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Research article

New exact solutions for the Kaup-Kupershmidt equation

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Abstract: We present new exact solutions for the (1+1)-dimensional Kaup-Kupershmidt (KK) equation by employing method of double $(G'/G, 1/G)$-expansion. We express solutions by hyperbolic, trigonometric and rational functions explicitly. Computational results indicate the efficiency and applicability potential of the method.

Keywords: Kaup-Kupershmidt equation; the method of double $(G'/G, 1/G)$-expansion; exact solutions
Mathematics Subject Classification: 35A09, 35E05

1. Introduction

Nonlinear evolution equations (NLEEs) model many complex phenomena in physics including plasma, solid state, chemical and optical fibers, nonlinear optics, fluid mechanics, etc. Exploring exact traveling wave solutions plays a significant role in nonlinear physics. For this purpose, a number of techniques were developed including method of modified Khater [1, 2], first integral [3, 4], functional variable [5], expansions [6, 7] of new generalized (G'/G) [8–10], new Φ6-model [11], Jacobi elliptic function [12, 13], sine-Gordon [14], bifurcation [15, 16], exp-function [17, 18], new auxiliary equation [19], exp(-$\phi(\xi)$)-expansion [20, 21], fan sub-equation [22, 23], inverse scattering [24], generalized Kudryshov [25–27], Hirota's bilinear [28, 29], extended direct algebraic [30], Lie group [31].

Consider the (2+1)-dimensional KK equations [32]

$$\begin{gathered} 9u_t + u_{5x} + 15uu_{xxx} + \tfrac{75}{2}u_xu_{xx} + 45u^2u_x + 5\sigma u_{xxy} - 5\sigma\partial_x^{-1}u_{yy} + 15\sigma uu_y \\ +15\sigma u_x\partial_x^{-1}u_y = 0. \end{gathered} \tag{1.1}$$

where $\sigma^2 = 1$, $\partial_x^{-1} = \int dx$. This equation has been widely applied in many branches of physics like plasma physics, fluid dynamics, nonlinear optics, and so forth. If we take $u(x, y, t) = u(x, t)$, Eq (1.1) becomes the (1+1)-dimensional KK equation [32]

$$9u_t + u_{5x} + 15uu_{xxx} + \frac{75}{2}u_xu_{xx} + 45u^2u_x = 0, \tag{1.2}$$

In [33], method of exp-function was applied to Eq (1.2). In [32], symmetric method was applied to the nonlinear (2+1)-KK equation.

The method of the present paper, a candid, succinct and efficient technique, considered as a generalization of (G'/G)-expansion technique [34–37] was developed in [38–45]. Main purpose of this paper is to investigate the applicability of the method to (1+1)-dimensional KK equation which was not considered in the history of research so far.

2. Double $(G'/G, 1/G)$-expansion technique

We shortly overview the method in such a fashion that maintains four remarks and five basic postulates:

Remark I. If we set up

$$\phi = G'/G, \psi = 1/G, \tag{2.1}$$

in

$$G''(\xi) + \lambda G(\xi) = \beta, \tag{2.2}$$

then we must have the relations

$$\phi' = -\phi^2 + \beta\psi - \lambda, \qquad \varphi' = -\varphi\psi, \tag{2.3}$$

wherein λ and β are parameters.

Remark II. If λ is negative, general solution of (2.2) is:

$$G(\xi) = D_1 \sinh(\xi\sqrt{-\lambda}) + D_2 \cosh(\xi\sqrt{-\lambda}) + \frac{\beta}{\lambda}. \tag{2.4}$$

and we receive the following relation

$$\psi^2 = \frac{-\lambda}{\lambda^2\alpha_1 + \beta^2}\left(\varphi^2 - 2\beta\psi + \lambda\right), \tag{2.5}$$

wherein D_1 and D_2 are arbitrary constants and $\alpha_1 = D_1^2 - D_2^2$.
Remark III. If λ is positive, general solution of (2.2) is:

$$G(\xi) = D_1 \ \sin(\xi\sqrt{\lambda}) + D_2 \ \cos(\xi\sqrt{\lambda}) + \frac{\beta}{\lambda}, \tag{2.6}$$

consequently, we obtain

$$\psi^2 = \frac{\lambda}{\lambda^2\alpha_2 - \beta^2}(\varphi^2 - 2\beta\psi + \lambda), \tag{2.7}$$

wherein $\alpha_2 = D_1^2 + D_2^2$.
Remark IV. If $\lambda = 0$, the general solution of (2.2),

$$G(\xi) = \frac{\beta}{2}\xi^2 + D_1\xi + D_2, \tag{2.8}$$

and therefore we get,

$$\psi^2 = \frac{\varphi^2 - 2\beta\psi}{D_1^2 - 2\beta D_2}. \tag{2.9}$$

Now let us consider:

$$R(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, u_{xt}, \cdots) = 0, \tag{2.10}$$

wherein R is a polynomial function in u and $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{yy} = \frac{\partial^2 u}{\partial y^2}$, $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$ and so on.
Postulate 1. Consider:

$$u(x, y, t) = u(\xi), \quad \text{and} \quad \xi = \eta x + \omega y + ct, \tag{2.11}$$

wherein η, ω and c are parameters. By traveling wave transformations (2.11), the Eq.(2.10) can be reduced to:

$$T(u, cu', \eta u', \omega u', c^2u'', \eta^2u'', \omega^2u'', \eta\omega u'', c\eta u'', \cdots) = 0, \tag{2.12}$$

wherein T is a polynomial.
Postulate 2. Let us assume that the following relation is the general solution expressed by a polynomial:

$$u(\xi) = a_0 + \sum_{i=1}^{N}\left(a_i\varphi^i(\xi) + b_i\varphi^{i-1}(\xi)\psi(\xi)\right), \tag{2.13}$$

wherein a_0, a_i and $b_i(i = 1, 2, 3, ..., N)$ are the constant coefficients such that $a_N^2 + b_N^2 \neq 0$.
Postulate 3. By homogeneous balance, we determine N in Eq (2.13).
Postulate 4. To convert the left-hand-side of Eq (2.12) into a polynomial function in ψ and ϕ, we write Eq (2.13) into Eq (2.12) with Eq (2.3) and Eq (2.5). By solving polynomial, we obtain the system: in a_0, a_i, $b_i(i = 1, 2, 3, ..., N)$, $\lambda(< 0)$, β, η, ω, c, D_1 and D_2. We solve this system with Mathematica. Setting values of above algebraic constants in Eq (2.13), solutions by hyperbolic functions in Eq (2.12) are obtained.
Postulate 5. Similar to Postulate 4, substituting Eq (2.13) into Eq (2.12), using Eq (2.3) and Eq (2.5) (or Eq (2.3) and Eq (2.7)), we obtain the exact traveling wave solutions of Eq (2.12) demonstrated by trigonometric functions.

3. Applications

Let us consider transformation:

$$u(x,t) = u(\xi), \qquad \xi = x + ct, \tag{3.1}$$

wherein c is a parameter, which reduces Eq (1.2) to:

$$9cu' + u^{(5)} + 15uu''' + \frac{75}{2}u'u'' + 45u^2u' = 0. \tag{3.2}$$

According to postulate 2, the positive number $N = 2$ is obtained by balancing between $u^{(5)}$ and u^2u', thus general solutions of Eq (3.2) is:

$$u(\xi) = a_0 + a_1\varphi(\xi) + a_2\varphi^2(\xi) + b_1\psi(\xi) + b_2\varphi(\xi)\psi(\xi), \tag{3.3}$$

whereina_0, a_i and $b_i(i = 1,2)$ are constant coefficients such that$a_N^2 + b_N^2 \neq 0(N = 1,2)$, $\phi(\xi)$ and $\psi(\xi)$ are satisfied by the Eq (2.3). Now, there are three categories of solutions of Eq (3.2):

Category 1: When $\lambda < 0$ (solutions by hyperbolic functions):

Writing Eq (3.3) with Eq (2.3) and Eq (2.5) into Eq (3.2), Eq (3.2) forms a polynomial in $\psi(\xi)$ and $\phi(\xi)$. Solving this polynomial, we obtain a system: a_0, a_1, a_2, b_1, b_2, $\lambda(<0)$, β, c and α_1. Solving this system with Mathematica, we obtain the values of a_0 a_1, a_2, b_1, b_2, β and c as:

Result 1:

$$a_0 = \frac{-10\lambda}{3}, \quad a_1 = 0, \quad a_2 = -4, \quad b_1 = 4\beta, \quad b_2 = \frac{\pm 4\sqrt{\beta^2 + \lambda^2\alpha_1}}{\sqrt{-\lambda}}, \quad c = \frac{-11\lambda^2}{9}, \quad \beta = \beta. \tag{3.4}$$

Writing these constants from Eq (3.4) into (3.3) and by Eq (2.1) and Eq (2.4), we obtain explicit solutions of Eq (1.2):

$$u(\xi) = \frac{-10\lambda}{3} + \frac{4\lambda\left\{D_1\cosh(\xi\sqrt{-\lambda})D_2\sinh(\xi\sqrt{-\lambda})\right\}^2}{\left\{D_1\sinh(\xi\sqrt{-\lambda}+D_2\cosh(\xi\sqrt{-\lambda})+\frac{\beta}{\lambda}\right\}^2} + \frac{4\beta}{\left\{D_1\sinh(\xi\sqrt{-\lambda}+D_2\cosh(\xi\sqrt{-\lambda})+\frac{\beta}{\lambda}\right\}}$$
$$\pm\frac{4\sqrt{\beta^2+\lambda^2\alpha_1}\lambda\left\{D_1\cosh(\xi\sqrt{-\lambda})D_2\sinh(\xi\sqrt{-\lambda})\right\}^2}{\left\{D_1\sinh(\xi\sqrt{-\lambda}+D_2\cosh(\xi\sqrt{-\lambda})+\frac{\beta}{\lambda}\right\}^2} \tag{3.5}$$

wherein $\xi = x - \frac{11\lambda^2 t}{9}$ and $\alpha_1 = D_1^2 - D_2^2$.

In particular, if we choose $D_1 \neq 0$, $D_2 = 0$ and $\beta = 0$ in Eq (3.5), we get:

$$u(x,t) = \frac{-10\lambda}{3} + 4\lambda\coth\left(\sqrt{-\lambda}(x - \frac{11\lambda^2 t}{9})\right)\left\{\coth\left(\sqrt{-\lambda}(x - \frac{11\lambda^2 t}{9})\right)\right.$$
$$\left.\pm\operatorname{csc}h\left(\sqrt{-\lambda}(x - \frac{11\lambda^2 t}{9})\right)\right\}. \tag{3.6}$$

Similarly, if we choose $D_2 \neq 0$, $D_1 = 0$ and $\beta = 0$ in Eq (3.5), we get:

$$u(x,t) = \frac{-10\lambda}{3} + 4\lambda\tanh\left(\sqrt{-\lambda}(x - \frac{11\lambda^2 t}{9})\right)\left\{\tanh\left(\sqrt{-\lambda}(x - \frac{11\lambda^2 t}{9})\right)\right.$$
$$\left.\pm i\sec h\left(\sqrt{-\lambda}(x - \frac{11\lambda^2 t}{9})\right)\right\}, \tag{3.7}$$

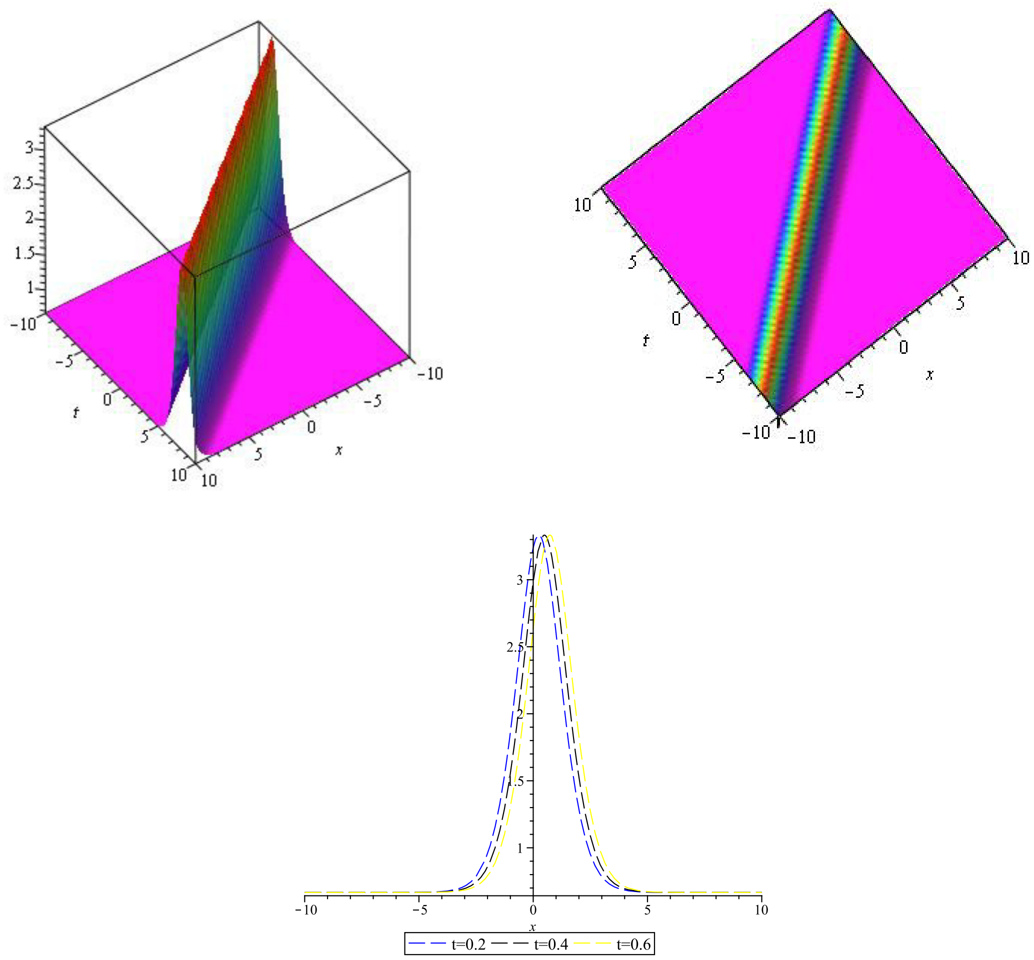

Figure 1. 3D, contour and 2D surfaces of absolute Eq (3.7) when $\lambda = -1$.

wherein $i = \sqrt{-1}$.

Result 2:

$$a_0 = \frac{-5\lambda}{12},\quad a_1 = 0, a_2 = \frac{-1}{2},\quad b_1 = \frac{\beta}{2},\quad b_2 = \frac{\pm\sqrt{\beta^2+\lambda^2\alpha_1}}{2\sqrt{-\lambda}},\quad c = \frac{-\lambda^2}{144},\quad \beta = \beta. \tag{3.8}$$

Explicit solutions of Eq (1.2) are given by:

$$\begin{aligned} u(\xi) = {} & \tfrac{-5\lambda}{12} + \tfrac{\lambda\left\{D_1\ \cosh(\xi\sqrt{-\lambda})+D_2\sinh(\xi\sqrt{-\lambda})\right\}^2}{2\left\{D_1\sinh(\xi\sqrt{-\lambda})+D_2\cosh(\xi\sqrt{-\lambda})+\frac{\beta}{\lambda}\right\}^2} + \tfrac{\beta}{2\left\{D_1\sinh(\xi\sqrt{-\lambda})+D_2\ \cosh(\xi\sqrt{-\lambda})+\frac{\beta}{\lambda}\right\}} \\ & \pm\tfrac{\sqrt{\beta^2+\lambda^2\alpha_1}\left\{D_1\cosh(\xi\sqrt{-\lambda})+D_2\sinh(\xi\sqrt{-\lambda})\right\}}{2\left\{D_1\sinh(\xi\sqrt{-\lambda})+D_2\cosh(\xi\sqrt{-\lambda})+\frac{\beta}{\lambda}\right\}^2}, \end{aligned} \tag{3.9}$$

wherein $\xi = x - \frac{\lambda^2 t}{144}$ and $\alpha_1 = D_1^2 - D_2^2$.

In particular, if we choose $D_1 \neq 0$, $D_2 = 0$ and $\beta = 0$ in Eq (3.9), we get:

$$\begin{aligned} u(x,t) = {} & \tfrac{-5\lambda}{12} + \tfrac{\lambda}{2}\coth\left(\sqrt{-\lambda}(x-\tfrac{\lambda^2 t}{144})\right)\left\{\coth\left(\sqrt{-\lambda}(x-\tfrac{\lambda^2 t}{144})\right)\right. \\ & \left.\pm\csc h\left(\sqrt{-\lambda}(x-\tfrac{\lambda^2 t}{144})\right)\right\}. \end{aligned} \tag{3.10}$$

Similarly, if we choose $D_2 \neq 0$, $D_1 = 0$ and $\beta = 0$ in Eq (3.9), we get:

$$\begin{aligned} u(x,t) = {} & \tfrac{-5\lambda}{12} + \tfrac{\lambda}{2}\tanh\left(\sqrt{-\lambda}(x-\tfrac{\lambda^2 t}{144})\right)\left\{\tanh\left(\sqrt{-\lambda}(x-\tfrac{\lambda^2 t}{144})\right)\right. \\ & \left.\pm i\sec h\left(\sqrt{-\lambda}(x-\tfrac{\lambda^2 t}{144})\right)\right\}, \end{aligned} \tag{3.11}$$

wherein $i = \sqrt{-1}$.

Result3:

$$\begin{aligned} & a_0 = -\tfrac{11\lambda\beta^2+8\lambda^3\alpha_1}{12(\beta^2+\lambda^2\alpha_1)},\quad a_1 = 0, a_2 = -1,\quad b_1 = \beta, \\ & b_2 = 0,\quad c = \tfrac{-\lambda^2(\beta^4-28\lambda^2\beta^2\alpha_1+16\lambda^4\alpha_1^2)}{144(\beta^2+\lambda^2\alpha_1)^2},\quad \beta = \beta. \end{aligned} \tag{3.12}$$

wherein $\beta^2 + \lambda^2\alpha_1 \neq 0$.

We get explicit solutions of Eq (1.2) as:

$$\begin{aligned} u(\xi) = {} & -\tfrac{11\lambda\beta^2+8\lambda^3\alpha_1}{12(\beta^2+\lambda^2\alpha_1)} + \tfrac{\lambda\left\{D_1\cosh(\xi\sqrt{-\lambda})+D_2\sinh(\xi\sqrt{-\lambda})\right\}^2}{\left\{D_1\sinh(\xi\sqrt{-\lambda})+D_2\cosh(\xi\sqrt{-\lambda})+\frac{\beta}{\lambda}\right\}^2} \\ & +\tfrac{\beta}{\left\{D_1\sinh(\xi\sqrt{-\lambda})+D_2\cosh(\xi\sqrt{-\lambda})+\frac{\beta}{\lambda}\right\}}, \end{aligned} \tag{3.13}$$

wherein $\xi = x - \frac{\lambda^2 t(\beta^4-28\lambda^2\beta^2\alpha_1+16\lambda^4\alpha_1^2)}{144(\beta^2+\lambda^2\alpha_1)^2}$ and $\alpha_1 = D_1^2 - D_2^2$.

In particular, if we choose $D_1 \neq 0$, $D_2 = 0$ and $\beta = 0$ in Eq (3.13), we get:

$$u(x,t) = \frac{-2\lambda}{3} + \lambda\coth^2\left(\sqrt{-\lambda}(x-\frac{\lambda^2 t}{9})\right). \tag{3.14}$$

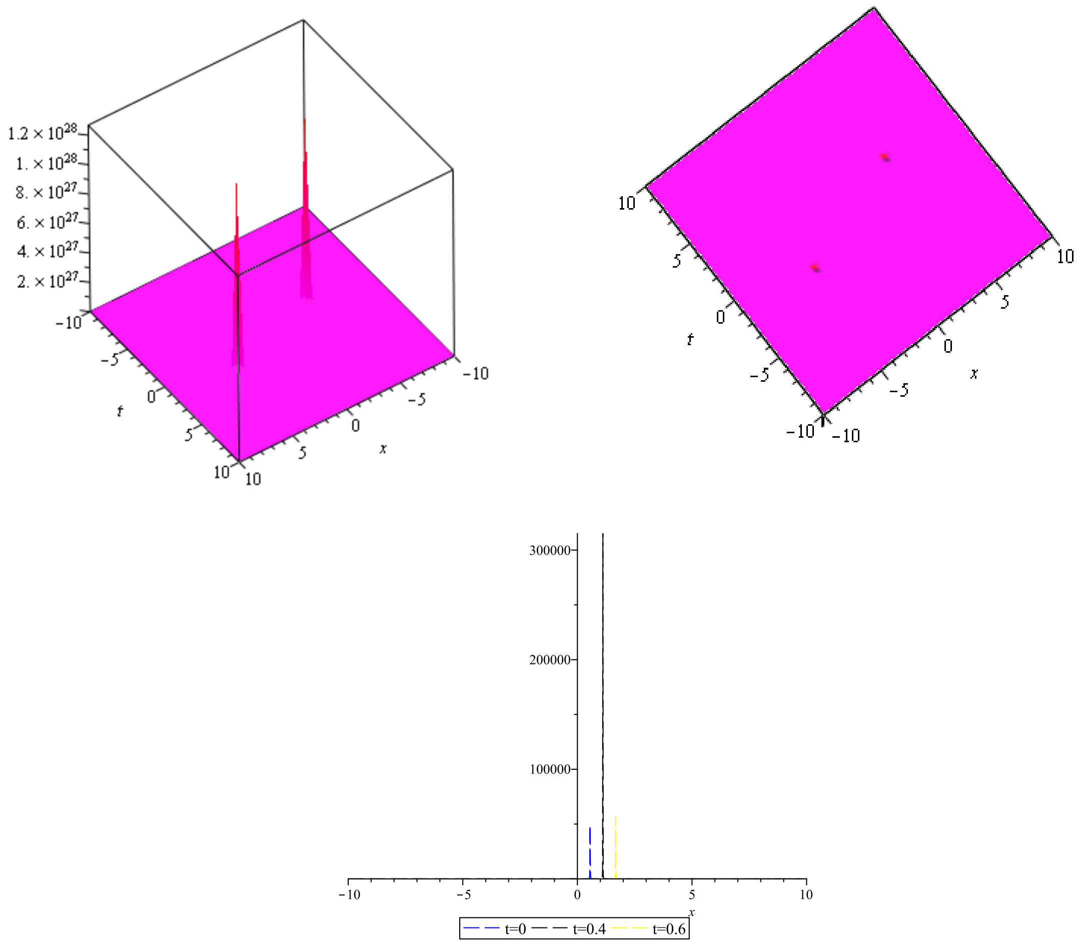

Figure 2. 3D, contour and 2D surfaces of absolute Eq (3.14) when $\lambda = -5$.

Similarly, if we choose $D_2 \neq 0$, $D_1 = 0$ and $\beta = 0$ in Eq (3.14), we get:

$$u(x,t) = \frac{-2\lambda}{3} + \lambda \tanh^2\left(\sqrt{-\lambda}(x - \frac{\lambda^2 t}{9})\right). \tag{3.15}$$

Category 2: For $\lambda > 0$, (i.e. trigonometric functions),
According to Postulate 5, if we execute as the category 1, we attain the values of a_0, a_1, a_2, b_1, b_2, β and c as the following results:
Result 1:

$$\begin{aligned} a_0 &= \tfrac{-10\lambda}{3}, \quad a_1 = 0, a_2 = -4, \quad b_1 = 4\beta, \quad b_2 = \tfrac{\pm 4\sqrt{-\beta^2+\lambda^2\alpha_1}}{\sqrt{\lambda}}, \\ c &= \tfrac{-11\lambda^2}{9}, \quad \beta = \beta. \end{aligned} \tag{3.16}$$

Writing constants in Eq (3.16) into Eq (3.3) and by Eq (2.1) and Eq (2.6), we get explicit solutions of Eq (1.2):

$$\begin{aligned} u(\xi) = \tfrac{-10\lambda}{3} &- \tfrac{4\lambda\left\{D_1\cos(\xi\sqrt{\lambda})-D_2\sin(\xi\sqrt{\lambda})\right\}^2}{\left\{D_1\sin(\xi\sqrt{\lambda})+D_2\ \cos(\xi\sqrt{\lambda})+\frac{\beta}{\lambda}\right\}^2} + \tfrac{4\beta}{D_1\sin(\xi\sqrt{\lambda})+D_2\cos(\xi\sqrt{\lambda})+\frac{\beta}{\lambda}} \\ &\pm \tfrac{4\sqrt{-\beta^2+\lambda^2\alpha_2}\left\{D_1\cos(\xi\sqrt{\lambda})-D_2\sin(\xi\sqrt{\lambda})\right\}}{\left\{D_1\sin(\xi\sqrt{\lambda})+D_2\cos(\xi\sqrt{\lambda})+\frac{\beta}{\lambda}\right\}^2}, \end{aligned} \tag{3.17}$$

wherein $\xi = x - \frac{11\lambda^2 t}{9}$ and $\alpha_2 = D_1^2 + D_2^2$.
Result 2:

$$a_0 = \frac{-5\lambda}{12}, \quad a_1 = 0, \quad a_2 = \frac{-1}{2}, \quad b_1 = \frac{\beta}{2}, \quad b_2 = \frac{\pm\sqrt{-\beta^2+\lambda^2\alpha_1}}{2\sqrt{\lambda}}, \quad c = \frac{-\lambda^2}{144}, \quad \beta = \beta. \tag{3.18}$$

We get explicit solutions of Eq (1.2) as:

$$\begin{aligned} u(\xi) = \tfrac{-5\lambda}{12} &- \tfrac{\lambda\left\{D_1\cos(\xi\sqrt{\lambda})-D_2\sin(\xi\sqrt{\lambda})\right\}^2}{2\left\{D_1\sin(\xi\sqrt{\lambda})+D_2\cos(\xi\sqrt{\lambda})+\frac{\beta}{\lambda}\right\}^2} + \tfrac{\beta}{2\left\{D_1\sin(\xi\sqrt{\lambda})+D_2\cos(\xi\sqrt{\lambda})+\frac{\beta}{\lambda}\right\}} \\ &\pm \tfrac{\sqrt{-\beta^2+\lambda^2\alpha_2}\left\{D_1\cos(\xi\sqrt{\lambda})-D_2\sin(\xi\sqrt{\lambda})\right\}}{2\left\{D_1\sin(\xi\sqrt{\lambda})+D_2\cos(\xi\sqrt{\lambda})+\frac{\beta}{\lambda}\right\}^2}, \end{aligned} \tag{3.19}$$

wherein $\xi = x - \frac{\lambda^2 t}{144}$ and $\alpha_2 = D_1^2 + D_2^2$.
Result 3:

$$\begin{aligned} a_0 &= \tfrac{11\lambda\beta^2-8\lambda^3\alpha_2}{12(-\beta^2+\lambda^2\alpha_2)}, \quad a_1 = 0, a_2 = -1, \quad b_1 = \beta, \\ b_2 &= 0, \quad c = \tfrac{-\lambda^2(\beta^4+28\lambda^2\beta^2\alpha_2+16\lambda^4\alpha_2^2)}{144(-\beta^2+\lambda^2\alpha_2)^2}, \quad \beta = \beta. \end{aligned} \tag{3.20}$$

wherein $-\beta^2 + \lambda^2\alpha_2 \neq 0$.

We get explicit solutions of Eq (1.2) as:

$$\begin{aligned} u(\xi) = \tfrac{11\lambda\beta^2-8\lambda^3\alpha_2}{12(-\beta^2+\lambda^2\alpha_2)} &- \tfrac{\lambda\left\{D_1 cos(\xi\sqrt{\lambda})-D_2\sin(\xi\sqrt{\lambda})\right\}^2}{\left\{D_1\sin(\xi\sqrt{\lambda})+D_2\cos(\xi\sqrt{\lambda})+\frac{\beta}{\lambda}\right\}^2} \\ &+ \tfrac{\beta}{\left\{D_1\sin(\xi\sqrt{\lambda})+D_2\cos(\xi\sqrt{\lambda})+\frac{\beta}{\lambda}\right\}}, \end{aligned} \tag{3.21}$$

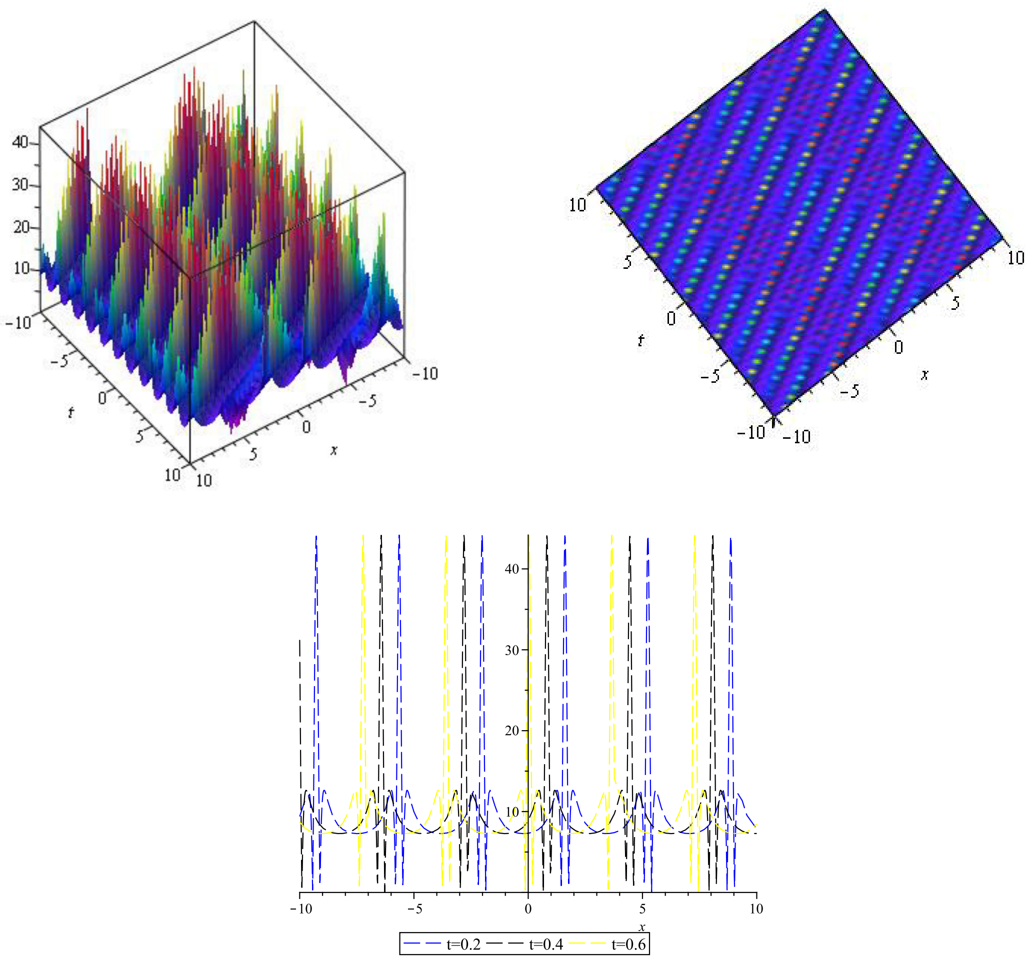

Figure 3. 3D, contour and 2D surfaces of Eq (3.21) when $\lambda = 3, D_1 = 0.8, D_2 = 0.5, \beta = 3$.

wherein $\xi = x - \frac{\lambda^2 t(\beta^4+28\lambda^2\beta^2\alpha_2+16\lambda^4\alpha_2^2)}{144(-\beta^2+\lambda^2\alpha_2)^2}$ and $\alpha_2 = D_1^2 + D_2^2$.
Category 3: For $\lambda = 0$, (i.e.rational functions),

According to Postulate 5, if we execute as the category 1, we attain the values of a_0, a_1, a_2, b_1, b_2, β and c as the following results:

$$a_0 = \frac{-\beta^2}{4(-D_1^2 + 2\beta D_2)}, \quad a_1 = 0, \quad a_2 = -1, \quad b_1 = \beta, \quad b_2 = 0, \quad c = \frac{-5\beta^4}{16(-D_1^2 + 2\beta D)^2}, \quad \beta = \beta. \tag{3.22}$$

We get explicit solutions of Eq (1.2) as:

$$u(\xi) = \frac{-\beta^2}{4(-D_1^2 + 2\beta D_2)} - \frac{(\beta\xi + D_1)^2}{\left(\frac{\beta}{2}\xi^2 + D_1\xi + D_2\right)^2} + \frac{\beta}{\left(\frac{\beta}{2}\xi^2 + D_1\xi + D_2\right)}, \tag{3.23}$$

wherein $\xi = x - \frac{5\beta^4 t}{16(-D_1^2+2\beta D)^2}$ and $-D_1^2 + 2\beta D \neq 0$.

If we set up the particular values of the arbitrary constants if we choose D_1, D_2 and βin the above Eq (3.17), Eq (3.19), Eq (3.21) and Eq (3.23), we attain abundant new explicit wave solutions of KK equation which are unexposed for minimalism of length of the paper.

4. Conclusions and outlook

We obtained new explicit solutions for the (1+1)-dimensional KK equation. We achieved solitary wave solutions for analogous traveling wave solutions of Eq (1.2). These affluent solutions including bell and anti-bell solitons, kink and anti-kink solitons, periodic and rational functions of KK equation indicate that double $(G'/G, 1/G)$-expansion technique is more powerful than the method of $(G'/G, 1/G)$-expansion. Comparing the solutions with the ones in [33], we presume that all the solutions are renewed which are un-indicted elsewhere. Our mentioned method is more powerful and also an offering method to demonstrate many higher order nonlinear PDEs. We will investigate the applicability of the method to (2+1)-dimensional KK equation in a future extension of the present work.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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