + 3a_3c_0,$$

and

$$2nb_{2n} = \sum_{p=1}^{2n} pa_pc_{2n-p}, \quad (2n+1)b_{2n+1} = \sum_{p=1}^{2n+1} pa_pc_{2n+1-p} \quad (a_1 = 1, n \geq 1).$$

By (2.1), we have

$$|b_3-\mu b_2^2| \leq \frac{1-\alpha^2}{3}\left\{1+\frac{3|\mu|(1-\alpha^2)}{4}+|a_2||2-3\mu b_1|\right\}+\alpha\left|a_3-\mu b_1 a_2^2\right|,$$

$$|b_{2n}-b_{2n-1}| \leq \begin{cases} \frac{1}{2}(1-\alpha^2)+\alpha(1+|a_2|), & n=1,\\ (1-\alpha^2)\left(\frac{1}{2n}\sum_{p=1}^{2n-1}p|a_p|+\frac{1}{2n-1}\sum_{p=1}^{2n-2}p|a_p|\right)+\alpha(|a_{2n}|+|a_{2n-1}|), & n\geq 2, \end{cases}$$

and

$$|b_{2n+1}-b_{2n}| \leq (1-\alpha^2)\left(\frac{1}{2n+1}\sum_{p=1}^{2n}p|a_p|+\frac{1}{2n}\sum_{p=1}^{2n-1}p|a_p|\right)+\alpha(|a_{2n+1}|+|a_{2n}|), \quad n\geq 1.$$

According to Lemma 3 and Lemma 4, we can compete the proof of Theorem 5. The estimates above are sharp. □

Paralleling the results of Zhu et al. [17], the corresponding results for functions of the classes defined in the paper can be obtained. For example, the estimates of distortion, growth of g and Jacobian of f and so on.

Theorem 6. *Let $|z| = r \in [0, 1)$.*

(1) If $f = h + \overline{g} \in HS^{*,\alpha}_{sc}(e)$, then

$$\frac{\max\{\alpha-r,0\}}{(1-\alpha r)}\phi_1(r) \leq |g'(z)| \leq \frac{\alpha+r}{(1+\alpha r)}\phi_2(r), \tag{3.15}$$

where $\phi_1(r)$ and $\phi_2(r)$ are given by (2.22).

Especially, let $\alpha = 0$, we have

$$|g'(z)| \leq r\exp\left(r+\int_0^r \frac{e^\eta-1}{\eta}d\eta\right).$$

(2) If $f = h + \overline{g} \in HK_{sc}^{\alpha}(e)$*, then*

$$\frac{\max\{\alpha - r, 0\}}{(1-\alpha r)}\psi_1(r) \le |g'(z)| \le \frac{(\alpha + r)}{(1+\alpha r)}\psi_2(r), \tag{3.16}$$

where $\psi_1(r)$ *and* $\psi_2(r)$ *are given by (2.23).*

Especially, let $\alpha = 0$*, we have*

$$|g'(z)| \le \int_0^r \exp(t + \int_0^t \frac{e^{\eta} - 1}{\eta} d\eta) dt.$$

Proof. According to the relation $g' = \omega h'$, it is easy to see $\omega(z)$ satisfying $|\omega(0)| = |g'(0)| = |b_1| = \alpha$ such that ([21]):

$$\left|\frac{\omega(z) - \omega(0)}{1 - \overline{\omega}(0)\omega(z)}\right| \le |z|.$$

It is easy to show

$$\left|\omega(z) - \frac{\omega(0)(1-r^2)}{1 - |\omega(0)|^2 r^2}\right| \le \frac{r(1 - |\omega(0)|^2)}{1 - |\omega(0)|^2 r^2}.$$

A tedious calculation gives

$$\frac{\max\{\alpha - r, 0\}}{1 - \alpha r} \le |\omega(z)| \le \frac{\alpha + r}{1 + \alpha r}, \ z \in \mathbb{U}. \tag{3.17}$$

Applying (3.17) and (2.22), we get (3.15). Similarly, applying (3.17) and (2.23), we get (3.16). Thus we complete the proof of Theorem 6. □

Using the method analogous to that in proof of Lemma 8, we can obtain the following results.

Theorem 7. *Let* $|z| = r \in [0, 1)$.

(1) If $f = h + \overline{g} \in HS_{sc}^{*,\alpha}(e)$*, then*

$$\int_0^r \frac{\max\{\alpha - \xi, 0\}}{(1 - \alpha\xi)}\phi_1(\xi) d\xi \le |g(z)| \le \int_0^r \frac{\alpha + \xi}{(1 + \alpha\xi)}\phi_2(\xi) d\xi,$$

where $\phi_1(\xi)$ *and* $\phi_2(\xi)$ *are given by (2.22).*

Especially, let $\alpha = 0$*, we have*

$$|g(z)| \le \int_0^r \xi \exp\left(\xi + \int_0^{\xi} \frac{e^{\eta} - 1}{\eta} d\eta\right) d\xi.$$

(2) If $f = h + \overline{g} \in HK_{sc}^{\alpha}(e)$*, then*

$$\int_0^r \frac{\max\{\alpha - \xi, 0\}}{(1 - \alpha\xi)}\psi_1(\xi) d\xi \le |g(z)| \le \int_0^r \frac{(\alpha + \xi)}{(1 + \alpha\xi)}\psi_2(\xi) d\xi,$$

where $\psi_1(\xi)$ *and* $\psi_2(\xi)$ *are given by (2.23).*

Especially, let $\alpha = 0$*, we have*

$$|g(z)| \le \int_0^r \int_0^{\xi} \exp(t + \int_0^t \frac{e^{\eta} - 1}{\eta} d\eta) dt d\xi.$$

Next, the Jacobian estimates and growth estimates of f are obtained.

Theorem 8. *Let* $|z| = r \in [0, 1)$.

(1) If $f = h + \overline{g} \in HS^{*,\alpha}_{sc}(e)$*, then*

$$\frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2}\phi_1^2(r) \le J_f(z) \le \begin{cases} \frac{(1-\alpha^2)(1-r^2)}{(1-\alpha r)^2}\phi_2^2(r), & r < \alpha, \\ \phi_2^2(r), & r \ge \alpha. \end{cases},$$

where $\phi_1(r)$ *and* $\phi_2(r)$ *are given by (2.22).*

(2) If $f = h + \overline{g} \in HK^{\alpha}_{sc}(e)$*, then*

$$\frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2}\psi_1^2(r) \le J_f(z) \le \begin{cases} \frac{(1-\alpha^2)(1-r^2)}{(1-\alpha r)^2}\psi_2^2(r), & r < \alpha, \\ \psi_2^2(r), & r \ge \alpha, \end{cases}$$

where $\psi_1(r)$ *and* $\psi_2(r)$ *are given by (2.23).*

Proof. It is well known that Jacobian of $f = h + \overline{g}$ is

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2(1 - |\omega(z)|^2), \tag{3.18}$$

where ω satisfying $g' = \omega h'$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{U}$.

Let $f \in HS^{*,\alpha}_{sc}(e)$, plugging (3.17) and (2.22) back into (3.18), we get

$$J_f(z) \ge \frac{(1-\alpha^2)(1-r^2)}{(1+\alpha r)^2}\exp\left(-2r + 2\int_0^r \frac{e^{-\eta}-1}{\eta}d\eta\right),$$

and

$$\begin{aligned} J_f(z) &\le \exp\left(2r + 2\int_0^r \frac{e^{\eta}-1}{\eta}d\eta\right)\left(1 - \frac{(\max\{(\alpha - r), 0\})^2}{(1-\alpha r)^2}\right) \\ &= \begin{cases} \exp\left(2r + 2\int_0^r \frac{e^{\eta}-1}{\eta}d\eta\right)\cdot\frac{(1-\alpha^2)(1-r^2)}{(1-\alpha r)^2}, & r < \alpha, \\ \exp\left(2r + 2\int_0^r \frac{e^{\eta}-1}{\eta}d\eta\right), & r \ge \alpha. \end{cases} \end{aligned}$$

Thus this completes the proof of (1). Plugging (3.17) and (2.23) back into (3.18), (2) of Theorem 8 can be proved by the same method as employed before. □

Theorem 9. *Let* $|z| = r,\ 0 \le r < 1$.

(1) If $f = h + \overline{g} \in HS^{*,\alpha}_{sc}(e)$*, then*

$$\int_0^r \frac{(1-\alpha)(1-\xi)}{(1+\alpha\xi)}\phi_1(\xi)d\xi \le |f(z)| \le \int_0^r \frac{(1+\alpha)(1+\xi)}{(1+\alpha\xi)}\phi_2(\xi)d\xi, \tag{3.19}$$

where $\phi_1(\xi)$ *and* $\phi_2(\xi)$ *are given by (2.22).*

(2) If $f = h + \overline{g} \in HK^{\alpha}_{sc}(e)$*, then*

$$\int_0^r \frac{(1-\alpha)(1-\xi)}{(1+\alpha\xi)}\psi_1(\xi)d\xi \le |f(z)| \le \int_0^r \frac{(1+\alpha)(1+\xi)}{(1+\alpha\xi)}\psi_2(\xi)d\xi, \tag{3.20}$$

where $\psi_1(\xi)$ *and* $\psi_2(\xi)$ *are given by (2.23).*

Proof. For any point $z = re^{i\theta} \in \mathbb{U}$, let $\mathbb{U}_r = \{z \in \mathbb{U} : |z| < r\}$ and denote

$$d = \min_{z \in \mathbb{U}_r} |f(\mathbb{U}_r)|.$$

It is easy to see that $\mathbb{U}(0, d) \subseteq f(\mathbb{U}_r) \subseteq f(\mathbb{U})$. Thus, there is $z_r \in \partial\mathbb{U}_r$ satisfying $d = |f(z_r)|$. Let $L(t) = tf(z_r)$ for $t \in [0, 1]$, then $\ell(t) = f^{-1}(L(t))$ is a well-defined Jordan arc. For $f = h + \overline{g} \in HS^{*,\alpha}_{sc}(\beta)$, by (2.22) and (3.17), we get

$$\begin{aligned} d = |f(z_r)| &= \int_L |d\omega| = \int_\ell |df| = \int_\ell |h'(\rho)d\rho + \overline{g'(\rho)}d\bar{\rho}| \\ &\geq \int_\ell |h'(\rho)|(1 - |\omega(\rho)|)|d\rho| \\ &\geq \int_\ell \frac{(1-\alpha)(1-|\rho|)}{(1+\alpha|\rho|)} \exp(-|\rho| + \int_0^{|\rho|} \frac{e^{-\eta}-1}{\eta} d\eta)|d\rho|, \\ &= \int_0^1 \frac{(1-\alpha)(1-|\ell(t)|)}{(1+\alpha|\ell(t)|)} \exp(-|\ell(t)| + \int_0^{|\ell(t)|} \frac{e^{-\eta}-1}{\eta} d\eta)dt, \\ &\geq \int_0^r \frac{(1-\alpha)(1-\xi)}{(1+\alpha\xi)} \exp(-\xi + \int_0^\xi \frac{e^{-\eta}-1}{\eta} d\eta)d\xi. \end{aligned}$$

Using (2.22) and (3.17), the right side of (3.19) is obtained. The remainder of proofs is similar to that in (3.20) and so we omit. □

According to (3.19) and (3.20), it follows that the covering theorems of f.

Theorem 10. *Let $f = h + \overline{g} \in S_H$.*
(1) If $f \in HS^{*,\alpha}_{sc}(e)$, then $\mathbb{U}_{R_1} \subset f(\mathbb{U})$, where

$$R_1 = \int_0^1 \frac{(1-\alpha)(1-\xi)}{(1+\alpha\xi)} \exp(-\xi + \int_0^\xi \frac{e^{-\eta}-1}{\eta} d\eta)d\xi.$$

(2) If $f \in HK^{\alpha}_{sc}(e)$, then $\mathbb{U}_{R_2} \subset f(\mathbb{U})$, where

$$R_2 = \int_0^1 \frac{(1-\alpha)(1-\xi)}{\xi(1+\alpha\xi)} \int_0^\xi \exp(-t + \int_0^t \frac{e^{-\eta}-1}{\eta} d\eta)dtd\xi.$$

4. Conclusion

In this paper, with the help of the analytic part h satisfying certain conditions, we obtain the coefficients estimates of the co-analytic part g and the geometric properties of harmonic functions. Applying the methods in the paper, the geometric properties of the co-analytic part and harmonic function with the analytic part satisfying other conditions can be obtained, which can enrich the research field of univalent harmonic mapping.

Acknowledgments

This work was supported by Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region(Grant No. NJZY20198; Grant No. NJZZ19209), Natural Science Foundation of Inner Mongolia Autonomous Region of China (Grant No. 2020MS01011; Grant No. 2019MS01023; Grant No. 2018MS01026) and the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region Under Grant (Grant No. NJYT-18-A14).

Conflict of interest

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

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