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Research article

A unified treatment for the restricted solutions of the matrix equation $AXB = C$

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Abstract: In this paper, the Hermitian, skew-Hermitian, Re-nonnegative definite, Re-positive definite, Re-nonnegative definite least-rank and Re-positive definite least-rank solutions of the matrix equation $AXB = C$ are considered. The necessary and sufficient condition for the existence of such type of solution to the equation is provided and the explicit expression of the general solution is also given.

Keywords: matrix equation; Hermitian (skew-Hermitian) solution; Re-nonnegative (Re-positive) definite solution; Re-nonnegative (Re-positive) definite least-rank solution
Mathematics Subject Classification: 15A09, 15A24

1. Introduction

Throughout this paper, the complex $m\times n$ matrix space is denoted by $\mathbb{C}^{m\times n}$. The conjugate transpose, the Moore-Penrose inverse, the range space and the null space of a complex matrix $A \in \mathbb{C}^{m\times n}$ are denoted by A^{H}, A^+, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. I_n denotes the $n\times n$ identity matrix. $P_{\mathcal{L}}$ stands for the orthogonal projector on the subspace $\mathcal{L} \subset \mathbb{C}^n$. Furthermore, for a matrix $A \in \mathbb{C}^{m\times n}$, E_A and F_A stand for two orthogonal projectors: $E_A = I_m - AA^+ = P_{\mathcal{N}(A^{\mathrm{H}})}$, $F_A = I_n - A^+A = P_{\mathcal{N}(A)}$.

A number of papers have been published for solving linear matrix equations. For example, Chen et al. [1] proposed LSQR iterative method to solve common symmetric solutions of matrix equations $AXB = E$ and $CXD = F$. Zak and Toutounian [2] studied the matrix equation of $AXB = C$ with nonsymmetric coefficient matrices by using nested splitting conjugate gradient (NSCG) iteration method. By applying a Hermitian and skew-Hermitian splitting (HSS) iteration method, Wang et al. [3] computed the solution of the matrix equation $AXB = C$. Tian et al. [4] obtained the solution of the matrix equation $AXB = C$ by applying the Jacobi and Gauss-Seidel-type iteration methods. Liu et al. [5] solved the matrix equation $AXB = C$ by employing stationary splitting iterative methods. In addition, some scholars studied matrix equations by direct methods. Yuan and Dai [6] obtained generalized reflexive solutions of the matrix equation $AXB = D$ and the optimal approximation solution by using the generalized singular value decomposition. Zhang et al. [7] provided the explicit expression of the minimal norm least squares Hermitian solution of the complex matrix equation $AXB+CXD = E$ by using the structure of the real representations of complex matrices and the Moore-Penrose inverse. By means of the definitions of the rank and inertias of matrices, Song and Yu [8] obtained the existence conditions and expressions of the nonnegative (positive) definite and the Re-nonnegative (Re-positive) definite solutions to the matrix equations $AXA^{\mathrm{H}} = C$ and $BXB^{\mathrm{H}} = D$.

In this paper, we will focus on the restricted solutions to the following well-known linear matrix equation

$$AXB = C, \tag{1}$$

where $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times p}$ and $C \in \mathbb{C}^{m\times p}$. We observe that the structured matrix, such as Hermitian matrix, skew-Hermitian matrix, Re-nonnegative definite matrix and Re-positive definite matrix, is of important applications in structural dynamics, numerical analysis, stability and robust stability analysis of control theory and so on [9–14]. Conditions for the existence of Hermitian solutions to Eq (1) were studied in [15–17]. A solvability criterion for the existence of Re-nonnegative definite solutions of Eq (1) by using generalized inner inverses was investigated by Cvetković-Ilić [19]. Recently, a direct method for solving Eq (1) by using the generalized inverses of matrices and orthogonal projectors was proposed by Yuan and Zuo [21]. In addition, the Re-nonnegative definite and Re-positive definite solutions to some special cases of Eq (1) were considered by Wu [22], Wu and Cain [23] and Groß [24]. In [25], necessary and sufficient conditions for the existence of common Re-nonnegative definite and Re-positive definite solutions to the matrix equations $AX = C, XB = D$ were discussed by virtue of the extremal ranks of matrix polynomials.

In this paper, necessary and sufficient conditions for the existence of Hermitian (skew-Hermitian), Re-nonnegative (Re-positive) definite, and Re-nonnegative (Re-positive) definite least-rank solutions to Eq (1) are deduced by using the Moore-Penrose inverse of matrices, and the explicit representations of the general solutions are given when the solvability conditions are satisfied. Compared with the approaches proposed in [18–21], the coefficient matrices of Eq (1) have no any constraints and the method in this paper is straightforward and easy to implement.

2. Preliminaries

Definition 1. *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be Re-nonnegative definite (Re-nnd) if $H(A) := \frac{1}{2}(A + A^{\mathrm{H}})$ is Hermitian nonnegative definite (i.e., $H(A) \geq 0$), and A is said to be Re-positive definite (Re-pd) if $H(A)$ is Hermitian positive definite (i.e., $H(A) > 0$). The set of all Re-nnd (Re-pd) matrices in $\mathbb{C}^{n\times n}$ is denoted by $\mathbb{RND}^{n\times n}$ ($\mathbb{RPD}^{n\times n}$).*

Lemma 1. *[26] Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times p}$ and $C \in \mathbb{C}^{m\times p}$. Then the matrix equation $AXB = C$ is solvable if and only if $AA^{+}CB^{+}B = C$. In this case, the general solution can be written in the following parametric form*

$$X = A^{+}CB^{+} + F_A L_1 + L_2 E_B,$$

where $L_1, L_2 \in \mathbb{C}^{n\times n}$ are arbitrary matrices.

Lemma 2. *[27, 28] Let $B_1 \in \mathbb{C}^{l\times q}, D_1 \in \mathbb{C}^{l\times l}$. Then the matrix equation*

$$YB_1^{\mathrm{H}} \pm B_1 Y^{\mathrm{H}} = D_1,$$

has a solution $Y \in \mathbb{C}^{l\times q}$ *if and only if*

$$D_1 = \pm D_1^{\mathrm{H}},\ E_{B_1} D_1 E_{B_1} = 0.$$

In which case, the general solution is

$$Y = \frac{1}{2} D_1 (B_1^{+})^{\mathrm{H}} + \frac{1}{2} E_{B_1} D_1 (B_1^{+})^{\mathrm{H}} + 2V - V B_1^{+} B_1 \mp B_1 V^{\mathrm{H}} (B_1^{+})^{\mathrm{H}} - E_{B_1} V B_1^{+} B_1,$$

where $V \in \mathbb{C}^{l\times q}$ *is an arbitrary matrix.*

Lemma 3. *[29] Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{m\times p}$*. Then the Moore-Penrose inverse of the matrix* $[A, B]$ *is*

$$[A, B]^{+} = \begin{bmatrix} (I + TT^{\mathrm{H}})^{-1}(A^{+} - A^{+}BC^{+}) \\ C^{+} + T^{\mathrm{H}}(I + TT^{\mathrm{H}})^{-1}(A^{+} - A^{+}BC^{+}) \end{bmatrix},$$

where $C = (I - AA^{+})B$ *and* $T = A^{+}B(I - C^{+}C)$.

Lemma 4. *[30] Suppose that a Hermitian matrix* M *is partitioned as*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathrm{H}} & M_{22} \end{bmatrix},$$

where $M_{11},\ M_{22}$ *are square. Then*
(i). *M is Hermitian nonnegative definite if and only if*

$$M_{11} \geq 0,\ M_{11}M_{11}^{+}M_{12} = M_{12},\ M_{22} - M_{12}^{\mathrm{H}}M_{11}^{+}M_{12} \triangleq H_2 \geq 0.$$

In which case, M *can be expressed as*

$$M = \begin{bmatrix} M_{11} & M_{11}H_1 \\ H_1^{\mathrm{H}}M_{11} & H_2 + H_1^{\mathrm{H}}M_{11}H_1 \end{bmatrix},$$

where H_1 *is an arbitrary matrix and* H_2 *is an arbitrary Hermitian nonnegative definite matrix.*
(ii). *M is Hermitian positive definite if and only if*

$$M_{11} > 0,\ M_{22} - M_{12}^{\mathrm{H}}M_{11}^{-1}M_{12} \triangleq H_3 > 0.$$

In the case, M *can be expressed as*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathrm{H}} & H_3 + M_{12}^{\mathrm{H}}M_{11}^{-1}M_{12} \end{bmatrix},$$

where H_3 *is an arbitrary Hermitian positive definite matrix.*

Lemma 5. *[31] Let*

$$M = \begin{bmatrix} C & A \\ B & 0 \end{bmatrix},\ N = \begin{bmatrix} 0 & I_n \end{bmatrix},\ S = \begin{bmatrix} 0 \\ I_n \end{bmatrix},\ N_1 = NF_M,\ S_1 = E_M S.$$

Then the general least-rank solution to Eq.(1) *can be written as*

$$X = -NM^{+}S + N_1R_1 + R_2S_1,$$

where $R_1 \in \mathbb{C}^{(p+n)\times n}, R_2 \in \mathbb{C}^{n\times(m+n)}$ *are arbitrary matrices.*

3. The Hermitian and skew-Hermitian solutions of Eq (1)

Theorem 1. *Eq* (1) *has a Hermitian solution X if and only if*

$$\begin{aligned} &AA^{+}CB^{+}B = C, \\ &P_{\mathcal{T}}\left(A^{+}CB^{+} - (A^{+}CB^{+})^{\mathrm{H}}\right)P_{\mathcal{T}} = 0, \end{aligned} \tag{2}$$

where $\mathcal{T} = \mathcal{R}(A^{\mathrm{H}}) \cap \mathcal{R}(B)$. *In which case, the general Hermitian solution of Eq* (1) *is*

$$X = A^{+}CB^{+} + F_A L_1 + L_2 E_B, \tag{3}$$

where

$$L_1 = -P_1D_1 + \frac{1}{2}P_1D_1W_1^{\mathrm{H}} + 2V_1^{\mathrm{H}} - 2F_AZ_1^{\mathrm{H}} + P_1(V_1F_A - V_2E_B) + F_AZ_1^{\mathrm{H}}W_1^{\mathrm{H}}, \tag{4}$$

$$L_2 = D_1Q_1^{\mathrm{H}} - \frac{1}{2}W_1D_1Q_1^{\mathrm{H}} + 2V_2 + 2Z_1E_B + (F_AV_1^{\mathrm{H}} - E_BV_2^{\mathrm{H}})Q_1^{\mathrm{H}} - W_1Z_1E_B, \tag{5}$$

$$\begin{aligned} &D_1 = A^{+}CB^{+} - (A^{+}CB^{+})^{\mathrm{H}},\ C_1 = -(I - F_AF_A^{+})E_B,\ T_1 = -F_A^{+}E_B(I - C_1^{+}C_1), \\ &P_1 = (I + T_1T_1^{\mathrm{H}})^{-1}(F_A^{+} + F_A^{+}E_BC_1^{+}),\ Q_1 = C_1^{+} + T_1^{\mathrm{H}}P_1, \\ &W_1 = F_AP_1 - E_BQ_1,\ Z_1 = V_1P_1 + V_2Q_1, \end{aligned}$$

and $V_1, V_2 \in \mathbb{C}^{n\times n}$ *are arbitrary matrices.*

Proof. By Lemma 1, if the first condition of (2) holds, then the general solution of Eq (1) is given by (3). Now, we will find L_1 and L_2 such that $AXB = C$ has a Hermitian solution, that is,

$$A^{+}CB^{+} + F_AL_1 + L_2E_B = (A^{+}CB^{+})^{\mathrm{H}} + L_1^{\mathrm{H}}F_A + E_BL_2^{\mathrm{H}}. \tag{6}$$

Clearly, Eq (6) can be equivalently written as

$$X_1A_1^{\mathrm{H}} - A_1X_1^{\mathrm{H}} = D_1, \tag{7}$$

where $A_1 = [F_A, -E_B],\ X_1 = [L_1^{\mathrm{H}}, L_2],\ D_1 = A^{+}CB^{+} - (A^{+}CB^{+})^{\mathrm{H}}$.
By Lemma 2, Eq (7) has a solution X_1 if and only if

$$D_1 = -D_1^{\mathrm{H}},\ E_{A_1}D_1E_{A_1} = 0. \tag{8}$$

The first condition of (8) is obviously satisfied. And note that

$$E_{A_1} = P_{\mathcal{N}(A_1^{\mathrm{H}})} = P_{\mathcal{N}(F_A)\cap\mathcal{N}(E_B)} = P_{\mathcal{R}(A^{\mathrm{H}})\cap\mathcal{R}(B)}.$$

Thus, the second condition of (8) is equivalent to

$$P_{\mathcal{T}}D_1P_{\mathcal{T}} = 0,$$

where $\mathcal{T} = \mathcal{R}(A^{\mathrm{H}}) \cap \mathcal{R}(B)$, which is the second condition of (2). In which case, the general solution of Eq (7) is

$$X_1 = D_1(A_1^{+})^{\mathrm{H}} - \frac{1}{2}A_1A_1^{+}D_1(A_1^{+})^{\mathrm{H}} + 2V - 2VA_1^{+}A_1 + A_1V^{\mathrm{H}}(A_1^{+})^{\mathrm{H}} + A_1A_1^{+}VA_1^{+}A_1, \tag{9}$$

where $V = [V_1, V_2]$ is an arbitrary matrix. By Lemma 3, we have

$$[F_A, -E_B]^+ = \begin{bmatrix} (I + T_1T_1^{\mathrm{H}})^{-1}(F_A^+ + F_A^+E_BC_1^+) \\ C_1^+ + T_1^{\mathrm{H}}(I + T_1T_1^{\mathrm{H}})^{-1}(F_A^+ + F_A^+E_BC_1^+) \end{bmatrix}, \tag{10}$$

where $C_1 = -(I - F_AF_A^+)E_B$ and $T_1 = -F_A^+E_B(I - C_1^+C_1)$. Upon substituting (10) into (9), we can get (4) and (5). □

Corollary 1. *Eq* (1) *has a skew-Hermitian solution X if and only if*

$$\begin{aligned} &AA^+CB^+B = C, \\ &P_{\mathcal{T}}\left(A^+CB^+ + (A^+CB^+)^{\mathrm{H}}\right)P_{\mathcal{T}} = 0, \end{aligned} \tag{11}$$

where $\mathcal{T} = \mathcal{R}(A^{\mathrm{H}}) \cap \mathcal{R}(B)$. *In which case, the general skew-Hermitian solution of Eq* (1) *is*

$$X = A^+CB^+ + F_AL_3 + L_4E_B, \tag{12}$$

where

$$L_3 = P_2D_2 - \frac{1}{2}P_2D_2W_2^{\mathrm{H}} + 2V_3^{\mathrm{H}} - 2F_AZ_2^{\mathrm{H}} - P_2(V_3F_A + V_4E_B) + F_AZ_2^{\mathrm{H}}W_2^{\mathrm{H}}, \tag{13}$$

$$L_4 = D_2Q_2^{\mathrm{H}} - \frac{1}{2}W_2D_2Q_2^{\mathrm{H}} + 2V_4 - 2Z_2E_B - (F_AV_3^{\mathrm{H}} + E_BV_4^{\mathrm{H}})Q_2^{\mathrm{H}} + W_2Z_2E_B, \tag{14}$$

$$\begin{aligned} &D_2 = -A^+CB^+ - (A^+CB^+)^{\mathrm{H}},\ C_2 = (I - F_AF_A^+)E_B,\ T_2 = F_A^+E_B(I - C_2^+C_2), \\ &P_2 = (I + T_2T_2^{\mathrm{H}})^{-1}(F_A^+ - F_A^+E_BC_2^+),\ Q_2 = C_2^+ + T_2^{\mathrm{H}}P_2, \\ &W_2 = F_AP_2 + E_BQ_2,\ Z_2 = V_3P_2 + V_4Q_2, \end{aligned}$$

and $V_3,\ V_4 \in \mathbb{C}^{n\times n}$ *are arbitrary matrices.*

4. The Re-nnd and Re-pd solutions of Eq (1)

Theorem 2. *Let* $A \in \mathbb{C}^{m\times n},\ B \in \mathbb{C}^{n\times p},\ C \in \mathbb{C}^{m\times p}$ *and* $\mathcal{T} = \mathcal{R}(A^{\mathrm{H}}) \cap \mathcal{R}(B)$. *Assume that the spectral decomposition of* $P_{\mathcal{T}}$ *is*

$$P_{\mathcal{T}} = U\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}U^{\mathrm{H}}, \tag{15}$$

where $U = [U_1, U_2] \in \mathbb{C}^{n\times n}$ *is a unitary matrix and* $s = dim(\mathcal{T})$. *Then*
(a). *Eq* (1) *has a Re-nnd solution if and only if*

$$AA^+CB^+B = C,\ U_1^{\mathrm{H}}A^+CB^+U_1 \in \mathbb{RND}^{s\times s}. \tag{16}$$

In which case, the general Re-nnd solution of (1) *is*

$$X = A^+CB^+ + F_AJ_1 + J_2E_B, \tag{17}$$

where

$$J_1 = P_3D_3 - \frac{1}{2}P_3D_3W_3^{\mathrm{H}} + 2V_5^{\mathrm{H}} - 2F_AZ_3^{\mathrm{H}} - P_3(V_5F_A + V_6E_B) + F_AZ_3^{\mathrm{H}}W_3^{\mathrm{H}}, \tag{18}$$

$$J_2 = D_3Q_3^{\mathrm{H}} - \frac{1}{2}W_3D_3Q_3^{\mathrm{H}} + 2V_6 - 2Z_3E_B - (F_AV_5^{\mathrm{H}} + E_BV_6^{\mathrm{H}})Q_3^{\mathrm{H}} + W_3Z_3E_B, \tag{19}$$

$$\begin{aligned}
&D_3 = K - A^+CB^+ - (A^+CB^+)^{\mathrm{H}},\ C_3 = (I - F_AF_A^+)E_B,\\
&T_3 = F_A^+E_B(I - C_3^+C_3),\ P_3 = (I + T_3T_3^{\mathrm{H}})^{-1}(F_A^+ - F_A^+E_BC_3^+),\ Q_3 = C_3^+ + T_3^{\mathrm{H}}P_3,\\
&W_3 = F_AP_3 + E_BQ_3,\ Z_3 = V_5P_3 + V_6Q_3,\ K_{11} = U_1^{\mathrm{H}}(A^+CB^+ + (A^+CB^+)^{\mathrm{H}})U_1,\\
&K = U\begin{bmatrix} K_{11} & K_{11}H_1 \\ H_1^{\mathrm{H}}K_{11} & H_2 + H_1^{\mathrm{H}}K_{11}H_1 \end{bmatrix}U^{\mathrm{H}},
\end{aligned}$$

$V_5, V_6 \in \mathbb{C}^{n\times n}, H_1 \in \mathbb{C}^{s\times(n-s)}$ are arbitrary matrices, and $H_2 \in \mathbb{C}^{(n-s)\times(n-s)}$ is an arbitrary Hermitian nonnegative definite matrix.
(*b*). *Eq* (1) *has a Re-pd solution if and only if*

$$AA^+CB^+B = C,\ U_1^{\mathrm{H}}A^+CB^+U_1 \in \mathbb{RPD}^{s\times s}. \tag{20}$$

In which case, the general Re-pd solution of (1) *is*

$$X = A^+CB^+ + F_AJ_1 + J_2E_B, \tag{21}$$

where

$$K = U\begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathrm{H}} & H_3 + K_{12}^{\mathrm{H}}K_{11}^{-1}K_{12} \end{bmatrix}U^{\mathrm{H}},$$

$J_1, J_2, D_3, C_3, T_3, P_3, Q_3, W_3, Z_3$ and K_{11} are given by (18) *and* (19), *$K_{12} \in \mathbb{C}^{s\times(n-s)}$ is an arbitrary matrix and $H_3 \in \mathbb{C}^{(n-s)\times(n-s)}$ is an arbitrary Hermitian positive definite matrix.*

Proof. By Lemma 1, if the first condition of (16) holds, then the general solution of Eq (1) is given by (17). Now, we will find J_1 and J_2 such that $AXB = C$ has a Re-nnd (Re-pd) solution, that is, we will choose suitable matrices J_1 and J_2 such that

$$A^+CB^+ + (A^+CB^+)^{\mathrm{H}} + F_AJ_1 + J_1^{\mathrm{H}}F_A + J_2E_B + E_BJ_2^{\mathrm{H}} \triangleq K \geq 0\ (K > 0). \tag{22}$$

Clearly, Eq (22) can be equivalently written as

$$X_3A_3^{\mathrm{H}} + A_3X_3^{\mathrm{H}} = D_3, \tag{23}$$

where $A_3 = [F_A, E_B],\ X_3 = [J_1^{\mathrm{H}}, J_2],\ D_3 = K - A^+CB^+ - (A^+CB^+)^{\mathrm{H}}$.
By Lemma 2, Eq (23) has a solution X_1 if and only if

$$D_3 = D_3^{\mathrm{H}},\ E_{A_3}D_3E_{A_3} = 0. \tag{24}$$

The first condition of (24) is obviously satisfied. And note that

$$E_{A_3} = P_{\mathcal{N}(A_3^{\mathrm{H}})} = P_{\mathcal{N}(F_A)\cap\mathcal{N}(E_B)} = P_{\mathcal{R}(A^{\mathrm{H}})\cap\mathcal{R}(B)}.$$

Then the second condition of (24) is equivalent to

$$P_{\mathcal{T}}D_3P_{\mathcal{T}} = 0,$$

where $\mathcal{T} = \mathcal{R}(A^{\mathrm{H}}) \cap \mathcal{R}(B)$. By (15), we can obtain

$$\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{H}} K U \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{H}}(A^+CB^+ + (A^+CB^+)^{\mathrm{H}})U \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}, \tag{25}$$

Let

$$U^{\mathrm{H}} K U = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathrm{H}} & K_{22} \end{bmatrix}, \tag{26}$$

where $U = [U_1, U_2]$. By (25), we can obtain

$$K_{11} = U_1^{\mathrm{H}}(A^+CB^+ + (A^+CB^+)^{\mathrm{H}})U_1, \tag{27}$$

it is easily known that $K \geq 0$ ($K > 0$) if and only if $U^{\mathrm{H}}KU \geq 0$ ($U^{\mathrm{H}}KU > 0$). And X is Re-nnd (Re-pd) if and only if $K \geq 0$ ($K > 0$). Thus, by (26) and (27), we can get

$$K \geq 0 \Longleftrightarrow K_{11} = U_1^{\mathrm{H}}(A^+CB^+ + (A^+CB^+)^{\mathrm{H}})U_1 \geq 0,$$
$$K > 0 \Longleftrightarrow K_{11} = U_1^{\mathrm{H}}(A^+CB^+ + (A^+CB^+)^{\mathrm{H}})U_1 > 0,$$

equivalently,

$$K \geq 0 \Longleftrightarrow U_1^{\mathrm{H}}A^+CB^+U_1 \in \mathbb{RND}^{s\times s},$$
$$K > 0 \Longleftrightarrow U_1^{\mathrm{H}}A^+CB^+U_1 \in \mathbb{RPD}^{s\times s},$$

which are the second conditions of (16) and (20). In which case, by Lemma 4,

$$K \geq 0 \Longleftrightarrow K = U \begin{bmatrix} K_{11} & K_{11}H_1 \\ H_1^{\mathrm{H}}K_{11} & H_2 + H_1^{\mathrm{H}}K_{11}H_1 \end{bmatrix} U^{\mathrm{H}},$$

$$K > 0 \Longleftrightarrow K = U \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathrm{H}} & H_3 + K_{12}^{\mathrm{H}}K_{11}^{-1}K_{12} \end{bmatrix} U^{\mathrm{H}},$$

where $H_1 \in \mathbb{C}^{s\times(n-s)}$ is an arbitrary matrix, $H_2 \in \mathbb{C}^{(n-s)\times(n-s)}$ is an arbitrary Hermitian nonnegative definite matrix and $H_3 \in \mathbb{C}^{(n-s)\times(n-s)}$ is an arbitrary Hermitian positive definite matrix. And the general solution of Eq (23) is

$$X_3 = D_3(A_3^+)^{\mathrm{H}} - \frac{1}{2}A_3A_3^+D_3(A_3^+)^{\mathrm{H}} + 2V - 2VA_3^+A_3 - A_3V^{\mathrm{H}}(A_3^+)^{\mathrm{H}} + A_3A_3^+VA_3^+A_3, \tag{28}$$

where $V = [V_5, V_6]$ is an arbitrary matrix. By Lemma 3, we have

$$[F_A, E_B]^+ = \begin{bmatrix} (I + T_3T_3^{\mathrm{H}})^{-1}(F_A^+ - F_A^+E_BC_3^+) \\ C_3^+ + T_3^{\mathrm{H}}(I + T_3T_3^{\mathrm{H}})^{-1}(F_A^+ - F_A^+E_BC_3^+) \end{bmatrix}, \tag{29}$$

where $C_3 = (I - F_AF_A^+)E_B$ and $T_3 = F_A^+E_B(I - C_3^+C_3)$. Upon substituting (29) into (28), we can get (18) and (19). □

5. The Re-nnd and Re-pd least-rank solutions of Eq (1)

Theorem 3. *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{m\times p}$ and $\tilde{\mathcal{T}} = \mathcal{N}(N_1^{\mathrm{H}}) \cap \mathcal{N}(S_1)$. Assume that the spectral decomposition of $P_{\tilde{\mathcal{T}}}$ is*

$$P_{\tilde{\mathcal{T}}} = \tilde{U}\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}\tilde{U}^{\mathrm{H}}, \tag{30}$$

where $\tilde{U} = [\tilde{U}_1, \tilde{U}_2] \in \mathbb{C}^{n\times n}$ is a unitary matrix and $k = dim(\tilde{\mathcal{T}})$, and M, N, S, N_1, S_1 are given by Lemma 5. *Then*
(a). *Eq* (1) *has a Re-nnd least-rank solution if and only if*

$$AA^{+}CB^{+}B = C,\ \tilde{U}_1^{\mathrm{H}}(-NM^{+}S)\tilde{U}_1 \in \mathbb{RND}^{k\times k}. \tag{31}$$

In which case, the general Re-nnd least-rank solution of Eq (1) *is*

$$X = -NM^{+}S + N_1R_1 + R_2S_1, \tag{32}$$

where

$$R_1 = P_4D_4 - \frac{1}{2}P_4D_4W_4^{\mathrm{H}} + 2V_7^{\mathrm{H}} - 2N_1^{\mathrm{H}}Z_4^{\mathrm{H}} - P_4(V_7N_1^{\mathrm{H}} + V_8S_1) + N_1^{\mathrm{H}}Z_4^{\mathrm{H}}W_4^{\mathrm{H}}, \tag{33}$$

$$R_2 = D_4Q_4^{\mathrm{H}} - \frac{1}{2}W_4D_4Q_4^{\mathrm{H}} + 2V_8 - 2Z_4S_1^{\mathrm{H}} - (N_1V_7^{\mathrm{H}} + S_1^{\mathrm{H}}V_8^{\mathrm{H}})Q_4^{\mathrm{H}} + W_4Z_4S_1^{\mathrm{H}}, \tag{34}$$

$$\begin{aligned}
&D_4 = \tilde{K} + NM^{+}S + (NM^{+}S)^{\mathrm{H}},\ C_4 = (I - N_1N_1^{+})S_1^{\mathrm{H}},\ T_4 = N_1^{+}S_1^{\mathrm{H}}(I - C_4^{+}C_4),\\
&P_4 = (I + T_4T_4^{\mathrm{H}})^{-1}(N_1^{+} - N_1^{+}S_1^{\mathrm{H}}C_4^{+}),\ Q_4 = C_4^{+} + T_4^{\mathrm{H}}P_4,\\
&W_4 = N_1P_4 + S_1^{\mathrm{H}}Q_4,\ Z_4 = V_7P_4 + V_8Q_4,\ \tilde{K}_{11} = \tilde{U}_1^{\mathrm{H}}(-NM^{+}S - (NM^{+}S)^{\mathrm{H}})\tilde{U}_1,\\
&\tilde{K} = \tilde{U}\begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{11}\tilde{H}_1 \\ \tilde{H}_1^{\mathrm{H}}\tilde{K}_{11} & \tilde{H}_2 + \tilde{H}_1^{\mathrm{H}}\tilde{K}_{11}\tilde{H}_1 \end{bmatrix}\tilde{U}^{\mathrm{H}},
\end{aligned}$$

$V_7 \in \mathbb{C}^{n\times(p+n)}$, $V_8 \in \mathbb{C}^{n\times(m+n)}$, $\tilde{H}_1 \in \mathbb{C}^{k\times(n-k)}$ *are arbitrary matrices, and* $\tilde{H}_2 \in \mathbb{C}^{(n-k)\times(n-k)}$ *is an arbitrary Hermitian nonnegative definite matrix.*
(b). *Eq* (1) *has a Re-pd least-rank solution if and only if*

$$AA^{+}CB^{+}B = C,\ \tilde{U}_1^{\mathrm{H}}(-NM^{+}S)\tilde{U}_1 \in \mathbb{RPD}^{k\times k}. \tag{35}$$

In which case, the general Re-nnd least-rank solution of Eq (1) *is*

$$X = -NM^{+}S + N_1R_1 + R_2S_1, \tag{36}$$

where

$$\tilde{K} = \tilde{U}\begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{12}^{\mathrm{H}} & \tilde{H}_3 + \tilde{K}_{12}^{\mathrm{H}}\tilde{K}_{11}^{-1}\tilde{K}_{12} \end{bmatrix}\tilde{U}^{\mathrm{H}},$$

$R_1, R_2, D_4, C_4, T_4, P_4, Q_4, W_4, Z_4$ *and* $\tilde{K}_{11}$ *are given by* (33) *and* (34), $\tilde{K}_{12} \in \mathbb{C}^{k\times(n-k)}$ *is an arbitrary matrix and* $\tilde{H}_3 \in \mathbb{C}^{(n-k)\times(n-k)}$ *is an arbitrary Hermitian positive definite matrix.*

Proof. By Lemmas 1 and 5, if the first condition of (31) holds, then the least-rank solution of Eq (1) is given by (32). Now, we will find R_1 and R_2 such that $AXB = C$ has a Re-nnd (Re-pd) least-rank solution, that is, we will choose suitable matrices R_1 and R_2 such that

$$-NM^{+}S-(NM^{+}S)^{\mathrm{H}}+N_1R_1+R_1^{\mathrm{H}}N_1^{\mathrm{H}}+R_2S_1+S_1^{\mathrm{H}}R_2^{\mathrm{H}}\triangleq\tilde{K}\geq 0\ (\tilde{K}>0). \tag{37}$$

Clearly, Eq (37) can be equivalently written as

$$X_4A_4^{\mathrm{H}}+A_4X_4^{\mathrm{H}}=D_4, \tag{38}$$

where $A_4=[N_1,S_1^{\mathrm{H}}],\ X_4=[R_1^{\mathrm{H}},R_2],\ D_4=\tilde{K}+NM^{+}S+(NM^{+}S)^{\mathrm{H}}$.
By Lemma 2, Eq (38) has a solution X_4 if and only if

$$D_4=D_4^{\mathrm{H}},\ E_{A_4}D_4E_{A_4}=0. \tag{39}$$

The first condition of (39) is obviously satisfied. And note that

$$E_{A_4}=P_{\mathcal{N}(A_4^{\mathrm{H}})}=P_{\mathcal{N}(N_1^{\mathrm{H}})\cap\mathcal{N}(S_1)}.$$

Thus, the second condition of (39) is equivalent to

$$P_{\tilde{\mathcal{T}}}D_4P_{\tilde{\mathcal{T}}}=0,$$

where $\tilde{\mathcal{T}}=\mathcal{N}(N_1^{\mathrm{H}})\cap\mathcal{N}(S_1)$. By (30), we can obtain

$$\begin{bmatrix} I_k & 0\\ 0 & 0\end{bmatrix}\tilde{U}^{\mathrm{H}}\tilde{K}\tilde{U}\begin{bmatrix} I_k & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix} I_k & 0\\ 0 & 0\end{bmatrix}\tilde{U}^{\mathrm{H}}(-NM^{+}S-(NM^{+}S)^{\mathrm{H}})\tilde{U}\begin{bmatrix} I_k & 0\\ 0 & 0\end{bmatrix}, \tag{40}$$

Let

$$\tilde{U}^{\mathrm{H}}\tilde{K}\tilde{U}=\begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12}\\ \tilde{K}_{12}^{\mathrm{H}} & \tilde{K}_{22}\end{bmatrix}, \tag{41}$$

where $\tilde{U}=[\tilde{U}_1,\tilde{U}_2]$. By (40), we can obtain

$$\tilde{K}_{11}=\tilde{U}_1^{\mathrm{H}}(-NM^{+}S-(NM^{+}S)^{\mathrm{H}})\tilde{U}_1, \tag{42}$$

it is easily known that $\tilde{K}\geq 0$ ($\tilde{K}>0$) if and only if $\tilde{U}^{\mathrm{H}}\tilde{K}\tilde{U}\geq 0$ ($\tilde{U}^{\mathrm{H}}\tilde{K}\tilde{U}>0$). And X is Re-nnd (Re-pd) least-rank solution if and only if $\tilde{K}\geq 0$ ($\tilde{K}>0$). Thus, by (41) and (42), we can get

$$\begin{aligned}\tilde{K}\geq 0 &\Longleftrightarrow \tilde{K}_{11}=\tilde{U}_1^{\mathrm{H}}(-NM^{+}S-(NM^{+}S)^{\mathrm{H}})\tilde{U}_1\geq 0,\\ \tilde{K}>0 &\Longleftrightarrow \tilde{K}_{11}=\tilde{U}_1^{\mathrm{H}}(-NM^{+}S-(NM^{+}S)^{\mathrm{H}})\tilde{U}_1>0,\end{aligned}$$

equivalently,

$$\begin{aligned}\tilde{K}\geq 0 &\Longleftrightarrow \tilde{U}_1^{\mathrm{H}}(-NM^{+}S)\tilde{U}_1\in\mathbb{RND}^{k\times k},\\ \tilde{K}>0 &\Longleftrightarrow \tilde{U}_1^{\mathrm{H}}(-NM^{+}S)\tilde{U}_1\in\mathbb{RPD}^{k\times k},\end{aligned}$$

which are the second conditions of (31) and (35). In which case, by Lemma 4,

$$\tilde{K}\geq 0\Longleftrightarrow \tilde{K}=\tilde{U}\begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{11}\tilde{H}_1\\ \tilde{H}_1^{\mathrm{H}}\tilde{K}_{11} & \tilde{H}_2+H_1^{\mathrm{H}}\tilde{K}_{11}\tilde{H}_1\end{bmatrix}\tilde{U}^{\mathrm{H}},$$

$$\tilde{K} > 0 \Longleftrightarrow \tilde{K} = \tilde{U} \begin{bmatrix} \tilde{K}_{11} & \tilde{K}_{12} \\ \tilde{K}_{12}^{\mathrm{H}} & \tilde{H}_3 + \tilde{K}_{12}^{\mathrm{H}} \tilde{K}_{11}^{-1} \tilde{K}_{12} \end{bmatrix} \tilde{U}^{\mathrm{H}},$$

where $\tilde{H}_1 \in \mathbb{C}^{k\times(n-k)}$ is an arbitrary matrix, $\tilde{H}_2 \in \mathbb{C}^{(n-k)\times(n-k)}$ is an arbitrary Hermitian nonnegative definite matrix and $\tilde{H}_3 \in \mathbb{C}^{(n-k)\times(n-k)}$ is an arbitrary Hermitian positive definite matrix. And the general solution of Eq (38) is

$$X_4 = D_4(A_4^+)^{\mathrm{H}} - \frac{1}{2}A_4A_4^+D_4(A_4^+)^{\mathrm{H}} + 2V - 2VA_4^+A_4 - A_4V^{\mathrm{H}}(A_4^+)^{\mathrm{H}} + A_4A_4^+VA_4^+A_4, \tag{43}$$

where $V = [V_7, V_8]$ is an arbitrary matrix. By Lemma 3, we have

$$[N_1, S_1^{\mathrm{H}}]^+ = \begin{bmatrix} (I + T_4T_4^{\mathrm{H}})^{-1}(N_1^+ - N_1^+S_1^{\mathrm{H}}C_4^+) \\ C_4^+ + T_4^{\mathrm{H}}(I + T_4T_4^{\mathrm{H}})^{-1}(N_1^+ - N_1^+S_1^{\mathrm{H}}C_4^+) \end{bmatrix}, \tag{44}$$

where $C_4 = (I - N_1N_1^+)S_1^{\mathrm{H}}$ and $T_4 = N_1^+S_1^{\mathrm{H}}(I - C_4^+C_4)$. Upon substituting (44) into (43), we can get (33) and (34). □

6. Numerical examples

The following example comes from [9].

Example 1. Consider a 7-DOF system modelled analytically with the first three measured modal data given by

$$\Lambda = \mathrm{diag}(3.5498,\ 101.1533,\ 392.8443),\ X = \begin{bmatrix} 0.5585 & 0.4751 & -0.4241 \\ -0.0841 & -0.2353 & 0.2838 \\ 0.3094 & -0.1717 & 0.2512 \\ -0.0800 & -0.1646 & 0.0852 \\ 0.0996 & -0.3562 & -0.0508 \\ -0.0553 & 0.0404 & -0.2105 \\ 0.0084 & -0.1788 & -0.4113 \end{bmatrix}.$$

and the corrected symmetric mass matrix M and symmetric stiffness matrix K should satisfy the orthogonality conditions, that is,

$$X^\top MX = I_3, X^\top KX = \Lambda.$$

Since

$$\|X^\top(X^\top)^+X^+X - I_3\| = 1.5442 \times 10^{-15},$$
$$\|P_{\mathcal{T}}\left((X^\top)^+X^+ - ((X^\top)^+X^+)^\top\right)P_{\mathcal{T}}\| = 0,$$

which means that the conditions of (2) are satisfied. Choose $V_1 = 0, V_2 = 0$. Then, by the equation of (3), we can obtain a corrected mass matrix given by

$$M = \begin{bmatrix} 1.1968 & -0.1073 & 0.8201 & -0.1678 & 0.1977 & -0.2079 & -0.1439 \\ -0.1073 & 0.3057 & 0.6292 & 0.0998 & 0.2953 & -0.2547 & -0.2095 \\ 0.8201 & 0.6292 & 2.2748 & 0.1347 & 1.1398 & -0.7290 & -0.3249 \\ -0.1678 & 0.0998 & 0.1347 & 0.0998 & 0.3427 & 0.0177 & 0.2804 \\ 0.1977 & 0.2953 & 1.1398 & 0.3427 & 1.8775 & 0.0370 & 1.4878 \\ -0.2079 & -0.2547 & -0.7290 & 0.0177 & 0.0370 & 0.3742 & 0.6461 \\ -0.1439 & -0.2095 & -0.3249 & 0.2804 & 1.4878 & 0.6461 & 2.1999 \end{bmatrix},$$

and

$$\|X^\top MX - I_3\| = 1.5016 \times 10^{-15},$$

which implies that M is a symmetric solution of $X^\top MX = I_3$.
Since

$$\|X^\top (X^\top)^+ \Lambda X^+ X - \Lambda\| = 4.0942 \times 10^{-13},$$
$$\|P_{\mathcal{T}} \left((X^\top)^+ \Lambda X^+ - ((X^\top)^+ \Lambda X^+)^\top\right) P_{\mathcal{T}}\| = 1.5051 \times 10^{-14},$$

which means that the conditions of (2) are satisfied. Choose $V_1 = 0, V_2 = 0$. Then, by the equation of (3), we obtain a corrected stiffness matrix given by

$$K = \begin{bmatrix} 50.0364 & -47.8369 & -66.1052 & 1.4621 & 21.6558 & 62.0904 & 44.8915 \\ -47.8369 & 93.9748 & 169.4007 & 41.9423 & 49.9653 & -78.9100 & -244.1315 \\ -66.1052 & 169.4007 & 176.9957 & -3.6625 & -103.7349 & -173.8782 & -169.8170 \\ 1.4621 & 41.9423 & -3.6625 & 25.7528 & -2.6730 & -189.4986 & 98.3110 \\ 21.6558 & 49.9653 & -103.7349 & -2.6730 & 95.3473 & -51.6395 & 446.8062 \\ 62.0904 & -78.9100 & -173.8782 & -189.4986 & -51.6395 & 56.3200 & 448.6900 \\ 44.8915 & -244.1315 & -169.8170 & 98.3110 & 446.8062 & 448.6900 & 394.1690 \end{bmatrix},$$

and

$$\|X^\top KX - \Lambda\| = 3.5473 \times 10^{-13},$$

which implies that K is a symmetric solution of $X^\top KX = \Lambda$.
Example 2. Given matrices

$$A = \begin{bmatrix} 7.9482 & 9.7975 & 1.3652 & 6.6144 & 5.8279 & 2.2595 \\ 9.5684 & 2.7145 & 0.1176 & 2.8441 & 4.2350 & 5.7981 \\ 5.2259 & 2.5233 & 8.9390 & 4.6922 & 5.1551 & 7.6037 \\ 8.8014 & 8.7574 & 1.9914 & 0.6478 & 3.3395 & 5.2982 \\ 1.7296 & 7.3731 & 2.9872 & 9.8833 & 4.3291 & 6.4053 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.9343 & 3.7837 & 8.2163 & 3.4119 & 3.7041 \\ 6.8222 & 8.6001 & 6.4491 & 5.3408 & 7.0274 \\ 3.0276 & 8.5366 & 8.1797 & 7.2711 & 5.4657 \\ 5.4167 & 5.9356 & 6.6023 & 3.0929 & 4.4488 \\ 1.5087 & 4.9655 & 3.4197 & 8.3850 & 6.9457 \\ 6.9790 & 8.9977 & 2.8973 & 5.6807 & 6.2131 \end{bmatrix},$$

$$C = \begin{bmatrix} 745.6317 & 1194.5543 & 1060.3913 & 995.6379 & 1010.8200 \\ 535.5044 & 831.5304 & 791.3676 & 684.6897 & 711.7632 \\ 845.4324 & 1338.1065 & 1123.7380 & 1077.0615 & 1096.2601 \\ 629.0768 & 1006.4762 & 928.6566 & 804.5134 & 824.7220 \\ 868.4158 & 1299.0559 & 995.0786 & 1012.7046 & 1054.9264 \end{bmatrix}.$$

Since $\|AA^+CB^+B - C\| = 9.3907 \times 10^{-12}$, and the eigenvalues of K_{11} are

$$\Lambda_1 = \text{diag}\{0.3561, 11.0230, 6.8922, 5, 3274\},$$

which means that the conditions of (16) are satisfied. Choose V, H_1 and H_2 as

$$V = [I_6, I_6],\ H_1 = \begin{bmatrix} 0.5211 & 0.6791 \\ 0.2316 & 0.3955 \\ 0.4889 & 0.3674 \\ 0.6241 & 0.9880 \end{bmatrix},\ H_2 = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, by the equation of (17), we get a solution of Eq (1):

$$X = \begin{bmatrix} 7.9305 & -1.8515 & -1.1364 & 0.5615 & 0.5411 & 1.2825 \\ -1.7228 & 2.3425 & 2.3029 & 1.4677 & 0.4975 & -0.4920 \\ 0.8582 & 1.6817 & 2.5285 & 1.8325 & 1.0895 & -0.8095 \\ 3.0514 & -0.8250 & -0.7922 & 3.3741 & 1.5249 & 1.6918 \\ -6.1787 & 3.8549 & 2.5294 & -2.6945 & 2.4568 & 0.6923 \\ -0.1968 & 1.0600 & 0.3214 & 0.6383 & -0.3125 & 3.2860 \end{bmatrix}$$

with corresponding residual

$$\|AXB - C\| = 6.4110 \times 10^{-12}.$$

Furthermore, it can be computed that the eigenvalues of $X + X^{\mathrm{H}}$ are

$$\Lambda_2 = \mathrm{diag}\{0,\ 0.3555,\ 3.9363,\ 7.0892,\ 11.1802,\ 21.2758\},$$

which implies that X is a Re-nnd solution of Eq (1).

Example 3. Let the matrices A and B be the same as those in Example 2 and the matrix C be given by

$$C = \begin{bmatrix} 1841.2323 & 2726.8415 & 2555.2926 & 2172.7688 & 2365.3939 \\ 1281.0788 & 1957.5932 & 1843.7252 & 1614.6211 & 1699.3985 \\ 1826.1325 & 3077.6141 & 2600.8168 & 2539.0181 & 2467.3558 \\ 1583.3543 & 2422.3169 & 2153.6625 & 1910.0156 & 2035.2123 \\ 2065.8761 & 2950.7783 & 2362.0060 & 2197.7129 & 2389.1173 \end{bmatrix}.$$

Since $\|AA^{+}CB^{+}B - C\| = 1.7944 \times 10^{-11}$, and the eigenvalues of K_{11} are

$$\Lambda_1 = \mathrm{diag}\{25.3998,\ 20.3854,\ 19.4050,\ 19.0160\},$$

which means that the conditions of (20) are satisfied. If select V, K_{12} and H_3 as

$$V = [I_6, I_6],\ K_{12} = \begin{bmatrix} 0.5211 & 0.6791 \\ 0.2316 & 0.3955 \\ 0.4889 & 0.3674 \\ 0.6241 & 0.9880 \end{bmatrix},\ H_3 = \begin{bmatrix} 6 & 0 \\ 0 & 8 \end{bmatrix}.$$

Then, by the equation of (21), we can achieve a Re-pd solution of Eq (1):

$$X = \begin{bmatrix} 7.0701 & 1.2500 & 0.8677 & 1.7685 & 2.7735 & -1.4689 \\ 1.0152 & 9.9032 & 0.1535 & -0.4383 & -0.0571 & 1.2297 \\ 0.8063 & 0.4874 & 10.0878 & -0.5956 & 2.0235 & 1.1426 \\ 2.1662 & 0.0136 & -0.8364 & 8.2937 & 1.8721 & 2.5887 \\ 1.6627 & 0.7279 & 1.2887 & 1.2198 & 5.6682 & 0.4925 \\ -1.6028 & 1.1737 & 1.8004 & 2.6718 & 0.2729 & 8.1489 \end{bmatrix},$$

and the eigenvalues of $X + X^{\mathrm{H}}$ are

$$\Lambda_2 = \mathrm{diag}\{5.9780,\ 7.9395,\ 19.1387,\ 19.4286,\ 20.4585,\ 25.4004\}.$$

Furthermore, it can be computed that

$$\|AXB - C\| = 1.7808 \times 10^{-11}.$$

7. Concluding remarks

In this paper, we mainly consider some special solutions of Eq (1). By imposing some constraints on the expression $X = A^{+}CB^{+} + F_A L_1 + L_2 E_B$, we succeed in obtaining a set of necessary and sufficient conditions for the existence of the Hermitian, skew-Hermitian, Re-nonnegative definite, Re-positive definite, Re-nonnegative definite least-rank and Re-positive definite least-rank solutions of Eq (1), respectively. Moreover, we give the explicit expressions for these special solutions, when the consistent conditions are satisfied.

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Conflict of interest

The authors declare no conflict of interest.

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