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Research article

Relation-theoretic fixed point theorems under a new implicit function with applications to ordinary differential equations

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Abstract: In this paper, we introduce a new implicit function without any continuity requirement and utilize the same to prove unified relation-theoretic fixed point results. We adopt some examples to exhibit the utility of our implicit function. Furthermore, we use our results to derive some multidimensional fixed point results. Finally, as applications of our results, we study the existence and uniqueness of solution for a first-order ordinary differential equations.

Keywords: fixed point; multidimensional fixed point; implicit function; binary relation; ordinary differential equations
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1. Introduction

The celebrated Banach contraction principle [7] is an indispensable result of metric fixed point theory. This fundamental result has been extended and generalized in various directions. This principle is utilized in diverse applications in the domain of mathematics and outside it as well. In recent years, several researchers attempted to unify the existing extensions and generalizations of Banach contraction principle employing varies methods. A very simple and effective method of carrying out such unifications is essentially due to Popa [26] wherein the author initiated the idea of implicit functions.

In Section 3 of this manuscript, we have a new class of implicit functions which is general enough to deduce several known fixed point theorems in one go besides being general enough to yield new but unknown contractions. Some examples are also given to support this view point.

The branch of related metric fixed point theory is a relatively new branch was initially studied by Turinici [35]. Now a days, this direction of research becomes very active especially after the existence of the fantastic articles due to Ran and Reurings [29] and Nieto and Rodriguez-lopez [24,25] which also contain fruitful applications. Recently, this direction of research is undertaken by several researchers such as: Bhaskar and Lakshmikantham [11], Samet and Turinici [33], Ben-El-Mechaiekh [8], Imdad et al. [14, 17], Mursaleen et al. [23] and some others.

In Section 4, we prove some relation-theoretic fixed point theorems utilizing our newly introduced implicit function. Some corollaries are deduced which cover several known as well as unknown fixed point results.

On the other hand, the extensions of coupled fixed point up to higher dimensional product set carried out by several authors are not unique ($c.f$ [3]). The first attempt to unify the multi-tupled fixed point notions was due to Berzig and Samet [10], wherein they defined a unified notion of N-tupled fixed point. Thereafter, the notion of N-tupled fixed point was extended by Roldán et al. [31] by introducing Υ-fixed point. Soon, Alam et al. [3] modified the notion of Υ-fixed point by introducing $*$-fixed point.

In Section 5, we apply Theorems 4.1 and 4.2 to deduce some multidimensional fixed point results utilizing the notion of $*$-fixed point. The proved results unify numerous multidimensional fixed point results of the existing literature especially those contained in [3, 9–11].

The existing literature contains numerous results on the existence of solutions for ordinary differential equations (in short ODE) in the presence of lower as well as upper solutions of the ODE problems under consideration. In Section 6, inspired by [24, 25], we establish the existence and uniqueness of the solution of the problem described by (6.1).

From now on, $\mathbb{N}, \mathbb{N}_0, \mathbb{R}_+$ and $\mathbb{R}$, respectively, refer to the set of: natural numbers, whole numbers, non-negative real numbers and real numbers. Also, M is a nonempty set, $f : M \to M$, $Fix(f) = \{x \in M : x = fx\}$ and (M, d) is a metric space. For brevity, we write fx instead of $f(x)$ and $\{x_n\} \to x$ whenever $\{x_n\}$ converges to x. Let $x_0 \in M$, a sequence $\{x_n\} \subseteq M$ defined by $x_{n+1} = f^n x_0 = fx_n$, for all n, is called a *Picard sequence* based on x_0.

2. Relation theoretic notions and auxiliary results

A binary relation $\mathcal{S}$ on M is a subset of $M \times M$. $M \times M$ is always a binary relation on M known as universal relation. We write $x\mathcal{S}y$ whenever $(x, y) \in \mathcal{S}$ and $x\mathcal{S}^{\neq}y$ whenever $x\mathcal{S}y$ and $x \neq y$. Observe that $\mathcal{S}^{\neq}$ is also a binary relation on M such that $\mathcal{S}^{\neq} \subseteq \mathcal{S}$. The points x and y are said to be $\mathcal{S}$-*comparable* if $x\mathcal{S}y$ or $y\mathcal{S}x$ which is often denoted by $[x, y] \in \mathcal{S}$. Throughout this work, $\mathcal{S}$ stands for a binary relation defined on M, $\mathcal{S}_M$ stands for the universal relation on M and $M(f, \mathcal{S}) = \{x \in M : x\mathcal{S}fx\}$.

Definition 2.1. (see [1, 12, 20, 21]) A binary relation $\mathcal{S}$ is said to be:

(i) *amorphous* if it is an arbitrary relation;

(ii) *reflexive* if $x\mathcal{S}x$, $\forall x \in M$;

(iii) *transitive* if $x\mathcal{S}z$ whenever $x\mathcal{S}y$ and $y\mathcal{S}z$, $\forall x, y, z \in M$;

(iv) *antisymmetric* if $x\mathcal{S}y$ and $y\mathcal{S}x$ imply $x = y$, $\forall x, y \in M$;

(v) *partial order* if it satisfies (ii), (iii) and (iv);

(vi) *complete or connected* if $[x, y] \in \mathcal{S}$, $\forall x, y \in M$;

(vii) f-*closed* if $x\mathcal{S}y$ implies $fx\mathcal{S}fy$, $\forall x, y \in M$.

Definition 2.2. [2] A sequence $\{x_n\} \subseteq M$ is called *$\mathcal{S}$-preserving sequence* if $x_n \mathcal{S} x_{n+1}$, $\forall n$.

Definition 2.3. [2] A mapping $f : M \to M$ is called *$\mathcal{S}$-continuous at $x \in M$* if for any $\mathcal{S}$-preserving sequence $\{x_n\} \subseteq M$ such that $\{x_n\} \to x$, we have $\{f x_n\} \to f x$. Furthermore, f is called *$\mathcal{S}$-continuous* if it is *$\mathcal{S}$-continuous* at each point of M.

Definition 2.4. [34] A subset $B \subseteq M$ is called *precomplete* if each Cauchy sequence $\{x_n\} \subseteq B$ converges to some $x \in M$.

Definition 2.5. [16] Let $B \subseteq M$. If each $\mathcal{S}$-preserving Cauchy sequence $\{x_n\} \subseteq B$ converges to some $x \in M$, then B is said to be *$\mathcal{S}$-precomplete*.

Remark 2.1. Every precomplete subset of M is $\mathcal{S}$-precomplete, for an arbitrary binary relation $\mathcal{S}$.

Definition 2.6. [1] A binary relation $\mathcal{S}$ on M is called *d-self-closed* if for any $\mathcal{S}$-preserving sequence $\{x_n\}$ converging to x, $\exists \{x_{n_k}\} \subseteq \{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{S}$, $\forall k \in \mathbb{N}$.

3. An implicit function

A natural and simple way to present unified fixed point results was possible via implicit function when Popa [26] initiated the idea of implicit functions wherein he proved some common fixed point theorems using continuous implicit function which is general enough to deduce several known fixed point theorems in one go besides being general enough to deduce several fixed point results under new contractions. Thereafter, several authors used the idea of implicit functions assuming several suitable assumptions (e.g., [4–6, 15, 18, 22, 27, 28, 30] and references therein). We are not familiar with any article dealing with implicit functions without continuity assumption deducing contraction mappings in complete metric spaces but in this paper we endeavor to do so. With this idea in mind, we introduce a new implicit function without continuity with merely two requirements.

Definition 3.1. Let $\mathcal{E}$ be the class of all functions $E : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying:

(E_1) E is non-increasing in the 4^{th}, 5^{th} and 6^{th} variables;

(E_2) $\exists\ \lambda \in [0, 1)$ such that

$$E(u, v, w, u + v + w, u + w, v + w) \leq 0 \text{ implies } u \leq \lambda v \ \ \forall\ u, v, w \in [0, \infty).$$

Example 3.1. Define $E : \mathbb{R}_+^6 \to \mathbb{R}$ by: $E(t_1, t_2, ..., t_6) = t_1 - \lambda t_2$, where $\lambda \in [0, 1)$, then $E \in \mathcal{E}$.

Example 3.2. Define $E : \mathbb{R}_+^6 \to \mathbb{R}$ by:

$$E(t_1, t_2, ..., t_6) = \begin{cases} t_1 - \lambda \frac{t_2 t_6}{t_2 + t_3}, & \text{if} \quad t_2 + t_3 \neq 0; \\ t_1 - \lambda t_2, & \text{if} \quad t_2 + t_3 = 0, \end{cases}$$

where $\lambda \in [0, 1)$, then $E \in \mathcal{E}$.

Example 3.3. Define $E : \mathbb{R}_+^6 \to \mathbb{R}$ by:

$$E(t_1, t_2, ..., t_6) = \begin{cases} t_1 - \lambda t_6 \frac{t_4 - t_3}{t_2 + t_3}, & \text{if} \quad t_2 + t_3 \neq 0; \\ t_1, & \text{if} \quad t_2 + t_3 = 0, \end{cases}$$

where $\lambda \in [0, \frac{1}{2})$, then $E \in \mathcal{E}$.

Example 3.4. Define $E : \mathbb{R}_+^6 \to \mathbb{R}$ by:

$$E(t_1, t_2, ..., t_6) = \begin{cases} t_1 - \lambda t_2 \frac{t_1+t_2+t_4+t_5+t_6}{t_1+t_2+t_3}, & \text{if} \quad t_1 + t_2 + t_3 \neq 0; \\ t_1, & \text{if} \quad t_1 + t_2 + t_3 = 0, \end{cases}$$

where $\lambda \in [0, \frac{1}{3})$, then $E \in \mathcal{E}$.

Example 3.5. Define $E : \mathbb{R}_+^6 \to \mathbb{R}$ by:

$$E(t_1, t_2, ..., t_6) = \begin{cases} t_1^p - \lambda t_2^p - \left(\frac{t_6}{t_2+t_3}\right)^p + \left(\frac{t_1+t_2+t_3}{t_4}\right)^p, & \text{if} \quad t_2 + t_3 \neq 0 \text{ and } t_4 \neq 0; \\ t_1, & \text{if} \quad t_2 + t_3 = 0 \text{ or } t_4 = 0, \end{cases}$$

where $\lambda \in [0, 1)$ and $p \geq 1$, then $E \in \mathcal{E}$.

Example 3.6. Define $E : \mathbb{R}_+^6 \to \mathbb{R}$ by: $E(t_1, t_2, ..., t_6) = t_1 - \lambda \max\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\} + \lambda t_3$, where $\lambda \in [0, \frac{1}{2})$, then $E \in \mathcal{E}$.

Example 3.7. Define $E : \mathbb{R}_+^6 \to \mathbb{R}$ by: $E(t_1, t_2, ..., t_6) = t_1 - \lambda \min\{t_2, t_4, t_6\}$, where $\lambda \in [0, 1)$, then $E \in \mathcal{E}$.

Example 3.8. Define $E : \mathbb{R}_+^6 \to \mathbb{R}$ by: $E(t_1, t_2, ..., t_6) = t_1 - at_2 + 2bt_3 - b(t_4 + t_5)$, where $a, b \geq 0$ and $a + 3b < 1$, then $E \in \mathcal{E}$.

4. Fixed point results

Here, we provide unified relation-theoretic fixed point results via our newly introduced implicit function beginning with the following one.

Theorem 4.1. *Let (M, d) be a metric space, $\mathcal{S}$ a binary relation on M and $f : M \to M$. Suppose:*

(a) $M(f, \mathcal{S})$ is non-empty;

(b) $\mathcal{S}^{\varkappa}$ is f-closed;

(c) fM is $\mathcal{S}^{\varkappa}$-precomplete;

(d) f is $\mathcal{S}^{\varkappa}$-continuous;

(e) $\exists E \in \mathcal{E}$ such that ($\forall x, y \in M$ with $x\mathcal{S}^{\varkappa}y$),

$$E(d(fx, fy), d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)) \leq 0.$$

Then f has a fixed point.

Proof. Due to (a), $\exists\ x_0 \in M$ such that $x_0\mathcal{S}fx_0$. Let $\{x_n\} \subseteq M$ be given by $x_{n+1} = f^{n+1}x_0 = fx_n$, $\forall\ n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ (for $n_0 \in \mathbb{N}_0$), then we are done as $x_{n_0} = fx_{n_0}$. Now, suppose $x_n \neq x_{n+1}$, $\forall n \in \mathbb{N}_0$. As $x_0\mathcal{S}fx_0$ and $x_n \neq x_{n+1}$ ($\forall\ n \in \mathbb{N}_0$), we have $x_0\mathcal{S}^{\varkappa}x_1$ and in general $x_n\mathcal{S}^{\varkappa}x_{n+1}$ ($\forall n \in \mathbb{N}_0$) due to f-closedness of $\mathcal{S}^{\varkappa}$. Now, using (e), we get ($\forall n \in \mathbb{N}_0$)

$$E(d(fx_n, fx_{n+1}), d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), d(x_n, fx_{n+1}), d(x_{n+1}, fx_n)) \leq 0,$$

which together with triangle inequality and (E_1) give rise

$$E(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) + 2d(x_n, x_{n+1}),$$
$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}), 2d(x_n, x_{n+1})) \leq 0,$$

so that (in view of (E_2)) there exists $\lambda \in [0, 1)$ such that

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}),\ \forall\ n \in \mathbb{N}_0. \tag{4.1}$$

Using induction on n in (4.1), we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1),\ \forall\ n \in \mathbb{N}. \tag{4.2}$$

Letting $n \to \infty$ in (4.2), we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.3}$$

Let $n, m \in \mathbb{N}$ with $n < m$. Now, on using triangular inequality and (4.2), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq (\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1}) d(x_0, x_1) \\ &= d(x_0, x_1)\lambda^n \sum_{i=0}^{m-n-1} \lambda^i \\ &= d(x_0, x_1)\lambda^n \frac{1-\lambda^{m-n}}{1-\lambda} \\ &< d(x_0, x_1)\frac{\lambda^n}{1-\lambda} \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

Hence, $\{x_n\}$ is Cauchy. Since fM is $\mathcal{S}^{\varkappa}$-precomplete and $\{x_n\}_{n\geq 1} \subseteq fM$ is $\mathcal{S}^{\varkappa}$-preserving Cauchy sequence, therefore $\exists x \in M$ in which $\{x_n\} \to x$.

As f is $\mathcal{S}^{\varkappa}$-continuous and $\{x_n\}$ is $\mathcal{S}^{\varkappa}$-preserving sequence converges to x, we have $\{x_{n+1} = fx_n\} \to fx$. Therefore, we have $fx = x$ (as the limit is unique). The end. □

Next, we present an analogous of Theorem 4.1 utilizing the d-self-closedness.

Theorem 4.2. *Theorem 4.1 holds true if the condition* (d) *is replaced by:*
(d') $\mathcal{S}^{\varkappa}$ *is d-self-closed.*

Proof. As in the proof of Theorem 4.1, one can see that $\{x_n\}$ is $\mathcal{S}^{\varkappa}$-preserving Cauchy sequence converging to x. In view (d'), $\exists \{x_{n_k}\} \subseteq \{x_n\}$ such that $[x, x_{n_k}] \in \mathcal{S}^{\varkappa}$. This implies that either $x\mathcal{S}^{\varkappa}x_{n_k}$ or $x_{n_k}\mathcal{S}^{\varkappa}x$. Assume that $x\mathcal{S}^{\varkappa}x_{n_k}$. On using condition (e), $\exists E \in \mathcal{E}$ satisfying

$$E(d(fx, fx_{n_k}), d(x, x_{n_k}), d(x, fx), d(x_{n_k}, fx_{n_k}), d(x, fx_{n_k}), d(x_{n_k}, fx)) \leq 0,$$

which together with triangle inequality and (E_1) give rise

$$E(d(fx, x_{n_k+1}), d(x, x_{n_k}), d(x, fx), d(x_{n_k}, x) + d(x, fx) + d(fx, x_{n_k+1}),$$
$$d(x, fx) + d(fx, x_{n_k+1}), d(x_{n_k}, x) + d(x, fx)) \leq 0,$$

so that (in view of (E_2)), $\exists \lambda \in [0, 1)$ in which $d(fx, x_{n_k+1}) \leq \lambda d(x, x_{n_k})$, which on making $k \to \infty$ gives rise $\{x_{n_k}\} \to fx$ (as $\{x_{n_k}\} \to x$). So, $fx = x$ (as the limit is unique). The proof of the case $x_{n_k}\mathcal{S}^{\varkappa}x$ is similar. The end. □

The following condition is useful in the next result:

(U) for each $x, y \in Fix(f)$ $\exists z \in M$ such that z is $\mathcal{S}$-comparable to both x and y.

Theorem 4.3. *Adding the condition (U) to the assumptions of Theorem 4.1 (or Theorem 4.2) ensures the uniqueness of the fixed point of f.*

Proof. Theorem 4.1 (or Theorem 4.2) ensures that the set $Fix(f)$ is not empty. Now, let $x, y \in Fix(f)$. Due to the condition (U), there is $z_0 \in M$ such that $[x, z_0] \in \mathcal{S}$ and $[y, z_0] \in \mathcal{S}$. Let $\{z_n\}$ be the sequence given by $z_{n+1} = fz_n$, $\forall n \in \mathbb{N}_0$. Now, we show $x = y$ by proving $\{z_n\} \to x$ and $\{z_n\} \to y$.

As $[x, z_0] \in \mathcal{S}$, either $x\mathcal{S}z_0$ or $z_0\mathcal{S}x$. Suppose that $x\mathcal{S}z_0$. If $x = z_{n_0}$, for $n_0 \in \mathbb{N}_0$, then $x = z_n$, for all $n \geq n_0$. Thus, $\{z_n\} \to x$. If $x \neq z_n$ $(\forall n \in \mathbb{N}_0)$, then $x\mathcal{S}^{\varkappa}z_0$. As $\mathcal{S}^{\varkappa}$ is f-closed, we have $x\mathcal{S}^{\varkappa}z_n$, for all $n \in \mathbb{N}_0$. Using condition (e), we have

$$E\big(d(fx, fz_n), d(x, z_n), d(x, fx), d(z_n, fz_n), d(x, fz_n), d(z_n, fx)\big) \leq 0,$$

which on using triangle inequality and (E_1), gives rise

$$E\big(d(x, z_{n+1}), d(x, z_n), 0, d(z_n, x) + d(x, z_{n+1}), d(x, z_{n+1}), d(z_n, x)\big) \leq 0,$$

so that (in view of (E_2)) $\exists$ $\lambda \in [0, 1)$ such that $d(x, z_{n+1}) \leq \lambda d(x, z_n)$. By induction on n, we have $d(x, z_{n+1}) \leq \lambda^{n+1} d(x, z_0)$, $\forall$ $n \in \mathbb{N}_0$. Making $n \to \infty$, we have

$$\lim_{n\to\infty} z_n = x.$$

The proof of the case $z_0\mathcal{S}x$ is similar. Also, by the same argument one can show that $\{z_n\} \to y$. This accomplishes the proof. □

Combining Examples 3.1–3.8 with Theorems 4.1–4.3, we can deduce several corollaries as follows.

Corollary 4.1. *Theorems 4.1–4.3 hold true if ($\forall x, y \in M$ with $x\mathcal{S}^{\varkappa}y$) the implicit function in the condition (e) is substantiated by any one of the following:*

(i) $d(fx, xy) \leq \lambda d(x, y)$, $\lambda \in [0, 1)$;

(ii)

$$d(fx, fy) \leq \begin{cases} \lambda \frac{d(x,y)d(y,fx)}{d(x,y)+d(x,fx)}, & \text{if} \quad \Delta \neq 0; \\ \lambda d(x, y), & \text{if} \quad \Delta = 0, \end{cases}$$

where $\lambda \in [0, 1)$ *and* $\Delta = d(x, y) + d(x, fx)$.

(iii)

$$d(fx, fy) \leq \begin{cases} \lambda d(y, fx)^{\frac{d(y,fy)-d(x,fx)}{d(x,y)+d(x,fx)}}, & \text{if} \quad \Delta \neq 0; \\ 0, & \text{if} \quad \Delta = 0, \end{cases}$$

where $\lambda \in [0, \frac{1}{2})$ *and* $\Delta = d(x, y) + d(x, fx)$.

(iv)

$$d(fx, fy) \leq \begin{cases} \lambda d(y, x)^{\frac{d(fx,fy)+d(x,y)+d(y,fy)+d(x,fy)+d(y,fx)}{d(fx,fy)+d(x,y)+d(x,fx)}}, & \text{if} \quad \Delta \neq 0; \\ 0, & \text{if} \quad \Delta = 0, \end{cases}$$

where $\lambda \in [0, \frac{1}{3})$ *and* $\Delta = d(fx, fy) + d(x, y) + d(x, fx)$.

(v)

$$d(fx,fy)^p \leq \begin{cases} \lambda d(x,y)^p + \left(\frac{d(y,fx)}{d(x,y)+d(x,fx)}\right)^p - \left(\frac{d(fx,fy)+d(x,y)+d(x,fx)}{d(y,fy)}\right)^p, & \text{if } \Delta \neq 0 \text{ and } d(y,fy) \neq 0; \\ 0, \quad \text{if} & \Delta = 0 \text{ or } d(y,fy) = 0, \end{cases}$$

where $\lambda \in [0, \frac{1}{2})$ *and* $\Delta = d(x,y) + d(x,fx)$*.*

(vi) $d(fx,fy) \leq \lambda \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\} - \lambda d(x,fx),\ \lambda \in [0, \frac{1}{2})$;

(vii) $d(fx,fy) \leq \lambda \min\{d(x,y), d(y,fy), d(y,fx)\},\ \lambda \in [0,1)$;

(viii) $d(fx,fy) \leq ad(x,y) - 2bd(x,fx) + bd(y,fy) + bd(x,fy)$*, where* a *and* b *are non-negative reals such that* $a + 3b < 1$*;*

Remark 4.1. Corollary 4.1 (corresponding to the contraction condition (i)) unifies and generalizes the Theorems of [1, 7, 8, 13, 19, 24, 25, 29, 33, 36, 37].

Remark 4.2. All relation-theoretic results embodied in the above corollary corresponding to various contraction conditions are new results in their own right except (i) which presents a weaker version of the main result of [1].

On setting $\mathcal{S} = \mathcal{S}_M$ in Theorem 4.3, we deduce the following unified fixed point result in metric spaces:

Corollary 4.2. *Let* (M,d) *be a complete metric space and* $f : M \to M$*. If there exists* $E \in \mathcal{E}$ *such that (for all* $x, y \in M$ *with* $x \neq y$*)*

$$E(d(fx,fy), d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)) \leq 0,$$

then f *has a unique fixed point.*

Remark 4.3. A corollary similar to Corollary 4.1 can be deduced corresponding to Corollary 4.2.

5. Corresponding multidimensional results

As consequences of Theorems 4.1–4.3, we provide here some existence and uniqueness multidimensional fixed point results.

Let B be a non-empty set. On the lines of [3], recall that a binary operation $*$ on B is a mapping from $B \times B$ to B and a permutation π on B is a one-one mapping defined on B. In what follows, the following notations are useful:

(i) N denotes a natural number ≥ 2.

(ii) $*(i,k)$ is denoted by i_k, for any $(i,k) \in I_N \times I_N$, where $I_N = \{1, 2, ..., N\}$.

(iii) a binary operation $*$ on I_N can be represented by an $N \times N$ matrix throughout its ordered image in such a way that the first and second components run over rows and columns respectively, *i.e.*, $* = [m_{ik}]_{N\times N}$ where $m_{ik} = i_k\ \forall i, k \in I_N$.

(iv) a permutation π on I_N can be represented by an N-tuple throughout its ordered image, *i.e.*, $\pi = (\pi(1), \pi(2), ..., \pi(N))$.

(v) $\mathfrak{B}_N$ stands for the class of all binary operations $*$ on I_N, *i.e.*, $\mathfrak{B}_N = \{*|* : I_N \times I_N \to I_N\}$.

(vi) let $U = (x_1, x_2, ..., x_N) \in M^N$, for $* \in \mathfrak{B}_N$ and for $i \in I_N$, U_i^* denotes the ordered element $(x_{i_1}, x_{i_2}, ..., x_{i_N})$ of M^N. A map $F : M^N \to M$ induces an associated map $F_* : M^N \to M^N$ defined by:

$$F_*(U) = (FU_1^*, FU_2^*, ..., FU_N^*), \text{ for all } U \in M^N.$$

Remark 5.1. It is clear that for each $i \in I_N$, $\{i_1, i_2, ..., i_N\} \subseteq I_N$.

Definition 5.1. Define a binary relation $\mathcal{S}^N$ on M^N as follows:

$$(x_1, x_2, ..., x_N)\mathcal{S}^N(y_1, y_2, ..., y_N) \Leftrightarrow x_i\mathcal{S}y_i, \; i = 1, 2, ..., N.$$

If $F : M^N \to M$ is a mapping, then $M^N(F, \mathcal{S}^N)$ is the set of all $U = (x_1, x_2, ..., x_N) \in M^N$ such that $U\mathcal{S}^N(FU_1^*, FU_2^*, ..., FU_N^*)$.

Definition 5.2. Let $F : M^N \to M$. Then $\mathcal{S}$ is called F_{N^*}*-closed* if for any $(x_1, ..., x_N), (y_1, ..., y_N) \in M^N$,

$$\left\{\begin{array}{c} (x_1, y_1) \in \mathcal{S} \\ (x_2, y_2) \in \mathcal{S} \\ \vdots \\ (x_N, y_N) \in \mathcal{S} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (F(x_{1_1}, x_{1_2}, ..., x_{1_N}), F(y_{1_1}, y_{1_2}, ..., y_{1_N})) \in \mathcal{S} \\ (F(x_{2_1}, x_{2_2}, ..., x_{2_N}), F(y_{2_1}, y_{2_2}, ..., y_{2_N})) \in \mathcal{S} \\ \vdots \\ (F(x_{N_1}, x_{N_2}, ..., x_{N_N}), F(y_{N_1}, y_{N_2}, ..., y_{N_N})) \in \mathcal{S} \end{array}\right\}$$

Definition 5.3. [3] Let $* \in \mathfrak{B}_N$ and $F : M^N \to M$. Then $(x_1, ..., x_N) \in M^N$ is called an N*-tupled* fixed point (in short, $*$*-fixed point*) of F w.r.t. $*$ if

$$F(x_{i_1}, ..., x_{i_N}) = x_i, \quad \text{for each } i \in I_N.$$

Example 5.1. The following selection of $* \in \mathfrak{B}_N$ represent the concept of fixed point of order N given by Berzig and Samet [32]:

$$\begin{bmatrix} 1 & 2 & ... & N \\ 2 & 3 & ... & 1 \\ \vdots & \vdots & & \vdots \\ N & 1 & ... & N-1 \end{bmatrix}$$

For more examples, one can see [3].

Definition 5.4. [3] Let $F : M^N \to M$ and $(x_1, x_2, ..., x_N) \in M^N$. Then F is called *continuous* at $(x_1, x_2, ..., x_N)$ if given $\{x_1^{(n)}\}, \{x_2^{(n)}\}, ..., \{x_N^{(n)}\} \subseteq M$, we have

$$\left[\{x_1^{(n)}\} \to x_1, \{x_2^{(n)}\} \to x_2, ..., \{x_N^{(n)}\} \to x_N\right] \Rightarrow \{F(x_1^{(n)}, x_2^{(n)}, ..., x_N^{(n)})\} \to F(x_1, x_2, ..., x_N).$$

That is, if given $\{U^{(n)}\} \subseteq M^N$, we have $\{U^{(n)}\} \to U \Rightarrow \{FU^{(n)}\} \to FU$. Furthermore, F is called *continuous* on M^N if it is continuous at each point of M^N.

Definition 5.5. A sequence $\{U^{(n)}\} \subseteq M^N$ is called $\mathcal{S}^N$*-preserving sequence* if $U^{(n)}\mathcal{S}U^{(n+1)}$, for all $n \in \mathbb{N}$.

Remark 5.2. A sequence $\{U^{(n)}\}_{n\geq 1} \subseteq M^N$ is $\mathcal{S}^N$-preserving if and only if $\{x_i^{(n)}\}_{n\geq 1} \subseteq M$ (for each $i \in \{1, 2, ..., N\}$) is $\mathcal{S}$-preserving, where $U^{(n)} = (x_1^{(n)}, x_2^{(n)}, ..., x_N^{(n)})$, for all $n \geq 1$.

Definition 5.6. Let $F : M^N \to M$ and $U \in M^N$. Then F is $\mathcal{S}^N$*-continuous* at U if given any $\mathcal{S}^N$-preserving sequence $\{U^{(n)}\} \subseteq M^N$ with $\{U^{(n)}\} \to U$, we have $\{FU^{(n)}\} \to FU$. Furthermore, F is $\mathcal{S}^N$*-continuous* if it is $\mathcal{S}^N$-continuous at each point of M^N.

The following auxiliary results exhibit that multidimensional concepts can be interpreted in terms of F_*.

Lemma 5.1. *Let $* \in \mathfrak{B}_N$ and $F : M^N \to M$ a mapping. A point $U = (x_1, x_2, ..., x_N) \in M^N$ is a $*$-fixed point of F w.r.t. $*$ iff it is a fixed point of F_*.*

Proof. Observe that

$$\begin{aligned} & U = (x_1, x_2, ..., x_N) \text{ is a } *\text{-fixed point of } F \Leftrightarrow F(x_{i_1}, ..., x_{i_N}) = x_i, \forall i \in I_N \\ \Leftrightarrow\ & (F(x_{1_1}, x_{1_2}, ..., x_{1_N}), F(x_{2_1}, ..., x_{2_N}), ..., F(x_{N_1}, ..., x_{N_N})) = (x_1, x_2, ..., x_N) \\ \Leftrightarrow\ & (FU_1^*, FU_2^*, ..., FU_N^*) = U \Leftrightarrow F_*U = U. \end{aligned}$$

□

Lemma 5.2. *Let $F : M^N \to M$ and $U = (x_1, x_2, ..., x_N) \in M^N$. Then $M^N(F, \mathcal{S}^N)$ is non-empty iff $M^N(F_*, \mathcal{S}^N)$ is also non-empty.*

Proof. Observe that

$$U = (x_1, x_2, ..., x_N) \in M^N(F, \mathcal{S}^N) \Leftrightarrow U\mathcal{S}^N(FU_1^*, FU_2^*, ..., FU_N^*) \Leftrightarrow U\mathcal{S}^N F_*U \Leftrightarrow U \in M^N(F_*, \mathcal{S}^N).$$

□

Lemma 5.3. *Let $F : M^N \to M$. Then $\mathcal{S}$ is F_{N^*}-closed on M iff $\mathcal{S}^N$ is F_*-closed on M^N.*

Proof. Observe that $\mathcal{S}$ is F_{N^*}-closed

$$\begin{aligned} \Leftrightarrow\ & \left[\left\{\begin{array}{c} (x_1, y_1) \in \mathcal{S} \\ (x_2, y_2) \in \mathcal{S} \\ \vdots \\ (x_N, y_N) \in \mathcal{S} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (F(x_{1_1}, ..., x_{1_N}), F(y_{1_1}, ..., y_{1_N})) \in \mathcal{S} \\ (F(x_{2_1}, ..., x_{2_N}), F(y_{2_1}, ..., y_{2_N})) \in \mathcal{S} \\ \vdots \\ (F(x_{N_1}, ..., x_{N_N}), F(y_{N_1}, ..., y_{N_N})) \in \mathcal{S} \end{array}\right\}\right] \\ \Leftrightarrow\ & \Big[((x_1, x_2, ..., x_N), (y_1, y_2, ..., y_N)) \in \mathcal{S}^N \Rightarrow \\ & \quad \Big((F(x_{1_1}, x_{1_2}, ..., x_{1_N}), F(x_{2_1}, ..., x_{2_N}), ..., F(x_{N_1}, ..., x_{N_N})), \\ & \qquad (F(y_{1_1}, y_{1_2}, ..., y_{1_N}), F(y_{2_1}, ..., y_{2_N}), ..., F(y_{N_1}, ..., y_{N_N}))\Big) \in \mathcal{S}^N\Big] \\ \Leftrightarrow\ & \Big[((x_1, ..., x_N), (y_1, ..., y_N)) \in \mathcal{S}^N \Rightarrow (F_*(x_1, ..., x_N), F_*(y_1, ..., y_N)) \in \mathcal{S}^N\Big] \\ \Leftrightarrow\ & \mathcal{S}^N \text{ is } F_*\text{-closed.} \end{aligned}$$

□

Lemma 5.4. *Let (M, d) be a metric space and $N \in \mathbb{N}$. Consider the product space M^N and define a metric Δ_N on M^N as follows:*

$$\Delta_N(U, V) = \frac{1}{N} \sum_{i=1}^{N} d(x_i, y_i), \text{ for all } U = (x_1, x_2, ..., x_N), V = (y_1, y_2, ..., y_N) \in M^N.$$

Then:

(i) (M^N, Δ_N) *is a metric space;*

(ii) *if* $\{U^{(n)} = (x_1^{(n)}, x_2^{(n)}, ..., x_N^{(n)})\}$ *is a sequence in* M^N *and* $U = (x_1, x_2, ..., x_N)\} \in M^N$*, then* $\{U^{(n)}\} \xrightarrow{\Delta_N} U$ *iff* $\{x_i^{(n)}\} \xrightarrow{d} x_i$*,* $\forall i \in \{1, 2, ..., N\}$*;*

(iii) *if* $\{U^{(n)} = (x_1^{(n)}, x_2^{(n)}, ..., x_N^{(n)})\}$ *is a sequence, then* $\{U^{(n)}\}$ *is Cauchy in* (M^N, Δ_N) *iff* $\{x_i^{(n)}\}$ *is Cauchy in* (M, d)*,* $\forall i \in \{1, 2, ..., N\}$*;*

(iv) *if* $W \subseteq M$*, then* (W, d) *is* $\mathcal{S}$*-precomplete iff* (W^N, Δ_N) *is* $\mathcal{S}^N$*-precomplete;*

(v) $\mathcal{S}$ *is d-self-closed iff* $\mathcal{S}^N$ *is* Δ_N*-self-closed;*

Proof. The proofs of $(i), (ii)$ and (iii) are trivial. To prove (iv) assume that W is $\mathcal{S}$-precomplete. Let $\{U^{(n)} = (x_1^{(n)}, ..., x_N^{(n)})\}$ be an $\mathcal{S}^N$-preserving Cauchy sequence in W^N. Then $\{x_i^{(n)}\}$ (for all $i \in \{1, 2, ..., N\}$) is an $\mathcal{S}$-preserving Cauchy sequence in W. By our hypothesis there is $x_i \in M$ such that $\{x_i^{(n)}\} \xrightarrow{d} x_i$. Thus, $U = (x_1, x_2, ..., x_N) \in M^N$ is such that $\{U^{(n)}\} \xrightarrow{\Delta_N} U$. Hence, (W^N, Δ_N) is $\mathcal{S}^N$-precomplete. On the other hand, assume that (W^N, Δ_N) is $\mathcal{S}^N$-precomplete. Let $\{x^{(n)}\}$ be an $\mathcal{S}$-preserving Cauchy sequence in W. Then $\{U^{(n)} = (x^{(n)}, ..., x^{(n)})(N - times)\}$ is an $\mathcal{S}$-preserving Cauchy sequence in W^N. Thus our assumption ensures the existence of a point $U = (x, x, ..., x) \in W^N$ such that $\{U^{(n)}\} \xrightarrow{\Delta_N} U$. This implies that $\{x^{(n)}\} \xrightarrow{d} x$ with $x \in W$. Hence, W is $\mathcal{S}$-precomplete. The proof of (v) is similar to (iv). □

Lemma 5.5. *[3] Let* $* \in \mathfrak{B}_N$*. Then, for any* $U = (x_1, x_2, ..., x_N)$*,* $V = (y_1, y_2, ..., y_N) \in M^N$ *and for each* $i \in I_N$*,*
$\frac{1}{N}\sum_{k=1}^{N} d(x_{i_k}, y_{i_k}) = \frac{1}{N}\sum_{j=1}^{N} d(x_j, y_j) = \Delta_n(U, V)$*, provided* $*$ *is permuted.*

Lemma 5.6. *If* F *is* $\mathcal{S}^N$*-continuous, then* F_* *is* $\mathcal{S}^N$*-continuous.*

Proof. Let $\{U^{(n)}\}$ be an $\mathcal{S}^N$-preserving sequence such that $\{U^{(n)}\} \xrightarrow{\Delta_N} U$, for some $U \in M^N$, where $U^{(n)} = (x_1^{(n)}, x_2^{(n)}, ..., x_N^{(n)})$ and $U = (x_1, x_2, ..., x_N)$. This (in view of Remark 5.2 and part (ii) of Lemma 5.4) implies that $x_i^{(n)} \mathcal{S} x_i^{(n+1)}$ and $\{x_i^{(n)}\} \xrightarrow{d} x_i$, for all $i \in I_N$. It follows (for each $i \in I_N$) that
$x_{i_1}^{(n)} \mathcal{S} x_{i_1}^{(n+1)}, x_{i_2}^{(n)} \mathcal{S} x_{i_2}^{(n+1)}, ..., x_{i_N}^{(n)} \mathcal{S} x_{i_N}^{(n+1)}, \{x_{i_1}^{(n)}\} \xrightarrow{d} x_{i_1}, \{x_{i_2}^{(n)}\} \xrightarrow{d} x_{i_2}, ..., \{x_{i_N}^{(n)}\} \xrightarrow{d} x_{i_N}$. Which (again in view of Remark 5.2 and part (ii) of Lemma 5.4) implies that $U_i^{(n)*} \mathcal{S} U_i^{(n+1)*}$ and $\{U_i^{(n)*}\} \xrightarrow{\Delta_N} U_i^*$, for all $i \in I_N$. As F is $\mathcal{S}^N$-continuous, we obtain (for each $i \in I_N$) $\{FU_i^{(n)*}\} \xrightarrow{d} FU_i^*$, which (in view of part (ii) of Lemma 5.4 and definition of F_*) gives rise $\{F_* U^{(n)}\} \xrightarrow{\Delta_N} F_* U$. Hence, F_* is $\mathcal{S}^N$-continuous. □

Now, we are equipped to present a multidimensional fixed point results beginning with the following existence result using Theorem 4.1.

Theorem 5.1. *Let* (M, d) *be a metric space,* $\mathcal{S}$ *a binary relation on* M*,* $* \in \mathfrak{B}_N$ *and* $F : M^N \to M$*. Assume that:*

(a) $M^N(F, \mathcal{S}^N)$ *is non-empty;*

(b) $\mathcal{S}^{\varkappa^N}$ *is* F_{N^*}*-closed;*

(c) FM^N *is* $\mathcal{S}^{\varkappa}$*-precomplete;*

(d) F *is* $\mathcal{S}^{\star^N}$*-continuous;*

(e) there exists $E \in \mathcal{E}$ *such that, for all* $i \in I_N$ *and for all* $U, V \in M^N$ *(with* $U\mathcal{S}^{\star^N}V$*),*

$$E\Big(\Delta_N((FU_1^*, ..., FU_N^*), (FV_1^*, ..., FV_N^*)), \Delta_N(U, V), \Delta_N(U, (FU_1^*, ..., FU_N^*)),$$
$$\Delta_N(V, (FV_1^*, ..., FV_N^*)), \Delta_N(U, (FV_1^*, ..., FV_N^*)), \Delta_N(V, (FU_1^*, ..., FU_N^*))\Big) \leq 0. \quad (5.1)$$

Then F *has a* $*$*-fixed point w.r.t.* $*$*.*

Proof. Observe that

1) the part (i) of Lemma 5.4 implies (M^N, Δ_N) is metric space;

2) the condition (a) together with Lemma 5.2 imply that $M^N(F_*, \mathcal{S}^N)$ is non-empty;

3) the condition (b) together with Lemma 5.3 imply that $\mathcal{S}^{\star^N}$ is F_*-closed;

4) the condition (c) together with part (iv) of Lemma 5.4 imply F_*M^N is $\mathcal{S}^{\star^N}$-precomplete;

5) the condition (d) together with Lemma 5.6 imply that F_* is $\mathcal{S}^{\star^N}$-continuous.

Therefore, F_* has a fixed point $U = (x_1, x_2, ..., x_N) \in M^N$ (due to Theorem 4.1). In view of Lemma 5.1, U is a $*$-fixed point of F. This concludes the proof. □

Next, we apply Theorem 4.2 to deduce a multidimensional fixed point existence result avoiding the continuity assumption which runs as follows:

Theorem 5.2. *The conclusion of Theorem 5.1 remains true if the condition* (d) *is replaced by:*

(d') $\mathcal{S}^{\star^N}$ *is* Δ_N*-self-closed.*

Proof. Follows in view of Theorem 4.2 and part (v) of Lemma 5.4. □

The following condition is useful to prove the uniqueness of the $*$-fixed point:

(U') for each $U, V \in Fix(F)$ there exists $W \in M^N$ such that W is $\mathcal{S}^N$-comparable to both U and V,

Theorem 5.3. *Adding the condition* (U') *to the assumptions of Theorem 5.1 (or Theorem 5.2) ensures the uniqueness of the* $*$*-fixed point of* F*.*

Remark 5.3. With suitable definitions of E and $*$ in Theorems 5.1–5.3, one can deduce Theorems of [9–11] and Theorem 8 of [3].

Remark 5.4. A corollary similar to Corollary 4.1 can be deduced corresponding to Theorems 5.1–5.3.

6. Applications to ordinary differential equations

As applications of our main results, we will examine in this section the existence and of a unique solution for the first-order periodic boundary value problem:

$$\begin{cases} x'(t) = f(t, x(t)), \quad t \in I = [0, T]; \\ x(0) = x(T), \end{cases} \quad (6.1)$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $T > 0$.

In what follows, $C(I, \mathbb{R})$ denotes the space of all real valued continuous functions defined on I.

Now, we recall the following definition which will be useful in the sequel:

Definition 6.1. (i) A function $x \in C^1(I, \mathbb{R})$ is said to be a solution for (6.1) if it satisfies (6.1).

(ii) A function $\alpha \in C^1(I, \mathbb{R})$ is said to be a lower solution of (6.1) if

$$\alpha'(t) \leq f(t, \alpha(t)),\ t \in I \text{ and } \alpha(0) \leq \alpha(T).$$

(iii) A function $\beta \in C^1(I, \mathbb{R})$ is said to be an upper solution of (6.1) if

$$\beta'(t) \geq f(t, \beta(t)),\ t \in I \text{ and } \beta(0) \geq \beta(T).$$

In the following results Nieto and Rodriguez-Lopez described some suitable conditions to ensure the existence of a unique solution of (6.1).

Theorem 6.1. *[24] Consider problem (6.1) such that f is continuous and there exist $\gamma > 0$ and $\delta > 0$ with $\gamma < \delta$ such that*

$$0 \leq f(t, y) + \delta y - [f(t, x) + \delta x] \leq \gamma(y - x),\ \textit{for all } x, y \in \mathbb{R} \textit{ with } x < y. \tag{6.2}$$

If (6.1) has a lower (or an upper) solution, then it has a unique solution.

Theorem 6.2. *[25] Consider problem (6.1) such that f is continuous and there exist $\gamma > 0$ and $\delta > 0$ with $\gamma < \delta$ such that*

$$-\gamma(y - x) \leq f(t, y) + \delta y - [f(t, x) + \delta x] \leq 0,\ \textit{for all } x, y \in \mathbb{R} \textit{ with } x < y. \tag{6.3}$$

If (6.1) has a lower (or an upper) solution, then it has a unique solution.

Now, under a new condition which unify conditions (6.2) and (6.3), we prove the existence of a unique solution for the first-order periodic problem (6.1) in the presence of a lower solution.

Theorem 6.3. *Consider problem (6.1) such that f is continuous and non-decreasing in the second variable and there exist $\gamma > 0$ and $\delta > 0$ with $\gamma < \delta$ such that*

$$-\gamma(y - x) \leq f(t, y) + \delta y - [f(t, x) + \delta x] \leq \gamma(y - x),\ \textit{for all } x, y \in \mathbb{R} \textit{ with } x < y. \tag{6.4}$$

If (6.1) has a lower solution, then it has a unique solution.

Proof. Observe that problem (6.1) can be written in the following form:

$$\begin{cases} x'(t) + \delta x(t) = f(t, x(t)) + \delta x(t), & t \in I = [0, T]; \\ x(0) = x(T), \end{cases}$$

which is equivalent to the following integral equation:

$$x(t) = \int_0^T G(t, s)[f(s, x(s)) + \delta x(s)]ds,$$

where

$$G(t, s) = \begin{cases} \frac{e^{\delta(T+s-t)}}{e^{\delta T}-1}, & 0 \leq s < t \leq T, \\ \frac{e^{\delta(s-t)}}{e^{\delta T}-1}, & 0 \leq t < s \leq T. \end{cases}$$

Let us define d on M by: $d(x, y) = \sup_{t\in I} |x(t) - y(t)|,\ \forall x, y \in M$. Then the pair (M, d) forms a metric space which is complete so that every subspace of M is precomplete.
Define a binary relation $\mathcal{S}$ on $M = C(I, \mathbb{R})$ as follows:

$$x\mathcal{S}y \Leftrightarrow [x(t) \leq y(t),\ \text{for all } t \in I],\ \text{for all } x, y \in M.$$

Now, define a mapping $\mathcal{K} : M \to M$ by:

$$[\mathcal{K}x](t) = \int_0^T G(t, s)[f(s, x(s)) + \delta x(s)]ds,\ t \in I. \tag{6.5}$$

Notice that $x \in M$ is a fixed point of $\mathcal{K}$ iff it is a solution of (6.1).

Since every subspace of M is precomplete and since every precomplete space is $\mathcal{S}^{\varkappa}$-precomplete, therefore $\mathcal{K}M$ is $\mathcal{S}^{\varkappa}$-precomplete.

Let $\{x_n\} \subseteq M$ be an $\mathcal{S}^{\varkappa}$-preserving sequence converging to $x \in M$. Then, for each $t \in I$, we have

$$x_1(t) < x_2(t) < ... < x_n(t) < .... \tag{6.6}$$

Since $\{x_n(t)\} \subseteq \mathbb{R}$ is $\mathcal{S}^{\varkappa}$-preserving sequence converging to $x(t)$, therefore (6.6) implies that $x_n(t) < x(t)$, $\forall\ t \in I,\ n \in \mathbb{N}$. Observe that $x_n(t) \neq x(t),$ for all $t \in I,\ n \in \mathbb{N}$. As if $x_{n_0}(t) = x(t),$ for all $t \in I$ and some $n_0 \in \mathbb{N}$, then $x_n = x_{n+1}$, $\forall\ n \geq n_0$, a contradiction. Thus, $x_n\mathcal{S}^{\varkappa}x$, $\forall n \in \mathbb{N}$. Thus, $\mathcal{S}^{\varkappa}$ is d-self-closed.

Next, we prove that $\mathcal{K}$ is $\mathcal{S}^{\varkappa}$-closed. Let $x, y \in M$ be such that $x\mathcal{S}^{\varkappa}y$. This amounts to saying that $x(t) < y(t)$, for all $t \in I$. As f is nondecreasing in the second variable, we get (for all $t \in I$)

$$f(t, y(t)) + \delta y(t) > f(t, x(t)) + \delta x(t). \tag{6.7}$$

As $G(t, s) > 0$, $\forall t, s \in I$, so (6.7) implies that

$$[\mathcal{K}x](t) = \int_0^T G(t, s)[f(s, x(s)) + \delta x(s)]ds < \int_0^T G(t, s)[f(s, y(s)) + \delta y(s)]ds = [\mathcal{K}y](t),$$

for all $t \in I$. That is, $\mathcal{K}x\mathcal{S}^{\varkappa}\mathcal{K}y$ so that $\mathcal{S}^{\varkappa}$ is $\mathcal{K}$-closed.

Now, let $x, y \in M$ with $x\mathcal{S}^{\varkappa}y$. Then $x(t) < y(t)$, for all $t \in I$. Observe that

$$\begin{aligned}
d(\mathcal{K}x, \mathcal{K}y) &= \sup_{t\in I} |[\mathcal{K}x](t) - [\mathcal{K}y](t)| \\
&\leq \sup_{t\in I} \int_0^T G(t, s)|f(s, x(s)) + \delta x(s) - f(s, y(s)) - \delta y(s)|ds \\
&\leq \sup_{t\in I} \int_0^T G(t, s)\gamma|x(s) - y(s)|ds \leq \gamma d(x, y) \sup_{t\in I} \int_0^T G(t, s)ds \\
&= \gamma d(x, y) \sup_{t\in I} \frac{1}{e^{\delta T} - 1}\left(\frac{1}{\delta}e^{\delta(T+s-t)}\Big]_0^t + \frac{1}{\delta}e^{\delta(s-t)}\Big]_t^T\right) \\
&= \frac{\gamma}{\delta}d(x, y)\frac{1}{e^{\delta T} - 1}(e^{\delta T} - 1) = \frac{\gamma}{\delta}d(x, y),
\end{aligned}$$

which shows that $\mathcal{K}$ satisfies the corresponding hypothesis (e) in Theorem 4.1 with $E(t_1, t_2, ..., t_6) = t_1 - \frac{\gamma}{\delta}t_2$.

Let $\alpha \in M$ be a lower solution of (6.1). Now, we show that $\alpha \mathcal{S} \mathcal{K}(\alpha)$, i.e., $\alpha \in M(\mathcal{K}, \mathcal{S})$. As α is a lower solution of (6.1), we have

$$\alpha'(t) + \delta\alpha(t) \leq f(t, \alpha(t)) + \delta\alpha(t),\ t \in I.$$

Multiplying both sides of this inequality by $e^{\delta t}$, we have

$$(\alpha(t)e^{\delta t})' \leq [f(t, \alpha(t)) + \delta\alpha(t)]e^{\delta t},\ t \in I,$$

or

$$\alpha(t)e^{\delta t} \leq \alpha(0) + \int_0^t [f(t, \alpha(t)) + \delta\alpha(t)]e^{\delta t} ds,\ t \in I, \tag{6.8}$$

yielding thereby (as $\alpha(0) \leq \alpha(T)$)

$$\alpha(0)e^{\delta T} \leq \alpha(T)e^{\delta T} \leq \alpha(0) + \int_0^T [f(s, \alpha(s)) + \delta\alpha(s)]e^{\delta s} ds,$$

so that

$$\alpha(0) \leq \int_0^T \frac{e^{\delta s}}{e^{\delta T} - 1}[f(s, \alpha(s)) + \delta\alpha(s)]ds,$$

which together with (6.8) imply that

$$\alpha(t)e^{\delta t} \leq \int_0^t \frac{e^{\delta(T+s)}}{e^{\delta T} - 1}[f(s, \alpha(s)) + \delta\alpha(s)]ds + \int_0^T \frac{e^{\delta s}}{e^{\delta T} - 1}[f(s, \alpha(s)) + \delta\alpha(s)]ds,\ t \in I,$$

or

$$\alpha(t) \leq \int_0^t \frac{e^{\delta(T+s-t)}}{e^{\delta T} - 1}[f(s, \alpha(s)) + \delta\alpha(s)]ds + \int_0^T \frac{e^{\delta s - t}}{e^{\delta T} - 1}[f(s, \alpha(s)) + \delta\alpha(s)]ds,\ t \in I,$$

i.e.,

$$\alpha(t) \leq \int_0^T G(t, s)[f(s, \alpha(s)) + \delta\alpha(s)]ds = [\mathcal{K}\alpha](t),\ t \in I,$$

so that $\alpha \mathcal{S} \mathcal{K}(\alpha)$. Hence, Theorem 4.2 ensures the existence of a solution of (6.1). Finally, if $x, y \in Fix(\mathcal{K})$, then $z = \max\{x, y\} \in M$. As $x \leq z$ and $y \leq z$, we have $x\mathcal{S}z$ and $y\mathcal{S}z$ so that Theorem 4.3 shows that the fixed point of $\mathcal{K}$ is unique. Hence, (6.1) has a unique solution. □

Finally, we present an analogous of Theorem 6.3 in the presence of an upper solution.

Theorem 6.4. *Consider problem (6.1) such that f is continuous and non-increasing in the second variable and there exist $\gamma > 0$ and $\delta > 0$ with $\gamma < \delta$ such that*

$$-\gamma(y - x) \leq f(t, y) + \delta y - [f(t, x) + \delta x] \leq \gamma(y - x),\ \textit{for all } x, y \in \mathbb{R} \textit{ with } x < y. \tag{6.9}$$

If (6.1) has an upper solution, then it has a unique solution.

Proof. Define a binary relation $\mathcal{S}$ on M as follows:

$$x\mathcal{S}y \Leftrightarrow [x(t) \geq y(t),\ \text{for all } t \in I],\ \text{for all } x, y \in M.$$

Using analogous procedure of the proof of Theorem 6.3, one can analogously show that all requirements of Theorem 4.2 are fulfilled. Hence, Theorem 4.2 ensures the existence of a fixed point of $\mathcal{K}$ which is unique (due to Theorem 4.3). Thus, (6.1) admits a unique solution. □

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