Existence of solutions for $q$-fractional differential equations with nonlocal Erdélyi-Kober $q$-fractional integral condition

Min Jiang$^{1,*}$ and Rengang Huang$^{2}$

$^1$ School of Data-Science and Information-Engineering, Guizhou Minzu University, Guiyang 550025, China
$^2$ College of Business, Guizhou Minzu University, Guiyang, 550025, China

* Correspondence: Email: minjiang0701@163.com.

Abstract: In this paper, we obtain sufficient conditions for the existence, uniqueness of solutions for a fractional $q$-difference equation with nonlocal Erdélyi-Kober $q$-fractional integral condition. Our approach is based on some classical fixed point techniques, as Banach contraction principle and Schauder’s fixed point theorem. Examples illustrating the obtained results are also presented.

Keywords: $q$-fractional differential equations; Erdélyi-Kober $q$-fractional integral; fixed point theorems; Riemann-Liouville fractional derivatives

Mathematics Subject Classification: 39A13, 34B18, 34A08

1. Introduction

The aim of this paper is to establish the existence and uniqueness of solutions for the following nonlinear Riemann-Liouville $q$-fractional differential equation subject to nonlocal Erdélyi-Kober $q$-fractional integral conditions

$$
\begin{align*}
D^\alpha_q x(t) + f(t, x(t), D^\delta_q x(t)) &= 0, \ t \in (0, T), \\
x(0) &= 0, \ ax(T) = \sum_{i=1}^n \lambda_i I^\eta_{\mu_i, \beta_i}_q x(\xi_i),
\end{align*}
(1.1)
$$

where $D^\alpha_q$ and $D^\delta_q$ are the fractional $q$-derivative of Riemann-Liouville type of order $\alpha$ and $\delta$ on $(0, T)$ respectively, $1 < \alpha < 2$, $0 < \delta < 1$, $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I^\eta_{\mu_i, \beta_i}_q$ denotes the Erdélyi-Kober fractional $q$-integral of order $\mu_i$ on $(0, T)$, $\mu_i > 0$, $\beta_i > 0$, $\eta_i \in \mathbb{R}$ and $\xi_i \in (0, T)$, $a$, $\lambda_i$ ($i = 1, 2, \ldots, n$) are some given constants.

The $q$-calculus or quantum calculus is an old subject that was initially developed by Jackson [1], basic definitions and properties of $q$-calculus can be found in [2]. The fractional $q$-calculus had its
origin in the works by Al-Salam [3] and Agarwal [4]. In recent years, considerable interest in $q$-fractional differential equations has been stimulated due to its applicability in mathematical modeling in different branches like engineering, physics and technical, etc. There are many papers and books dealing with the theoretical development of $q$-fractional calculus and the existence of solutions of boundary value problems for nonlinear $q$-fractional differential equations, for examples and details, one can see [5–22] and references therein.

In [9], Zhao, Chen and Zhang considered the following nonlocal $q$-integral boundary value problem of nonlinear fractional $q$-derivatives:

$$
\begin{align*}
D_q^q x(t) + f(t, x(t)) &= 0, \quad t \in (0, 1), \\
x(0) &= 0, \quad x(1) = \mu I_q^q x(\eta) = \mu \int_0^\eta \frac{(\eta - q\gamma)(\gamma - 1)}{\Gamma_q(\gamma)} x(s) ds,
\end{align*}
$$

where $q \in (0, 1)$, $1 < \alpha \leq 2$, $0 < \beta \leq 2$, $0 < \eta < 1$ and $\mu > 0$. $D_q^q$ is the fractional $q$-derivative of Riemann-Liouville type of order $\alpha$.

Inspired by the paper [23], we consider the existence and uniqueness for problem (1.1) by using Banach contraction principle, the monotone iterative method, and Krasnoselskiis fixed point theorem, the authors obtained some existence results of positive solutions to the above problem.

In [10], the authors investigated the $q$-integral boundary value problem for $q$-integro-difference equations involving Riemann-Liouville $q$-derivatives and a $q$-integral of different orders as follows:

$$
\begin{align*}
(\lambda D_q^\alpha + (1 - \lambda) D_q^\beta) x(t) &= a f(t, x(t)) + b I_q^\delta g(t, x(t)), \quad t \in [0, 1], \quad a, b \in \mathbb{R}^+, \\
\mu \int_0^1 \frac{(1-q\gamma)(\gamma-1)}{\Gamma_q(\gamma)} x(s) ds + (1 - \mu) \int_0^1 \frac{(q\gamma)(\gamma-1)}{\Gamma_q(\gamma)} x(s) ds &= 0, \\
x(0) &= 0,
\end{align*}
$$

where $q \in (0, 1)$, $1 < \alpha, \beta < 2$, $0 < \delta < 1$, $0 < \lambda \leq 1$ and $0 \leq \mu \leq 1$, $\alpha - \beta > 1$. $D_q^\alpha$ denotes the Riemann-Liouville fractional $q$-derivative of order $\alpha$ and $f, g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

In [23], the authors considered the existence of solutions for the following nonlinear Riemann-Liouville fractional differential equation with nonlocal Erdélyi-Kober fractional integral conditions

$$
\begin{align*}
(\lambda D_q^\alpha x(t) &= f(t, x(t)), \quad t \in (0, T), \\
x(0) &= 0, \quad x(T) = \sum_{i=1}^m \beta_i I_q^{\gamma_i, \delta_i} x(\xi_i),
\end{align*}
$$

where $1 < q \leq 2$, $D_q^\alpha$ is the Riemann-Liouville fractional derivative of order $\alpha$, $I_q^{\eta_i, \delta_i}$ is the Erdélyi-Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$ and $\gamma_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\alpha_i, \beta_i \in \mathbb{R}$, $\xi_i \in (0, T)$, $i = 1, 2, \ldots, m$ are given constants.

As we all know, few people solve the existence of solutions for a nonlinear Riemann-Liouville $q$-fractional differential equation subject to nonlocal Erdélyi-Kober $q$-fractional integral conditions. Inspired by the paper [23], we consider the existence and uniqueness for problem (1.1) by using Banach contraction principle and Schauder’s fixed point theorem.

2. Preliminaries on $q$-calculus and Lemmas

Here we recall some definitions and fundamental results on fractional $q$-integral and fractional $q$-derivative. See the references [4–7] for complete theory.
For \( q \in (0, 1) \), define \( [a]_q = \frac{1 - q^a}{1 - q} \), \( a \in \mathbb{R} \). The \( q \)-factorial function is defined as \( (a - b)^{(n)} = \prod_{k=0}^{n-1} (a - bq^k) \), \( a, b \in \mathbb{R} \). If \( n \) is a positive integer. If \( \nu \) is not a positive integer, then \( (a - b)^{(\nu)} = a^\nu \prod_{n=0}^{\infty} \frac{a - bq^\nu}{a - bq^n} \). If \( b = 0 \), then \( a^\nu = a^\nu \). The \( q \)-gamma function is defined by \( \Gamma_q(\alpha) = \frac{(1 - q)(\alpha - 1)}{(1 - q^{\alpha - 1})} \), \( \alpha > 0 \), and satisfies \( \Gamma_q(\alpha + 1) = [\alpha]_q \Gamma_q(\alpha) \).

The \( q \)-derivative of a function \( f \) is defined by \( (D_q f)(t) = \frac{(1 - (1 - f(t))/q)}{(1 - q^\nu)}, (D_q f)(0) = \lim_{t \to 0} (D_q f)(t) \). The \( q \)-integral of a function \( f \) defined on the interval \([0, \infty)\) is given by \( (I_q f)(t) = \int_0^t f(s)d_q s = (1 - q)^t \sum_{i=0}^{\infty} q^i f(q^i t), t \in [0, \infty) \).

Some results about operator \( D_q \) and \( I_q \) can be found in references [4]. Let us define fractional \( q \)-derivative and \( q \)-integral and outline some of their properties [4, 6, 8].

**Definition 1** ([4]) Let \( \alpha \geq 0 \) and \( f \) be a function. The fractional \( q \)-integral of Riemann-Liouville type is given by \( (I_q^\alpha f)(t) = f(t) \) and

\[
(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - q s)^{(\alpha - 1)} f(s) d_q s, \quad \alpha > 0, \quad t \in [0, \infty).
\]

**Definition 2** ([6]) The fractional \( q \)-derivative fractional of Riemann-Liouville type of order \( \nu \geq 0 \) is defined by \( (D_q^\nu f)(t) = f(t) \) and

\[
(D_q^\nu f)(t) = D_q^l (I_q^{\nu - l} f)(t), \quad \nu > 0,
\]

where \( l \) is the smallest integer greater than or equal to \( \nu \).

**Definition 3** ([24]) For \( 0 < q < 1 \), the Erdélyi-Kober fractional \( q \)-integral of order \( \mu > 0 \) with \( \beta > 0 \) and \( \eta \in \mathbb{R} \) of a continuous function \( f : (0, \infty) \to \mathbb{R} \) is defined by

\[
(I_q^\beta \mu \eta f)(t) = \frac{\beta^\nu}{\Gamma_q(\mu)} \int_0^\nu (t - s q)^{(\mu - 1)} q^\beta f(s) d_q s.
\]

provided the right side is pointwise defined on \( \mathbb{R}^+ \).

**Remark 1** For \( \beta = 1 \) the above operator is reduced to the \( q \)-analogue Kober operator

\[
(I_q^\mu f)(t) = \frac{1}{\Gamma_q(\mu)} \int_0^\nu (t - s q)^{(\mu - 1)} q^\mu f(s) d_q s
\]

that is given in [4]. For \( \eta = 0 \) the \( q \)-analogue Kober operator is reduced to the Riemann-Liouville fractional \( q \)-integral with a power weight:

\[
(I_q^\mu f)(t) = \frac{1}{\Gamma_q(\mu)} \int_0^\nu (t - s q)^{(\mu - 1)} f(s) d_q s, \quad \mu > 0.
\]

**Lemma 1** ([4]) Let \( \alpha, \beta \in \mathbb{R}^+ \) and \( f \) be a continuous function on \([0, \infty)\). The Riemann-Liouville fractional \( q \)-integral has the following semi-group property

\[
(I_q^\beta f)(t) = I_q^\beta I_q^\alpha f(t) = I_q^{\alpha + \beta} f(t).
\]

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Lemma 2 ([8]) Let $f$ be a $q$-integrable function on $[0, b]$. Then the following equality holds

$$D_q^\alpha I_q^\alpha f(t) = f(t), \text{ for } \alpha > 0, \ t \in [0, b].$$

Lemma 3 ([4]) Let $\alpha > 0$ and $p$ be a positive integer. Then for $t \in [0, b]$ the following equality holds

$$I_q^p D_q^p f(t) = D_q^p I_q^p f(t) - \sum_{k=0}^{p-1} \frac{p^{\alpha-p+k}}{\Gamma(q(\alpha + k-p + 1))} D_q^k f(0).$$

Lemma 4 ([24]) For $f(t) = t^k$ and $\beta > 0$, $\mu > 0$, $\eta, \lambda \in \mathbb{R}$, $0 < q < 1$, then

$$I_q^{\eta, \mu, \beta} t^k = \beta \left[ \frac{1}{\beta} \right] q \frac{\Gamma(\eta + 1 + \frac{\alpha}{\beta})}{\Gamma(\mu + \eta + 1 + \frac{\alpha}{\beta})} t^k.$$

3. Main results

In this section, we will give the main results of this paper. Let the space $E = \{ x \in C([0, T], \mathbb{R}), D_q^\alpha x \in C([0, T], \mathbb{R}) \}$ be endowed with the norm $\|x\| = \max_{t \in [0, T]} |x(t)| + \max_{t \in [0, T]} |D_q^\alpha x(t)|$. It is known that the space $E$ is a Banach space. To obtain our main results, we need the following lemma.

Lemma 5 Let $h(t) \in C([0, T], \mathbb{R})$. Then for any $t \in [0, T]$, the solution of the following problem

$$\begin{cases}
D_q^\alpha x(t) + h(t) = 0, \ t \in (0, T), \\
x(0) = 0, \ \alpha x(T) = \sum_{i=1}^n \lambda_i I_q^{\eta_i, \mu_i, \beta_i} x(\xi_i).
\end{cases} \tag{3.1}$$

is given by

$$x(t) = -I_q^\alpha h(t) + \frac{t^{\alpha-1}}{M} \left( aI_q^\alpha h(T) - \sum_{i=1}^n \lambda_i I_q^{\eta_i, \mu_i, \beta_i} I_q^\alpha h(\xi_i) \right), \tag{3.2}$$

where $M = aT^{\alpha-1} - \sum_{i=1}^n \lambda_i I_q^{\eta_i, \mu_i, \beta_i} \xi_i^{\alpha-1} = aT^{\alpha-1} - \sum_{i=1}^n \lambda_i \beta_i \left[ \frac{1}{\beta_i} \right] q \frac{\Gamma(\eta_i + 1 + \frac{\alpha}{\beta_i})}{\Gamma(\mu_i + \eta_i + 1 + \frac{\alpha}{\beta_i})} \xi_i^{\alpha-1} \neq 0$.

Proof. Applying the operator $I_q^\alpha$ on both sides of the first equation of (3.1) for $t \in (0, T)$ and using Lemma 1 and Lemma 2, we have

$$x(t) = -I_q^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \tag{3.3}$$

Applying the initial value condition $x(0) = 0$, we get $c_2 = 0$. By the boundary value condition, we have

$$- \sum_{i=1}^n \lambda_i I_q^{\eta_i, \mu_i, \beta_i} I_q^\alpha h(\xi_i) + c_1 \sum_{i=1}^n \lambda_i I_q^{\eta_i, \mu_i, \beta_i} \xi_i^{\alpha-1} = -aI_q^\alpha h(T) + c_1 aT^{\alpha-1},$$


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for each $x$ where $L$ and $\tau$ where.

Theorem 1 Suppose that there exists a function $L(t) : [0, T] \to \mathbb{R}^+$ $q$-integrable such that

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq L(t)(|x - \bar{x}| + |y - \bar{y}|),$$

for each $x, \bar{x}, y, \bar{y} \in \mathbb{R}$. Then problem (1.1) has an unique solution on $[0, T]$ if

$$L_1 + L_3 + \frac{T^{\alpha-1}}{|M|} \left( 1 + \frac{T^{-\delta} \Gamma_q(\alpha)}{\Gamma_q(\alpha - \delta)} \right) \left( |a| L_2 + \sum_{i=0}^{n} |\lambda_i| |\beta_i| \frac{1}{\beta_i} L_i \Gamma_q(\eta_i + 1) \right) < 1,$$

(3.5)

where $L_1 = \sup_{t \in [0,T]} I_q^\alpha L(t)$, $L_2 = \sup_{t \in [0,T]} I_q^{\alpha-1} L(t)$, $L_3 = \sup_{t \in [0,T]} I_q^{-\delta} L(t)$.

Proof. The conclusion will follow once we have shown that the operator $Q$ defined (3.4) is contractively with respect to a suitable norm on $E$.

For any functions $x, y \in E$, we have

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\(|(Qx)(t) - (Qy)(t)| \leq \left| \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha - 1)} (f(s, x(s), D_q^\delta x(s)) - f(s, y(s), D_q^\delta y(s))) ds \right| \\
\quad + \frac{r^{\alpha - 1}}{M} \left| a \right| \int_0^T (T - qs)^{(\alpha - 2)} (f(s, x(s), D_q^\delta x(s)) - f(s, y(s), D_q^\delta y(s))) ds \\
\quad + \sum_{i=0}^n |\lambda_i| \left| \beta_i \xi_i^{\beta_i(\eta_i + \mu)} \right| \int_0^\infty \left( \xi_i^{\beta_i} - s^{\beta_i} q^{(\alpha - 1)} s^{\beta_i(\eta_i + 1)} \right) \\
\quad + \frac{1}{\Gamma_q(\alpha)} \int_0^T (s - q\tau)^{(\alpha - 1)} (f(\tau, x(\tau), D_q^\delta x(\tau)) - f(\tau, y(\tau), D_q^\delta y(\tau))) d\tau ds \\
\leq \frac{\| x - y \|}{\Gamma_q(\alpha)} \int_0^T (t - qs)^{(\alpha - 1)} L(s) ds + \frac{T^{\alpha - 1}}{M} \left| a \right| \int_0^T (T - qs)^{(\alpha - 2)} L(s) ds \\
\quad + \sum_{i=0}^n |\lambda_i| \left| \beta_i \xi_i^{\beta_i(\eta_i + \mu)} \right| \int_0^\infty \left( \xi_i^{\beta_i} - s^{\beta_i} q^{(\alpha - 1)} s^{\beta_i(\eta_i + 1)} \right) \\
\quad + \left( L_1 + \frac{T^{\alpha - 1}}{|M|} |a|L_2 + L_1 \sum_{i=0}^n |\lambda_i| \left| \beta_i \right| \frac{1}{\Gamma_q(\mu_i)} \int_0^\infty \left( \xi_i^{\beta_i} - s^{\beta_i} q^{(\alpha - 1)} s^{\beta_i(\eta_i + 1)} \right) d\tau ds \\
\quad = \left( L_1 + \frac{T^{\alpha - 1}}{|M|} |a|L_2 + L_1 \sum_{i=0}^n |\lambda_i| \left| \beta_i \right| \frac{1}{\Gamma_q(\mu_i + \eta_i + 1)} \right) \| x - y \| .

On the other hand

\[ D_q^\delta (Qx)(t) = -D_q^\delta I_q^\alpha f(s, x(s), D_q^\delta x(s))(t) + \frac{1}{M} (aI_q^{\alpha - 1} f(s, x(s), D_q^\delta x(s))(T) \\
\quad - \sum_{i=1}^n \lambda_i I_q^{\alpha - \mu_i - \beta_i} I_q^\alpha f(s, x(s), D_q^\delta x(s))(\xi_i)) D_q^\delta I_q^\delta \\
\quad = -D_q^\delta I_q^\alpha f(s, x(s), D_q^\delta x(s))(t) + \frac{1}{M} (aI_q^{\alpha - 1} f(s, x(s), D_q^\delta x(s))(T) \\
\quad - \sum_{i=1}^n \lambda_i I_q^{\alpha - \mu_i - \beta_i} I_q^\alpha f(s, x(s), D_q^\delta x(s))(\xi_i)) \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha - \delta)} I_q^{\alpha - \delta} \\
\quad = -I_q^{\alpha - \delta} f(s, x(s), D_q^\delta x(s))(t) + \frac{1}{M} (aI_q^{\alpha - 1} f(s, x(s), D_q^\delta x(s))(T) \\
\quad - \sum_{i=1}^n \lambda_i I_q^{\alpha - \mu_i - \beta_i} I_q^\alpha f(s, x(s), D_q^\delta x(s))(\xi_i)) \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha - \delta)} I_q^{\alpha - \delta} .

Thus

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\[
|D_q^\delta(Qx)(t) - D_q^\delta(Qy)(t)| \leq \left| f_q^{\alpha - \delta}(f(s, x(s), D_q^\delta x(s))(t) - f(s, y(s), D_q^\delta y(s))(t)
\right. \\
+ \frac{\Gamma_q(\alpha)}{|M|}\Gamma_q(\alpha - \delta)\left(|f_q^{\alpha - 1}(f(s, x(s), D_q^\delta x(s))(T) - f(s, y(s), D_q^\delta y(s))(T)) + \sum_{i=1}^{n} |\lambda_i|p_i^{\mu_i.\beta_i} f_q^p(f(s, x(s), D_q^\delta x(s)) - f(s, y(s), D_q^\delta y(s)))(\xi_i))| \right. \\
+ \sum_{i=1}^{n} |\lambda_i|p_i^{\mu_i.\beta_i} f_q^p(f(s, x(s), D_q^\delta x(s)) - f(s, y(s), D_q^\delta y(s)))(\xi_i))| \\
\leq f_q^{\alpha - \delta}L(s)(t) + \frac{\Gamma_q(\alpha) T^{\alpha - 1 - \delta}}{|M|}\Gamma_q(\alpha - \delta)\left(|a| L_q f_q^{\alpha - 1}L(s)(T) + \sum_{i=0}^{n} |\lambda_i|p_i^{\mu_i.\beta_i} f_q^p L(\tau)(\xi_i)) \right) \\
\leq \left(L_3 + \frac{\Gamma_q(\alpha) T^{\alpha - 1 - \delta}}{|M|}\Gamma_q(\alpha - \delta)\left(|a| L_2 + \sum_{i=0}^{n} |\lambda_i|p_i^{\mu_i.\beta_i} \Gamma_q(\eta_i + 1) L_1 \Gamma_q(\mu_i + \eta_i + 1) \right) \right) \| x - y \| .
\]

which implies that
\[
\| Qx - Qy \| \leq \left( L_1 + L_3 + \frac{T^{\alpha - 1}}{|M|} \left( 1 + \frac{T^{\alpha - \delta}}{\Gamma_q(\alpha - \delta)} \right) \left(|a| L_2 + \sum_{i=0}^{n} |\lambda_i|p_i^{\mu_i.\beta_i} \Gamma_q(\eta_i + 1) L_1 \Gamma_q(\mu_i + \eta_i + 1) \right) \right) \| x - y \|. 
\]

Thus the operator \( Q \) is a contraction in view of the condition (3.5). By Banach’s contraction mapping principle, the problem (1.1) has a unique solution on \([0, T]\). This completes the proof.

**Corollary 1** Assume that there exists \( L_0 > 0 \) such that
\[
|f(t, x, y) - f(t, \tilde{x}, \tilde{y})| \leq L_0(|x - \tilde{x}| + |y - \tilde{y}|),
\]
for each \( t \in [0, T] \) and \( x, \tilde{x}, y, \tilde{y} \in \mathbb{R} \). Then the problem (1.1) has a unique solution whenever
\[
\left( \frac{T^{\alpha}}{\Gamma_q(\alpha + 1)} + \frac{T^{\alpha - \delta + 1}}{\Gamma_q(\alpha - \delta + 1)} + \frac{|a| T^{2\alpha - 2}}{|M|} \left( \frac{T^{2\alpha - 2}}{\Gamma_q(\alpha - \delta)} + \frac{T^{2\alpha - 2}}{\Gamma_q(\alpha - \delta)} \right) + \frac{1}{|M|} \left( \frac{T^{\alpha - 1}}{\Gamma_q(\alpha - \delta)} + \frac{T^{\alpha - 1}}{\Gamma_q(\alpha + 1)} \right) \sum_{i=0}^{n} |\lambda_i|p_i^{\mu_i.\beta_i} s^\alpha(\xi_i) \right) L_0 < 1,
\]
where \( I_q^{\eta_i,\mu_i.\xi}(\xi_i) = \beta_i \left( \Gamma_q(\eta_i + \frac{\mu_i}{\beta_i} + \frac{\alpha}{\beta_i}) \right) \Gamma_q(\mu_i + \eta_i + 1) \).

In the following Theorem, the existence results for nontrivial solution for problem (1.1) are presented. For convenience, we denote
\[
\zeta = \sum_{i=1}^{n} |\lambda_i|p_i^{\mu_i.\beta_i} s^\alpha(\xi_i) = \sum_{i=1}^{n} |\lambda_i|\beta_i \Gamma_q(\eta_i + 1) L_1 \Gamma_q(\mu_i + \eta_i + 1),
\]
\[
\sigma_i = \max_{t \in [0, T]} \frac{1}{\Gamma_q(\alpha)} \int_{0}^{t} (t - qs)^{\alpha - 1} l_i(s) ds, i = 1, 2, 3,
\]
\[
\sigma_i = \max_{t \in [0, T]} \frac{1}{\Gamma_q(\alpha - 1)} \int_{0}^{t} (t - qs)^{\alpha - 2} l_i(s) ds, i = 1, 2, 3,
\]
\[
\sigma_i = \max_{t \in [0, T]} \frac{1}{\Gamma_q(\alpha - \delta)} \int_{0}^{t} (t - qs)^{\alpha - \delta - 1} l_i(s) ds, i = 1, 2, 3,
\]
where \( l_i(t), i = 1, 2, 3 \) are defined in Theorem 2.

**Theorem 2** Let \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a continuous function. Assume there exist three nonnegative continuous functions \( l_i(t), i = 1, 2, 3 \) such that

\[
|f(t, x(t), D_q^\lambda x(t))| \leq l_1(t) + l_2(t)|x(t)|^{r_1} + l_3(t)|D_q^\lambda x(t)|^{r_2}, \quad 0 < r_i < 1, i = 1, 2,
\]

and \( f(t, 0, 0) \neq 0 \) for \( t \in [0, T] \). Then the problem (1.1) has a nontrivial solution.

**Proof.** We shall use Schauder’s fixed point theorem to prove our theorem. Define \( E = \{ x : x \in E, ||x|| \leq d \} \), where \( d \geq \max \{ \frac{3}{2} |1 + Q_1|, \frac{3}{2} |1 + Q_2|, \frac{3}{2} |1 + Q_3|, 3(\alpha + 1 + \rho)\} \). Note that \( E_d \) is a closed, bounded and convex subset of the Banach space \( E \). We now show that \( Q : E_d \to E_d \). In fact, for \( x \in E_d \) we have that

\[
||x(t)|| \leq \max_{t \in [0,T]}||x(t)|| \leq ||x|| \leq d,
\]

which implies

\[
|f(t, x(t), D_q^\lambda x(t))| \leq l_1(t) + l_2(t)d^{r_1} + l_3(t)d^{r_2}.
\]

Thus

\[
|Qx(t)| \leq \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - q\alpha)^{\alpha-1}(l_1(s) + l_2(s)d^{r_1} + l_3(s)d^{r_2})d_qs
\]

\[
+ \frac{a-1}{M} \left( \frac{|d|}{\Gamma_q(\alpha-1)} \int_0^T (T - q\alpha)^{\alpha-2}(l_1(s) + l_2(s)d^{r_1} + l_3(s)d^{r_2})d_qs
\]

\[
+ \sum_{i=1}^n \left| \beta_i \right| \left( \sum_{j=1}^{\eta_i} \frac{\Gamma_q(\mu_i)}{\Gamma_q(\alpha)} \int_0^{\xi_i} (\zeta_{i,j} - s^{\beta_i})^{|s| \beta_i - 1} \right) d_q\tau d_qs
\]

\[
\leq (Q_1 + Q_2d^{r_1} + Q_3d^{r_2})(1 + \xi) \frac{T^{\alpha-1}}{|M|} + \frac{T^{\alpha-1}|d|}{|M|} (\sigma_1 + \sigma_2d^{r_1} + \sigma_3d^{r_2}).
\]

\[
|D_q^\lambda Qx(t)| \leq \left[ (Q_1 + Q_2d^{r_1} + Q_3d^{r_2}) \xi + |d|(\sigma_1 + \sigma_2d^{r_1} + \sigma_3d^{r_2}) \right] \frac{T^{\alpha-\delta-1}}{|M|} \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha - \delta)}
\]

\[
+ (\xi_1 + \xi_2d^{r_1} + \xi_3d^{r_2}).
\]

From the two inequalities above, we get

\[
||Qx|| = \max_{t \in [0,T]} |x(t)| + \max_{t \in [0,T]} |D_q^\lambda x(t)|
\]

\[
\leq (Q_1 + Q_2d^{r_1} + Q_3d^{r_2}) \xi + |d|(\sigma_1 + \sigma_2d^{r_1} + \sigma_3d^{r_2})
\]

\[
\leq \frac{d}{3} + \frac{d}{3} + \frac{d}{3} = d.
\]
Hence, \( Q \) maps \( E_d \) into \( E_d \). Also, it is easy to check that \( Q \) is continuous, since \( f \) is continuous. For each \( x \in E_d \) and each \( 0 < t_1 < t_2 < T \), we have

\[
| (Qx)(t_2) - (Qx)(t_1) | \leq \frac{1}{\Gamma_q(\alpha)} \int_0^{t_1} (t_2 - QS)^{(\alpha-1)} - (t_1 - QS)^{(\alpha-1)} f(s, x(s), D^\delta_q x(s)) ds \\
+ \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - QS)^{(\alpha-1)} f(s, x(s), D^\delta_q x(s)) ds \\
= \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{M} \left( \left[ a \right] I^\delta_q f(s, x(s), D^\delta_q x(s))(T) \\
+ \sum_{i=1}^n \left| aI^\delta_q f(s, x(s), D^\delta_q x(s))(\xi_i) \right| \right).
\]

Let \( t_2 \to t_1 \), we get \( \| Qx(t_2) - Qx(t_1) \| \to 0 \). Thus, \( Q \) is uniformly bounded and equicontinuous. The theorem of Arzelà-Ascoli implies that \( Q \) is completely continuous. By Schauder’s fixed point theorem, \( Q \) has a fixed point in \( E_d \). Clearly \( x = 0 \) is not a fixed point because \( f(t, 0, 0) \neq 0 \) for \( t \in [0, T) \). Hence, the problem (1.1) has at least one nontrivial solution. This proves the theorem.

**Remark 2** In the Theorem 2, if \( r_i > 1, i = 1, 2 \), we may choose \( l_2(t) \), \( l_3(t) \) and \( d \) such that

\[
| f(t, x(t), D^\delta_q x(t)) | \leq l_2(t)|x(t)|^{r_1} + l_3(t)|D^\delta_q x(t)|^{r_2},
\]

and

\[
0 < d \leq \frac{1}{2(c_2 + |a|\sigma_2 \rho + g_2(1 + \rho)\zeta)} + \left( \frac{1}{2(c_3 + |a|\sigma_3 \rho + g_3(1 + \rho)\zeta)} \right)^{\frac{1}{r_2}}.
\]

For \( r_i = 1, i = 1, 2 \), we have the following theorem.
**Theorem 3** Let \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a continuous function. Assume there exist three nonnegative continuous functions \( l_i(t), i = 1, 2, 3 \) such that

\[
|f(t, x(t), D_q^x x(t))| \leq l_1(t) + l_2(t)|x(t)| + l_3(t)|D_q^x x(t)|,
\]

and \( \max \{ \varsigma_2 + |a|\sigma_2 \rho + \varphi_2(1 + \rho)\zeta, (\varsigma_3 + |a|\sigma_3 \rho + \varphi_3(1 + \rho)\zeta) \} < \frac{1}{\lambda} \). Further, assume that \( f(t, 0, 0) \neq 0 \) for \( t \in [0, T] \). Then the problem (1.1) has a nontrivial solution.

**Proof.** Let \( d > 3(\varsigma_1 + |a|\sigma_1 \rho + \varphi_1(1 + \rho)\zeta) \), The proof is similar to Theorem 2, so it is omitted. This completes the proof.

Although Theorems 2 and Theorem 3 provide some simple conditions on the existence of solution of problem (1.1) and Theorem 1 provides a condition on the existence and uniqueness on the solution of problem (1.1), the following theorem provides an easily verifiable condition for the existence of a nontrivial solution for the problems (1.1).

**Theorem 4** Assume that \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) with \( f(t, 0, 0) \neq 0 \) for \( t \in [0, T] \) and \( x \in E \). Suppose that

\[
\lim_{||x|| \to \infty} \max_{t \in [0, T]} \frac{|f(t, x(t), D_q^x x(t))|}{||x||} = 0
\]

(3.6)

holds. Then problem (1.1) has at least one nontrivial solution.

**Proof.** Choose a constant \( A \) such that

\[
A \left( \gamma_q \frac{T^\alpha}{\Gamma_q(\alpha + 1)} + \frac{|a|T^{2\alpha - 2}}{|M|\Gamma_q(\alpha)} + \frac{T^{\alpha - 1} \kappa}{|M|\Gamma_q(\alpha + 1)} + \frac{T^{\alpha - \delta}}{|M|\Gamma_q(\alpha - \delta)} + \frac{|a|T^{2\alpha - 2 - \delta}}{|M|\Gamma_q(\alpha - \delta)} + \frac{T^{\alpha - 1 - \delta} \kappa}{|M|\Gamma_q(\alpha - \delta)} \right) < 1,
\]

where \( \kappa = \sum_{i=1}^n |\lambda_i| \beta_i \left[ \frac{1}{\mu_i} \right] \Gamma_q(\eta_i + 1 + \frac{\gamma_i}{\mu_i}) \Gamma_q(\mu_i + 1 + \frac{\gamma_i}{\mu_i}) \delta_i ^{\gamma_i} \).

By the condition (3.6), there exists a constant \( c_1 \) such that

\[
|f(t, x(t), D_q^x x(t))| < A \parallel x \parallel \text{ for any } t \in [0, T] \text{ and } \parallel x \parallel \geq c_1.
\]

Since \( f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), we can find another constant \( A_1 > 0 \) such that \( |f(t, x(t), D_q^x x(t))| < A_1 \), for \( t \in [0, T] \) and \( \parallel x \parallel \leq c_1 \). Let \( c = \max \left\{ \frac{A_1}{A}, c_1 \right\} \), then for any \( \parallel x \parallel \leq c \), we have \( |f(t, x(t), D_q^x x(t))| < Ac \). Set

\[
E_c = \{ x \in E : \parallel x \parallel \leq c \}.
\]
Then for any \( x \in E_c \), we have

\[
|Qx(t)| \leq \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha - 1)} Acd_q s + \frac{\Gamma_q(\alpha) T^{\alpha - 1}}{|M| \Gamma_q(\alpha - 1)} \int_0^T (T - qs)^{(\alpha - 2)} Acd_q s + \sum_{i=1}^n |\lambda_i| \frac{\beta_i \xi_i^{\gamma(s_i - q\tau)} \gamma(\mu_i)}{\Gamma_q(\mu_i)} \int_0^T (T - qs)^{\alpha - 1} Acd_q s
\]

\[
\leq \left( \frac{T^\alpha}{\Gamma_q(\alpha + 1)} + \frac{T^{\alpha - 1} |a|}{|M| \Gamma_q(\alpha) + \Gamma_q(\alpha + 1)} \right) Ac.
\]

Thus

\[
||Qx|| = \max_{t \in [0, T]} |Qx(t)| + \max_{t \in [0, T]} |D_q^\alpha Qx(t)|
\]

\[
\leq \left( \frac{T^\alpha}{\Gamma_q(\alpha + 1)} + \frac{T^{\alpha - 1} |a|}{|M| \Gamma_q(\alpha) + \Gamma_q(\alpha + 1)} \right) Ac + \left( \frac{\Gamma_q(\alpha - \delta + 1)}{\Gamma_q(\alpha + 1)} + \frac{|a| T^{2\alpha - 2\delta}}{|M| \Gamma_q(\alpha) + \Gamma_q(\alpha + 1)} \right) Ac + \left( \frac{T^{\alpha - 1} \rho K}{\Gamma_q(\alpha + 1)} + \frac{|a| T^{2\alpha - 2\delta}}{|M| \Gamma_q(\alpha) + \Gamma_q(\alpha + 1)} \right) Ac
\]

\[
\leq \left( \frac{T^\alpha}{\Gamma_q(\alpha + 1)} + \frac{T^{\alpha - 1} |a|}{|M| \Gamma_q(\alpha) + \Gamma_q(\alpha + 1)} \right) Ac + \left( \frac{T^{\alpha - 1} \rho K}{\Gamma_q(\alpha + 1)} + \frac{|a| T^{2\alpha - 2\delta}}{|M| \Gamma_q(\alpha) + \Gamma_q(\alpha + 1)} \right) c < c.
\]

From (3.9), we obtain \( Q(E_c) \subseteq E_c \). By Schauder fixed point theorem, \( Q \) has at least one fixed point in \( E_c \). Clearly, \( x = 0 \) is not a fixed point because \( f(t, 0, 0) \neq 0 \). Therefore, problem (1.1) has at least one nontrivial solution, which completes the proof.

4. Example

In this section, we illustrate the results obtained in the last section.
Example 1 Consider the following fractional-order boundary value problem involving nonlocal Erdélyi-Kober fractional q-integral conditions:

\[
\begin{aligned}
D_q^{\frac{1}{z}} x(t) + \frac{t^{\frac{1}{z}}}{1 + t^{\frac{1}{z}}} \left( \frac{|x(t) + D_q^{\frac{1}{z}} x(t)|}{1 + 2|x(t) + D_q^{\frac{1}{z}} x(t)|} + \cos t + 2 \right) + 1 &= 0, \\
x(0) &= 0, \\
5x(1) &= \frac{1}{100} L_1^{\frac{1}{z} + \frac{1}{10} + \frac{1}{2}} x(\frac{1}{4}) + \frac{1}{100} L_1^{\frac{1}{z} + \frac{1}{10} + \frac{1}{2}} x(\frac{1}{4}),
\end{aligned}
\]

(4.1)

where \( q = \frac{1}{2}, \delta = \frac{1}{4}, T = 1, \alpha = \frac{3}{2}, a = 3, \lambda_1 = \frac{1}{100}, \lambda_2 = \frac{3}{100}, \beta_1 = \frac{5}{15}, \beta_2 = \frac{7}{5}, \eta_1 = \frac{3}{8}, \eta_2 = \frac{1}{8}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}, \mu_1 = \frac{5}{4}, \mu_1 = \frac{7}{4}, \) and \( f(t, x(t), D_q^\alpha x(t)) = \frac{t^2}{2} \left( \frac{|x(t) + y(t)|}{1 + 2|x(t) + y(t)|} + \cos t + 2 \right) + 1. \)

By computation, we deduce that

\[ |f(t, x(t), D_q^\alpha x(t)) - f(t, y(t), D_q^\alpha y(t))| \leq \frac{t^2}{2} \left( |x(t) - y(t)| + |D_q^\alpha x(t) - D_q^\alpha y(t)| \right), \]

then, the first condition is satisfied with \( L(t) = \frac{t^2}{2}. \)

\[
M = 5 - \frac{1}{100} \times 10^{[10]} \frac{\Gamma_q(\frac{1}{8} + \frac{1}{2} + \frac{3}{10})}{\Gamma_q(\frac{1}{8} + \frac{3}{10} + \frac{3}{10})} - \frac{3}{100} \times 5^{[5]} \frac{\Gamma_q(\frac{3}{8} + 1 + \frac{1}{2})}{\Gamma_q(\frac{3}{8} + \frac{3}{8} + 1 + \frac{1}{2})} = 4.986 \neq 0.
\]

\[
L_1 = \sup_{t \in [0,1]} \frac{1}{\Gamma_q(\frac{3}{8})} \int_0^t (t - qs)^\left(\frac{3}{8} - 1\right) s^2 \frac{d_q s}{2} = \frac{1}{2} \frac{\Gamma_q(3)}{\Gamma_q(4.5)}. \\
L_2 = \sup_{t \in [0,1]} \frac{1}{\Gamma_q(\frac{3}{8}) - 1} \int_0^t (t - qs)^\left(\frac{3}{8} - 2\right) s^2 \frac{d_q s}{2} = \frac{1}{2} \frac{\Gamma_q(3)}{\Gamma_q(4.5)}. \\
L_3 = \sup_{t \in [0,1]} \frac{1}{\Gamma_q(\frac{3}{8} - \frac{1}{4})} \int_0^t (t - qs)^\left(\frac{3}{8} - \frac{1}{4}\right) s^2 \frac{d_q s}{2} = \frac{1}{2} \frac{\Gamma_q(3)}{\Gamma_q(4.25)}. \\
L_1 + L_3 + \frac{T^\alpha - 1}{|M|} \left( 1 + \frac{T^\alpha - 1}{\Gamma_q(\alpha - \rho)} \right) \left( |a| L_2 + \sum_{i=0}^n |\beta_i| \beta_i^\alpha L_i \frac{\Gamma_q(\eta + 1)}{\Gamma_q(\mu_i + \eta_i + 1)} \right) \leq \frac{1}{2} \frac{\Gamma_q(3)}{\Gamma_q(4.5)} + \frac{1}{5} \frac{\Gamma_q(3)}{\Gamma_q(4.5)} + \frac{1}{5} \frac{\Gamma_q(3)}{\Gamma_q(4.5)} + \frac{1}{100} \times \frac{5}{2} \frac{\Gamma_q(3)}{\Gamma_q(3.5)} + \frac{1}{100} \times \frac{1}{2} \frac{\Gamma_q(3)}{\Gamma_q(4.5)} + \approx 0.894875 < 1.
\]
Hence, by Theorem 1, the boundary value problem (1.1) has an unique solution on [0, 1].

**Example 2** Consider the following fractional-order boundary value problem involving nonlocal Erdélyi-Kober fractional $q$-integral conditions:

\[
\begin{cases}
D^q_t x(t) + \frac{1}{2\pi} \sin(\pi|x|) \left(\frac{|x|+|D^\delta_q x|}{|x|+|D^\delta_q x|+1}\right)^{\frac{1}{2}} + 1 = 0, & t \in (0, 1) \\
x(0) = 0, \\
5x(1) = \frac{1}{2} I^{\frac{1}{2}, \frac{1}{2}}_\gamma x(\frac{1}{2}) + \frac{3}{10} I^{\frac{1}{2}, \frac{1}{2}}_\gamma x(\frac{1}{2}).
\end{cases}
\]

Here, $q = \frac{1}{2}, \delta = \frac{1}{4}, f(t, x(t), D^\delta_q x(t)) = \frac{1}{2\pi} \sin(\pi|x|) \left(\frac{|x|+|D^\delta_q x|}{|x|+|D^\delta_q x|+1}\right)^{\frac{1}{2}} + 1,$

\[
\frac{|f(t, x(t), D^\delta_q x(t))|}{\| x \|} = \frac{\frac{1}{2\pi} \sin(\pi|x|) \left(\frac{|x|+|D^\delta_q x|}{|x|+|D^\delta_q x|+1}\right)^{\frac{1}{2}} + 1}{\| x \|} \leq \frac{1}{2} \left(\frac{|x|+|D^\delta_q x|}{|x|} + 1\right) \rightarrow 0, \quad \text{as} \quad \| x \| \rightarrow \infty.
\]

Therefore, the conclusion of Theorem 4 implies that problem (1.1) has at least one solution on [0, 1].

5. Conclusions

In this work, we utilize Banach contraction principle and Schauder’s fixed point theorem to research the existence, uniqueness of solutions for a $q$-fractional differential equation with nonlocal Erdélyi-Kober $q$-fractional integral condition and in which the nonlinear term contains a fractional $q$-derivative of Riemann-Liouville type. Some existence and uniqueness results of solutions are obtained, we also provide an easily verifiable condition for the existence of nontrivial solution for the problem (1.1).

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Conflict of interest

The authors declare no conflict of interest.

References


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