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Research article

Existence of solutions for q-fractional differential equations with nonlocal Erdélyi-Kober q-fractional integral condition

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Abstract: In this paper, we obtain sufficient conditions for the existence, uniqueness of solutions for a fractional q-difference equation with nonlocal Erdélyi-Kober q-fractional integral condition. Our approach is based on some classical fixed point techniques, as Banach contraction principle and Schauder's fixed point theorem. Examples illustrating the obtained results are also presented.

Keywords: q-fractional differential equations; Erdélyi-Kober q-fractional integral; fixed point theorems; Riemann-Liouville fractional derivatives
Mathematics Subject Classification: 39A13, 34B18, 34A08

1. Introduction

The aim of this paper is to establish the existence and uniqueness of solutions for the following nonlinear Riemann-Liouville q-fractional differential equation subject to nonlocal Erdélyi-Kober q-fractional integral conditions

$$\begin{cases} D_q^{\alpha} x(t) + f(t, x(t), D_q^{\delta} x(t)) = 0, \ t \in (0, T), \\ x(0) = 0, \ ax(T) = \sum\limits_{i=1}^{n} \lambda_i I_q^{\eta_i, \mu_i, \beta_i} x(\xi_i), \end{cases} \tag{1.1}$$

where D_q^{α} and D_q^{ρ} are the fractional q-derivative of Riemann-Liouville type of order α and δ on $(0, T)$ respectively, $1 < \alpha < 2, \ 0 < \delta < 1, f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_q^{\eta_i, \mu_i, \beta_i}$ denotes the Erdélyi-Kober fractional q-integral of order μ_i on $(0, T)$, $\mu_i > 0, \ \beta_i > 0, \ \eta_i \in \mathbb{R}$ and $\xi_i \in (0, T)$, $a, \ \lambda_i(\ i = 1, 2, \cdots, \ n)$ are some given constants.

The q-calculus or quantum calculus is an old subject that was initially developed by Jackson [1], basic definitions and properties of q-calculus can be found in [2]. The fractional q-calculus had its origin in the works by Al-Salam [3] and Agarwal [4]. In recent years, considerable interest in q-fractional differential equations has been stimulated due to its applicability in mathematical modeling in different branches like engineering, physics and technical,etc. There are many papers and books dealing with the theoretical development of q-fractional calcaulus and the existence of solutions of boundary value problems for nonlinear q-fractional differential equations, for examples and details, one can see [5–22] and references therein.

In [9], Zhao, Chen and Zhang considered the following nonlocal q-integral boundary value problem of nonlinear fractional q-derivatives equation:

$$\begin{cases} D_q^\alpha x(t) + f(t, x(t)) = 0, \ t \in (0, 1), \\ x(0) = 0, \ x(1) = \mu I_q^\beta x(\eta) = \mu \int_0^\eta \frac{(\eta - qs)^{(\beta-1)}}{\Gamma_q(\beta)} x(s) d_q s, \end{cases}$$

where $q \in (0, 1)$, $1 < \alpha \leq 2$, $0 < \beta \leq 2$, $0 < \eta < 1$ and $\mu > 0$. D_q^α is the fractional q-derivative of Riemann-Liouville type of order α. By using the the generalized Banach contraction principle, the monotone iterative method, and Krasnoselskiis fixed point theorem, the authors obtained some existence results of positive solutions to the above problem.

In [10], the authors investigated the q-integral boundary value problem for q-integro-difference equations involving Riemann-Liouville q-derivatives and a q-integral of different orders as follows:

$$\begin{cases} (\lambda D_q^\alpha + (1 - \lambda) D_q^\beta) x(t) = a f(t, x(t)) + b I_q^\delta g(t, x(t)), \ t \in [0, 1], \ a, \ b \in \mathbb{R}^+, \\ \mu \int_0^1 \frac{(1-qs)^{(\gamma_1 - 1)}}{\Gamma_q(\gamma_1)} x(s) d_q s + (1 - \mu) \int_0^1 \frac{(1-qs)^{(\gamma_2 - 1)}}{\Gamma_q(\gamma_2)} x(s) d_q s, \\ x(0) = 0, \end{cases}$$

where, $q \in (0, 1)$, $1 < \alpha, \beta < 2$, $0 < \delta < 1$, $0 < \lambda \leq 1$ and $0 \leq \mu \leq 1$, $\alpha - \beta > 1$. D_q^α denotes the Riemann-Liouville fractional q-derivative of order α and $f, \ g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

In [23], the authors considered the existence of solutions for the following nonlinear Riemann-Liouville fractional differential equation with nonlocal Erdélyi-Kober fractional integral conditions

$$\begin{cases} (\lambda D^q x(t) = f(t, x(t)), \ t \in (0, T),, \\ x(0) = 0, \alpha x(T) = \sum_{i=1}^m \beta_i I_{\eta_i}^{\gamma_i, \delta_i} x(\xi_i), \end{cases}$$

where $1 < q \leq 2$, D^q is the Riemann-Liouville fractional derivative of order q, $I_{\eta_i}^{\gamma_i, \delta_i}$ is the Erdélyi-Kober fractional integral of order $\delta_i > 0$ with $\eta_i > 0$ and $\gamma_i \in \mathbb{R}$, $i = 1, 2, \cdots, m$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\alpha_i, \beta_i \in \mathbb{R}$, $\xi_i \in (0, T)$, $i = 1, 2, \cdots, m$ are given constants.

As we all know, few people solve the existence of solutions for a nonlinear Riemann-Liouville q-fractional differential equation subject to nonlocal Erdélyi-Kober q-fractional integral conditions. Inspired by the paper [23], we consider the existence and uniqueness for problem (1.1) by using Banach contraction principle and Schauder's fixed point theorem.

2. Preliminaries on q-calculus and Lemmas

Here we recall some definitions and fundamental results on fractional q-integral and fractional q-derivative. See the references [4–7] for complete theory.

For $q \in (0,1)$, define $[a]_q = \frac{1-q^a}{1-q}$, $a \in \mathbb{R}$. The q-factorial function is defined as $(a-b)^{(n)} = \prod_{k=0}^{n-1}(a-bq^k)$, $a, b \in \mathbb{R}$, If n is a positive integer. If ν is not a positive integer, then $(a-b)^{(\nu)} = a^\nu \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\nu+n}}$. If $b = 0$, then $a^{(\nu)} = a^\nu$. The q-gamma function is defined by $\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}$, $\alpha > 0$, and satisfies $\Gamma_q(\alpha+1) = [\alpha]_q\Gamma_q(\alpha)$.

The q-derivative of a function f is defined by $(D_q f)(t) = \frac{f(t)-f(qt))}{(1-q)t}$, $(D_q f)(0) = \lim_{t\to 0}(D_q f)(t)$. The q-integral of a function f defined on the interval $[0,b]$ is given by $(I_q f(t) = \int_0^t f(s)d_q s = (1-q)t\sum_{i=0}^{\infty} q^i f(q^i t)$, $t \in [0,b]$.

Some results about operator D_q and I_q can be found in references [4]. Let us define fractional q-derivative and q-integral and outline some of their properties [4, 6, 8].

Definition 1 ([4]) Let $\alpha \geq 0$ and f be a function. The fractional q-integral of Riemann-Liouville type is given by $(I_q^0 f)(t) = f(t)$ and

$$(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s)d_q s,\ \alpha > 0,\ t \in [0,b].$$

Definition 2 ([6]) The fractional q-derivative fractional of Riemann-Liouville type of order $\nu \geq 0$ is defined by $D_q^0 f(t) = f(t)$ and

$$D_q^\nu f(t) = D_q^l I_q^{l-\nu} f(t),\ \nu > 0,$$

where l is the smallest integer greater than or equal to ν.

Definition 3 ([24]) For $0 < q < 1$, the Erdélyi-Kober fractional q-integral of order $\mu > 0$ with $\beta > 0$ and $\eta \in \mathbb{R}$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$I_q^{\eta,\mu,\beta} f(t) = \frac{\beta t^{-\beta(\eta+\mu)}}{\Gamma_q(\mu)}\int_0^t (t^\beta - s^\beta q)^{(\mu-1)} s^\eta f(s)d_q s.$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Remark 1 For $\beta = 1$ the above operator is reduced to the q-analogue Kober operator

$$I_q^{\eta,\mu} f(t) = \frac{t^{-(\eta+\mu)}}{\Gamma_q(\mu)}\int_0^t (t-sq)^{(\mu-1)} s^\eta f(s)d_q s$$

that is given in [4]. For $\eta = 0$ the q-analogue Kober operator is reduced to the Riemann-Liouville fractional q-integral with a power weight:

$$I_q^\mu f(t) = \frac{t^{-\mu}}{\Gamma_q(\mu)}\int_0^t (t-sq)^{(\mu-1)} f(s)d_q s,\ \mu > 0.$$

Lemma 1 ([4]) Let $\alpha, \beta \in \mathbb{R}^+$ and f be a continuous function on $[0,b]$. The Riemann-Liouville fractional q-integral has the following semi-group property

$$I_q^\beta I_q^\alpha f(t) = I_q^\alpha I_q^\beta f(t) = I_q^{\alpha+\beta} f(t).$$

Lemma 2 ([8]) Let f be a q-integrable function on $[0,b]$. Then the following equality holds

$$D_q^\alpha I_q^\alpha f(t) = f(t), \text{ for } \alpha > 0,\ t \in [0,b].$$

Lemma 3 ([4]) Let $\alpha > 0$ and p be a positive integer. Then for $t \in [0,b]$ the following equality holds

$$I_q^\alpha D_q^p f(t) = D_q^p I_q^\alpha f(t) - \sum_{k=0}^{p-1} \frac{t^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} D_q^k f(0).$$

Lemma 4 ([24]) For $f(t) = t^\lambda$ and $\beta > 0,\ \mu > 0,\ \eta, \lambda \in \mathbb{R},\ 0 < q < 1$, then

$$I_q^{\eta,\mu,\beta} t^\lambda = \beta[\frac{1}{\beta}]_q \frac{\Gamma_q(\eta + 1 + \frac{\lambda}{\beta})}{\Gamma_q(\mu + \eta + 1 + \frac{\lambda}{\beta})} t^\lambda.$$

3. Main results

In this section, we will give the main results of this paper. Let the space $E = \{x \in C([0,T],\mathbb{R}), D_q^\delta x \in C([0,T],\mathbb{R})\}$ be endowed with the norm $\|x\| = \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |D_q^\delta x(t)|$. It is known that the space E is a Banach space. To obtain our main results, we need the following lemma.

Lemma 5 Let $h(t) \in C([0,T],\mathbb{R})$. Then for any $t \in [0,T]$, the solution of the following problem

$$\begin{cases} D_q^\alpha x(t) + h(t) = 0,\ t \in (0,T), \\ x(0) = 0,\ ax(T) = \sum_{i=1}^n \lambda_i I_q^{\eta_i,\mu_i,\beta_i} x(\xi_i). \end{cases} \tag{3.1}$$

is given by

$$x(t) = -I_q^\alpha h(t) + \frac{t^{\alpha-1}}{M}\Big(aI_q^\alpha h(T) - \sum_{i=1}^n \lambda_i I_q^{\eta_i,\mu_i,\beta_i} I_q^\alpha h(\xi_i)\Big), \tag{3.2}$$

where $M = aT^{\alpha-1} - \sum_{i=1}^n \lambda_i I_q^{\eta_i,\mu_i,\beta_i} \xi_i^{\alpha-1} = aT^{\alpha-1} - \sum_{i=1}^n \lambda_i\beta_i[\frac{1}{\beta_i}]_q \frac{\Gamma_q(\eta_i+1+\frac{\alpha-1}{\beta_i})}{\Gamma_q(\mu_i+\eta_i+1+\frac{\alpha-1}{\beta_i})} \xi_i^{\alpha-1} \neq 0$.

Proof. Applying the operator I_q^α on both sides of the first equation of (3.1) for $t \in (0,T)$ and using Lemma 1 and Lemma 2, we have

$$x(t) = -I_q^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \tag{3.3}$$

Applying the initial value condition $x(0) = 0$, we get $c_2 = 0$. By the boundary value condition, we have

$$-\sum_{i=1}^n \lambda_i I_q^{\eta_i,\mu_i,\beta_i} I_q^\alpha h(\xi_i) + c_1 \sum_{i=1}^n \lambda_i I_q^{\eta_i,\mu_i,\beta_i} \xi_i^{\alpha-1} = -aI_q^\alpha h(T) + c_1 aT^{\alpha-1},$$

that is

$$c_1 = \frac{aI_q^\alpha h(T) - \sum_{i=1}^{n} \lambda_i I_q^{\eta_i,\mu_i,\beta_i} I_q^\alpha h(\xi_i)}{aT^{\alpha-1} - \sum_{i=1}^{n} \lambda_i I_q^{\eta_i,\mu_i,\beta_i} I_q^\alpha \xi_i^{\alpha-1}} = \frac{aI_q^\alpha h(T) - \sum_{i=1}^{n} \lambda_i I_q^{\eta_i,\mu_i,\beta_i} I_q^\alpha h(\xi_i)}{M}.$$

Substituting c_1, c_2 to (3.3) , we obtain the solution (3.2). This completes the proof.

Using the Lemma 5, we can define an operator $Q : E \to E$ as follows:

$$\begin{aligned} Qx(t) = &- I_q^\alpha f(s, x(s), D_q^\delta x(s))(t) + \frac{t^{\alpha-1}}{M}\Big(aI_q^{\alpha-1} f(s, x(s), D_q^\rho x(s))(T) \\ &- \sum_{i=1}^{n} \lambda_i I_q^{\eta_i,\mu_i,\beta_i} I_q^\alpha f(s, x(s), D_q^\delta x(s))(\xi_i)\Big), \end{aligned} \tag{3.4}$$

where

$$I_q^\alpha f(s, x(s), D_q^\delta x(s))(\tau) = \frac{1}{\Gamma_q(\alpha)} \int_0^\tau (\tau - qs)^{(\alpha-1)} f(s, x(s), D_q^\delta x(s)) d_q s$$

and $\tau \in \{t, T, \xi_1, \xi_2 \cdots, \xi_n\}$. Then, the existence of solutions of system (1.1) is equivalent to the problem of fixed point of operator Q in (3.4).

In the following, we will use some classical fixed point techniques to give our main results.

Theorem 1 Suppose that there exists a function $L(t) : [0, T] \to \mathbb{R}^+$ q-integrable such that

$$|f(t, x, y) - f(t, \tilde{x}, \tilde{y})| \le L(t)(|x - \tilde{x}| + |y - \tilde{y}|),$$

for each $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}$. Then problem (1.1) has an unique solution on $[0, T]$ if

$$L_1 + L_3 + \frac{T^{\alpha-1}}{|M|}\Big(1 + \frac{T^{-\delta}\Gamma_q(\alpha)}{\Gamma_q(\alpha - \delta)}\Big)\Big(|a|L_2 + \sum_{i=0}^{n} |\lambda_i|\beta_i[\frac{1}{\beta_i}]_q \frac{L_1\Gamma_q(\eta_i + 1)}{\Gamma_q(\mu_i + \eta_i + 1)}\Big) < 1, \tag{3.5}$$

where $L_1 = \sup_{t\in[0,T]} I_q^\alpha L(t),\ L_2 = \sup_{t\in[0,T]} I_q^{\alpha-1} L(t),\ L_3 = \sup_{t\in[0,T]} I_q^{\alpha-\delta} L(t)$.

Proof. The conclusion will follow once we have shown that the operator Q defined (3.4) is contractively with respect to a suitable norm on E.

For any functions $x, y \in E$, we have

$$
\begin{aligned}
|(Qx)(t)-(Qy)(t)| \leq & \left|\frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}(f(s,x(s),D_q^\delta x(s))-f(s,y(s),D_q^\delta y(s))d_q s\right| \\
&+\frac{t^{\alpha-1}}{M}\Bigg(\left|\frac{|a|}{\Gamma_q(\alpha-1)}\int_0^T (T-qs)^{(\alpha-2)}(f(s,x(s),D_q^\delta x(s))-f(s,y(s),D_q^\delta y(s))d_q s\right| \\
&+\sum_{i=0}^{n}|\lambda_i|\Bigg|\frac{\beta_i\xi_i^{-\beta_i(\eta_i+\mu_i)}}{\Gamma_q(\mu_i)}\int_0^{\xi_i}(\xi_i^{\beta_i}-s^{\beta_i}q)^{(\mu_i-1)}s^{\beta_i(\eta_i+1)-1} \\
&\frac{1}{\Gamma_q(\alpha)}\int_0^s (s-q\tau)^{(\alpha-1)}(f(\tau,x(\tau),D_q^\delta x(\tau))-f(\tau,y(\tau),D_q^\delta y(\tau))d_q\tau d_q s\Bigg|\Bigg) \\
&\leq \frac{\parallel x-y\parallel}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}L(s)d_q s+\frac{T^{\alpha-1}}{M}\Bigg(\frac{|a|}{\Gamma_q(\alpha-1)}\int_0^T (T-qs)^{(\alpha-2)}L(s)d_q s \\
&+\sum_{i=0}^{n}|\lambda_i|\Bigg|\frac{\beta_i\xi_i^{-\beta_i(\eta_i+\mu_i)}}{\Gamma_q(\mu_i)\Gamma_q(\alpha)}\int_0^{\xi_i}(\xi_i^{\beta_i}-s^{\beta_i}q)^{(\mu_i-1)}s^{\beta_i(\eta_i+1)-1} \\
&\int_0^s (s-q\tau)^{(\alpha-1)}L(\tau)d_q\tau d_q s\Bigg)\parallel x-y\parallel \\
&\leq \parallel x-y\parallel\Bigg(L_1+\frac{T^{\alpha-1}}{|M|}\Big(|a|L_2+L_1\sum_{i=0}^{n}|\lambda_i|\Big|\frac{\beta_i\xi_i^{-\beta_i(\eta_i+\mu_i)}}{\Gamma_q(\mu_i)} \\
&\int_0^{\xi_i}(\xi_i^{\beta_i}-s^{\beta_i}q)^{(\mu_i-1)}s^{\beta_i(\eta_i+1)-1}d_q s\Big)\Bigg) \\
&=\Big(L_1+\frac{T^{\alpha-1}}{|M|}(|a|L_2+L_1\sum_{i=0}^{n}|\lambda_i|\beta_i[\frac{1}{\beta_i}]_q\frac{\Gamma_q(\eta_i+1)}{\Gamma_q(\mu_i+\eta_i+1)}\Big)\parallel x-y\parallel .
\end{aligned}
$$

on the other hand

$$
\begin{aligned}
D_q^\delta(Qx)(t)= & -D_q^\delta I_q^\alpha f(s,x(s),D_q^\delta x(s))(t)+\frac{1}{M}\Big(aI_q^{\alpha-1}f(s,x(s),D_q^\delta x(s))(T) \\
&-\sum_{i=1}^{n}\lambda_i I_q^{\eta_i,\mu_i,\beta_i}I_q^\alpha f(s,x(s),D_q^\delta x(s))(\xi_i)\Big)D_q^\delta t^{\alpha-1} \\
= & -D_q^\rho I_q^\delta I_q^{\alpha-\delta}f(s,x(s),D_q^\delta x(s))(t)+\frac{1}{M}\Big(aI_q^{\alpha-1}f(s,x(s),D_q^\delta x(s))(T) \\
&-\sum_{i=1}^{n}\lambda_i I_q^{\eta_i,\mu_i,\beta_i}I_q^\alpha f(s,x(s),D_q^\rho x(s))(\xi_i)\Big)\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\delta)}t^{\alpha-1-\delta} \\
= & -I_q^{\alpha-\delta}f(s,x(s),D_q^\delta x(s))(t)+\frac{1}{M}\Big(aI_q^{\alpha-1}f(s,x(s),D_q^\delta x(s))(T) \\
&-\sum_{i=1}^{n}\lambda_i I_q^{\eta_i,\mu_i,\beta_i}I_q^\alpha f(s,x(s),D_q^\delta x(s))(\xi_i)\Big)\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\delta)}t^{\alpha-1-\delta}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|D_q^\delta(Qx)(t)-D_q^\delta(Qy)(t)| \leq & \Big|I_q^{\alpha-\delta}(f(s,x(s),D_q^\delta x(s))(t)-f(s,y(s),D_q^\delta y(s)))(t)\Big| \\
& + \frac{\Gamma_q(\alpha)t^{\alpha-1-\delta}}{|M|\Gamma_q(\alpha-\delta)}\Big(|aI_q^{\alpha-1}(f(s,x(s),D_q^\delta x(s))(T)-f(s,y(s),D_q^\delta y(s)))(T)| \\
& + \sum_{i=1}^{n}|\lambda_i||I_q^{\eta_i,\mu_i,\beta_i}I_q^{\alpha}((f(s,x(s),D_q^\rho x(s))-f(s,y(s),D_q^\delta y(s)))(\xi_i)\Big)| \\
\leq & I_q^{\alpha-\delta}L(s)(t)+\frac{\Gamma_q(\alpha)T^{\alpha-1-\delta}}{|M|\Gamma_q(\alpha-\delta)}\Big(|a|I_q^{\alpha-1}L(s)(T)+\sum_{i=1}^{n}\lambda_i I_q^{\eta_i,\mu_i,\beta_i}I_q^{\alpha}L(\tau)(\xi_i)\Big) \\
\leq & \Big(L_3+\frac{\Gamma_q(\alpha)T^{\alpha-1-\delta}}{|M|\Gamma_q(\alpha-\delta)}\Big(|a|L_2+\sum_{i=0}^{n}|\lambda_i|\beta_i[\frac{1}{\beta_i}]_q\frac{L_1\Gamma_q(\eta_i+1)}{\Gamma_q(\mu_i+\eta_i+1)}\Big)\Big)\parallel x-y\parallel,
\end{aligned}
$$

which implies that

$$
\parallel Qx-Qy\parallel\leq\left\{L_1+L_3+\frac{T^{\alpha-1}}{|M|}\Big(1+\frac{T^{-\delta}\Gamma_q(\alpha)}{\Gamma_q(\alpha-\delta)}\Big)\Big(|a|L_2+\sum_{i=0}^{n}|\lambda_i|\beta_i[\frac{1}{\beta_i}]_q\frac{L_1\Gamma_q(\eta_i+1)}{\Gamma_q(\mu_i+\eta_i+1)}\Big)\right\}\parallel x-y\parallel.
$$

Thus the operator Q is a contraction in view of the condition (3.5). By Banach's contraction mapping principle, the problem (1.1) has an unique solution on $[0,T]$. This completes the proof.

Corollary 1 Assume that there exists $L_0>0$ such that

$$
|f(t,x,y)-f(t,\tilde{x},\tilde{y})|\leq L_0(|x-\tilde{x}|+|y-\tilde{y}|),
$$

for each $t\in[0,T]$ and $x,\tilde{x},y,\tilde{y}\in\mathbb{R}$. Then the problem (1.1) has an unique solution whenever

$$
\begin{aligned}
&\Big(\frac{T^\alpha}{\Gamma_q(\alpha+1)}+\frac{T^{\alpha-\delta+1}}{\Gamma_q(\alpha-\delta+1)}+\frac{|a|}{|M|}(\frac{T^{2\alpha-2}}{\Gamma_q\alpha}+\frac{T^{2\alpha-2}}{\Gamma_q(\alpha-\delta)})+ \\
&\frac{1}{|M|}(\frac{T^{\alpha-1-\delta}}{[\alpha]_q\Gamma_q(\alpha-\delta)}+\frac{T^{\alpha-1}}{\Gamma_q(\alpha+1)})\sum_{i=1}^{n}|\lambda_i|I_q^{\eta_i,\mu_i,\xi_i}s^\alpha(\xi_i)\Big)L_0<1,
\end{aligned}
$$

where $I_q^{\eta_i,\mu_i,\xi_i}s^\alpha(\xi_i)=\beta_i[\frac{1}{\beta_i}]_q\frac{\Gamma_q(\eta_i+1+\frac{\alpha-1}{\beta_i})}{\Gamma_q(\mu_i+\eta_i+1+\frac{\alpha-1}{\beta_i})}\xi_i^\alpha$.

In the following Theorem, the existence results for nontrivial solution for problem (1.1) are presented. For convenience, we denote

$$
\zeta=\sum_{i=1}^{n}|\lambda_i|I_q^{\eta_i,\mu_i,\xi_i}1(\xi_i)=\sum_{i=1}^{n}|\lambda_i|\beta_i[\frac{1}{\beta_i}]_q\frac{\Gamma_q(\eta_i+1)}{\Gamma_q(\mu_i+\eta_i+1)},
$$

$$
\varrho_i=\max_{t\in[0,T]}\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}l_i(s)d_qs,\ i=1,2,3,
$$

$$
\sigma_i=\max_{t\in[0,T]}\frac{1}{\Gamma_q(\alpha-1)}\int_0^t(t-qs)^{(\alpha-2)}l_i(s)d_qs,\ i=1,2,3,
$$

$$
\varsigma_i=\max_{t\in[0,T]}\frac{1}{\Gamma_q(\alpha-\delta)}\int_0^t(t-qs)^{(\alpha-\delta-1)}l_i(s)d_qs,\ i=1,2,3,
$$

where $l_i(t), i = 1, 2, 3$ are defined in Theorem 2.

Theorem 2 Let $f : [0, T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Assume there exist three nonnegative continuous functions $l_i(t), i = 1, 2, 3$ such that

$$|f(t, x(t), D_q^\delta x(t))| \le l_1(t) + l_2(t)|x(t)|^{r_1} + l_3(t)|D_q^\delta x(t)|^{r_2}, \quad 0 < r_i < 1, i = 1, 2,$$

and $f(t, 0, 0) \neq 0$ for $t \in [0, T]$. Then the problem (1.1) has a nontrivial solution.

Proof. We shall use Schauder's fixed point theorem to prove our theorem. Define $E_d = \{x : x \in E, \| x \|\le d\}$, where $d \ge \max\Big\{3(\varsigma_1 + |a|\sigma_1\rho + \varrho_1(1 + \rho)\zeta), 3(\varsigma_2 + |a|\sigma_2\rho + \varrho_2(1 + \rho)\zeta)^{\frac{1}{1-r_1}}, 3(\varsigma_3 + |a|\sigma_3\rho + \varrho_3(1 + \rho)\zeta)^{\frac{1}{1-r_2}}\Big\}$ and $\rho = \frac{T^{\alpha-1}}{|M|} + \frac{T^{\alpha-\delta-1}}{|M|}\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\delta-1)}$. Note that E_d is a closed, bounded and convex subset of the Banach space E. We now show that $Q : E_d \to E_d$. In fact, for $x \in E_d$ we have that

$$|x(t)| \le \max_{t\in[0,T]}|x(t)| \le\| x \|\le d,$$

$$|D_q^\rho x(t)| \le \max_{t\in[0,T]}|D_q^\rho x(t)| \le\| x \|\le d,$$

which implies

$$|f(t, x(t), D_q^\delta x(t)| \le l_1(t) + l_2(t)d^{r_1} + l_3(t)d^{r_2}.$$

Thus

$$\begin{aligned}|Qx(t)| \le&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t - qs)^{(\alpha-1)}(l_1(s) + l_2(s)d^{r_1} + l_3(s)d^{r_3}d_qs\\ &+\frac{t^{\alpha-1}}{M}\Big(\frac{|a|}{\Gamma_q(\alpha - 1)}\int_0^T(T - qs)^{(\alpha-2)}(l_1(s) + l_2(s)d^{r_1} + l_3(s)d^{r_3}d_qs\\ &+\sum_{i=1}^n|\lambda_i|\frac{\beta_i\xi_i^{-\beta_i(\eta_i+\mu_i)}}{\Gamma_q(\mu_i)}\int_0^{\xi_i}(\xi_i^{\beta_i} - s^{\beta_i}q)^{(\mu_i-1)}s^{\beta(\eta_i+1)-1}\\ &\frac{1}{\Gamma_q(\alpha)}\int_0^s(s - q\tau)^{(\alpha-1)}(l_1(\tau) + l_2(\tau)d^{r_1} + l_3(\tau)d^{r_3})d_q\tau d_qs\\ \le&(\varrho_1 + \varrho_2d^{r_1} + \varrho_3d^{r_2})(1 + \zeta)\frac{T^{\alpha-1}}{|M|} + \frac{T^{\alpha-1}|a|}{|M|}(\sigma_1 + \sigma_2d^{r_1} + \sigma_3d^{r_2}).\end{aligned}$$

$$\begin{aligned}|D_q^\rho Qx(t)| \le&\Big[(\varrho_1 + \varrho_2d^{r_1} + \varrho_3d^{r_2})\zeta + |a|(\sigma_1 + \sigma_2d^{r_1} + \sigma_3d^{r_2})\Big]\frac{T^{\alpha-\delta-1}}{|M|}\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\delta)}\\ &+ (\varsigma_1 + \varsigma_2d^{r_1} + \varsigma_3d^{r_2}).\end{aligned}$$

From the two inequalities above, we get

$$\begin{aligned}\| Qx \|=&\max_{t\in[0,T]}|x(t)| + \max_{t\in[0,T]}|D_q^\delta x(t)|\\ \le&(\varsigma_1 + |a|\sigma_1\rho + \varrho_1(1 + \rho)\zeta) + (\varsigma_2 + |a|\sigma_2\rho + \varrho_2(1 + \rho)\zeta)d^{r_1}+\\ &(\varsigma_3 + |a|\sigma_3\rho + \varrho_3(1 + \rho)\zeta)d^{r_2}\\ \le&\frac{d}{3} + \frac{d}{3} + \frac{d}{3} = d.\end{aligned}$$

Hence, Q maps E_d into E_d. Also, it is easy to check that Q is continuous, since f is continuous. For each $x \in E_d$ and each $0 < t_1 < t_2 < T$, we have

$$\begin{aligned}|(Qx)(t_2) - (Qx)(t_1)| \leq & \left|\frac{1}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}\right] f(s, x(s), D_q^\delta x(s)) d_q s\right| \\ & + \left|\frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)} f(s, x(s), D_q^\delta x(s)) d_q s\right| \\ & + \left|\frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{M} \Big(|a| I_q^{\alpha-1} f(s, x(s), D_q^\delta x(s))(T) \right. \\ & \left. + \sum_{i=1}^{n} |\lambda_i| I_q^{\eta_i,\mu_i,\beta_i} I_q^\alpha f(s, x(s), D_q^\delta x(s))(\xi_i)\Big)\right|.\end{aligned}$$

$$\begin{aligned}|(D_q^\rho Qx)(t_2) - (D_q^\delta Qx)(t_1)| \leq & \left|I_q^{\alpha-\delta} f(s, x(s), D_q^\delta x(s))(t_2) - I_q^{\alpha-\delta} f(s, x(s), D_q^\delta x(s))(t_1)\right| \\ & + \left|\frac{1}{M}\Big(a I_q^{\alpha-1} f(s, x(s), D_q^\delta x(s))(T) \right. \\ & \left. - \sum_{i=1}^{n} \lambda_i I_q^{\eta_i,\mu_i,\beta_i} I_q^\alpha f(s, x(s), D_q^\delta x(s))(\xi_i)\Big) \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\delta)} (t_2^{\alpha-1-\delta} - t_1^{\alpha-1-\delta})\right| \\ \leq & \left|\frac{1}{\Gamma_q(\alpha-\delta)} \int_0^{t_1} \left[(t_2 - qs)^{(\alpha-\delta-1)} - (t_1 - qs)^{(\alpha-\delta-1)}\right] f(s, x(s), D_q^\delta x(s)) d_q s\right| \\ & + \left|\frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1-\delta)} f(s, x(s), D_q^\delta x(s)) d_q s\right| \\ & + \left|\frac{t_2^{\alpha-1} - t_1^{\alpha-1-\delta}}{M} \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\delta)} \Big(|a| I_q^{\alpha-1} f(s, x(s), D_q^\delta x(s))(T) \right. \\ & \left. + \sum_{i=1}^{n} |\lambda_i| I_q^{\eta_i,\mu_i,\beta_i} I_q^\alpha f(s, x(s), D_q^\delta x(s))(\xi_i)\Big)\right|.\end{aligned}$$

Let $t_2 \to t_1$, we get $\| Qx(t_2) - Qx(t_2) \| \to 0$. Thus, Q is uniformly bounded and equicontinuous. The theorem of Arzelá-Ascoli implies that Q is completely continuous. By Schauder's fixed point theorem, Q has a fixed point in E_d. Clearly $x = 0$ is not a fixed point because $f(t, 0, 0) \neq 0$ for $t \in [0, T]$. Hence, the problem (1.1) has at least one nontrivial solution. This proves the theorem.

Remark 2 In the Theorem 2, if $r_i > 1, i = 1, 2$, we may choose $l_2(t),\ l_3(t)$ and d such that

$$|f(t, x(t), D_q^\delta x(t)| \leq l_2(t)|x(t)|^{r_1} + l_3(t)|D_q^\delta x(t)|^{r_2},$$

and

$$0 < d \leq \min\left\{\Big(\frac{1}{2(\varsigma_2 + |a|\sigma_2\rho + \varrho_2(1+\rho)\zeta)}\Big)^{\frac{1}{r_1-1}}, \Big(\frac{1}{2(\varsigma_3 + |a|\sigma_3\rho + \varrho_3(1+\rho)\zeta)}\Big)^{\frac{1}{r_2-1}}\right\}.$$

For $r_i = 1, i = 1, 2$, we have the following theorem.

Theorem 3 Let $f : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Assume there exist three nonnegative continuous functions $l_i(t), i = 1,2,3$ such that

$$|f(t,x(t),D_q^\delta x(t))| \leq l_1(t) + l_2(t)|x(t)| + l_3(t)|D_q^\delta x(t)|,$$

and $\max\Big\{\varsigma_2 + |a|\sigma_2\rho + \varrho_2(1+\rho)\zeta), (\varsigma_3 + |a|\sigma_3\rho + \varrho_3(1+\rho)\zeta)\Big\} < \frac{1}{3}$. Further, assume that $f(t,0,0) \neq 0$ for $t \in [0,T]$. Then the problem (1.1) has a nontrivial solution.

Proof. Let $d > 3\Big(\varsigma_1 + |a|\sigma_1\rho + \varrho_1(1+\rho)\zeta\Big)$, The proof is similar to Theorem 2, so it is omitted. This completes the proof.

Although Theorems 2 and Theorem 3 provide some simple conditions on the existence of solution of problem (1.1) and Theorem 1 provides a condition on the existence and uniqueness on the solution of problem (1.1), the following theorem provides an easily verifiable condition for the existence of a nontrivial solution for the problems (1.1).

Theorem 4 Assume that $f : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with $f(t,0,0) \neq 0$ for $t \in [0,T]$ and $x \in E$. Suppose that

$$\lim_{\|x\|\to\infty} \max_{t\in[0,T]} \frac{|f(t,x(t),D_q^\delta x(t))|}{\| x \|} = 0 \tag{3.6}$$

holds. Then problem (1.1) has at least one nontrivial solution.

Proof. Choose a constant A such that

$$A\Big(\frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{|a|T^{2\alpha-2}}{|M|\Gamma_q(\alpha)} + \frac{T^{\alpha-1}\kappa}{|M|\Gamma_q(\alpha+1)} + \frac{T^{\alpha-\delta}}{\Gamma_q(\alpha-\delta+1)} + \frac{|a|T^{2\alpha-2-\delta}}{|M|\Gamma_q(\alpha-\delta)} + \frac{T^{\alpha-1-\delta}\kappa}{|M|[\alpha]_q\Gamma_q(\alpha-\delta)}\Big) < 1,$$

where $\kappa = \sum_{i=1}^{n} |\lambda_i|\beta_i[\frac{1}{\beta_i}]_q \frac{\Gamma_q(\eta_i+1+\frac{\alpha}{\beta_i})}{\Gamma_q(\mu_i+1+\frac{\alpha}{\beta_i})}\xi_i^\alpha$.

By the condition (3.6), there exists a constant c_1 such that

$$|f(t,x(t),D_q^\delta x(t))| < A \| x \| \text{ for any } t \in [0,T] \text{and } \| x \| \geq c_1.$$

Since $f : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, we can find another constant $A_1 > 0$ such that $|f(t,x(t),D_q^\delta x(t))| < A_1$, for $t \in [0,T]$ and $\| x \| \leq c_1$. Let $c = \max\{\frac{A_1}{A}, c_1\}$, then for any $\| x \| \leq c$, we have $|f(t,x(t),D_q^\delta x(t))| < Ac$. Set

$$E_c = \{x \in E : \| x \| \leq c\}.$$

Then for any $x \in E_c$, we have

$$
\begin{aligned}
|Qx(t)| \leq & \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} Acd_q s + \frac{t^{\alpha-1}}{M}\Big(\frac{|a|}{\Gamma_q(\alpha-1)} \int_0^T (T-qs)^{(\alpha-2)} Acd_q s \\
& + \sum_{i=1}^{n} |\lambda_i| \frac{\beta_i \xi_i^{-\beta_i(\eta_i+\mu_i)}}{\Gamma_q(\mu_i)} \int_0^{\xi_i} (\xi_i^{\beta_i} - s^{\beta_i} q)^{(\mu_i-1)} s^{\beta(\eta_i+1)-1} \\
& \frac{1}{\Gamma_q(\alpha)} \int_0^s (s-q\tau)^{(\alpha-1)} Acd_q \tau d_q s \\
\leq & \Big(\frac{T^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{T^{\alpha-1}}{|M|}\Big(\frac{T^{\alpha-1}|a|}{\Gamma_q(\alpha)} + \frac{\kappa}{\Gamma_q(\alpha+1)}\Big)\Big)Ac.
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
|D_q^{\rho}(Qx)(t)| \leq & \frac{1}{\Gamma_q(\alpha-\delta)} \int_0^t (t-qs)^{(\alpha-\delta-1)} Acd_q s + \frac{\Gamma_q(\alpha) T^{\alpha-\delta-1}}{|M|\Gamma_q(\alpha-\delta)}\Big(\frac{|a|}{\Gamma_q(\alpha-1)} \\
& \int_0^T (T-qs)^{(\alpha-2)} Acd_q s + \sum_{i=1}^{n} |\lambda_i| \frac{\beta_i \xi_i^{-\beta_i(\eta_i+\mu_i)}}{\Gamma_q(\mu_i)} \int_0^{\xi_i} (\xi_i^{\beta_i} - s^{\beta_i} q)^{(\mu_i-1)} s^{\beta(\eta_i+1)-1} \\
& \frac{1}{\Gamma_q(\alpha)} \int_0^s (s-q\tau)^{(\alpha-1)} Acd_q \tau d_q s\Big) \\
= & \Big(\frac{T^{\alpha-\delta}}{\Gamma_q(\alpha-\delta+1)} + \frac{|a| T^{2\alpha-2-\delta}}{|M|\Gamma_q(\alpha-\delta)} + \frac{T^{\alpha-1-\delta}\kappa}{|M|[\alpha]_q \Gamma_q(\alpha-\delta)}\Big)Ac.
\end{aligned} \tag{3.8}
$$

Thus

$$
\begin{aligned}
\|Qx\| = & \max_{t\in[0,T]} |Qx(t)| + \max_{t\in[0,T]} |D_q^{\delta} Qx(t)| \\
\leq & \Big(\frac{T^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{T^{\alpha-1}}{|M|}\Big(\frac{T^{\alpha-1}|a|}{\Gamma_q(\alpha)} + \frac{\kappa}{\Gamma_q(\alpha+1)}\Big)\Big)Ac+ \\
& \Big(\frac{T^{\alpha-\delta}}{\Gamma_q(\alpha-\delta+1)} + \frac{|a| T^{2\alpha-2-\delta}}{|M|\Gamma_q(\alpha-\delta)} + \frac{T^{\alpha-1-\rho}\kappa}{|M|[\alpha]_q \Gamma_q(\alpha-\delta)}\Big)Ac \\
= & A\Big(\frac{T^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{|a| T^{2\alpha-2}}{|M|\Gamma_q(\alpha)} + \frac{T^{\alpha-1}\kappa}{|M|\Gamma_q(\alpha+1)} \\
& + \frac{T^{\alpha-\delta}}{\Gamma_q(\alpha-\delta+1)} + \frac{|a| T^{2\alpha-2-\delta}}{|M|\Gamma_q(\alpha-\delta)} + \frac{T^{\alpha-1-\delta}\kappa}{|M|[\alpha]_q \Gamma_q(\alpha-\delta)}\Big)c < c.
\end{aligned} \tag{3.9}
$$

From (3.9), we obtain $Q(E_c) \subset E_c$. By Schauder fixed point theorem, Q has at least one fixed point in E_c. Clearly, $x = 0$ is not a fixed point because $f(t, 0, 0) \neq 0$. Therefore, problem (1.1) has at least one nontrivial solution, which completes the proof.

4. Example

In this section, we illustrate the results obtained in the last section.

Example 1 Consider the following fractional-order boundary value problem involving nonlocal Erdélyi-Kober fractional q-integral conditions:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{3}{2}}x(t) + \frac{t^2}{1+e^t}\Big(\frac{|x(t)+D_{\frac{1}{2}}^{\frac{1}{4}}x(t)|}{1+2|x(t)+D_{\frac{1}{2}}^{\frac{1}{4}}x(t)|} + \cos t + 2\Big) + 1 = 0, \\ x(0) = 0, \\ 5x(1) = \frac{1}{100}I_{\frac{1}{2}}^{\frac{3}{8},\frac{5}{4},\frac{1}{10}}x(\frac{1}{4}) + \frac{3}{100}I_{\frac{1}{2}}^{\frac{1}{8},\frac{7}{4},\frac{1}{5}}x(\frac{1}{2}), \end{cases} \tag{4.1}$$

where $q = \frac{1}{2}, \delta = \frac{1}{4}, T = 1, \alpha = \frac{3}{2}, a = 3, \lambda_1 = \frac{1}{100}, \lambda_2 = \frac{3}{100}, \beta_1 = \frac{1}{10}, \beta_2 = \frac{1}{5}, \eta_1 = \frac{3}{8}, \eta_2 = \frac{1}{8}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}, \mu_1 = \frac{5}{4}, \mu_1 = \frac{7}{4}$, and $f(t,, x(t), D_q^{\delta}x(t)) = \frac{t^2}{1+e^t}\Big(\frac{|x(t)+D_{\frac{1}{2}}^{\frac{1}{2}}x(t)|}{1+2|x(t)+D_{\frac{1}{2}}^{\frac{1}{2}}x(t)|} + \cos t + 2\Big) + 1.$

By computation, we deduce that

$$|f(t,, x(t), D_q^{\delta}x(t)) - f(t,, y(t), D_q^{\rho}y(t))| \leq \frac{t^2}{2}\Big(|x(t) - y(t)| + |D_{\frac{1}{2}}^{\frac{1}{4}}x(t) - D_{\frac{1}{2}}^{\frac{1}{4}}y(t)|\Big),$$

then, the first condition is satisfied with $L(t) = \frac{t^2}{2}$.

$$\begin{aligned} M =& 5 - \frac{1}{100} \times \frac{1}{10}[10]_q \frac{\Gamma_{\frac{1}{2}}(\frac{3}{8} + 1 + \frac{30}{2})}{\Gamma_{\frac{1}{2}}(\frac{5}{4} + \frac{3}{8} + 1 + \frac{30}{2})} - \frac{3}{100} \times \frac{1}{5}[5]_q \frac{\Gamma_{\frac{1}{2}}(\frac{1}{8} + 1 + \frac{15}{2})}{\Gamma_{\frac{1}{2}}(\frac{7}{4} + \frac{1}{8} + 1 + \frac{15}{2})} \\ >& 5 - \frac{1}{100} \times \frac{1}{5}(1 - (\frac{1}{2})^{10})(\frac{1}{4})^{\frac{1}{2}} - \frac{3}{100} \times \frac{2}{5}(1 - (\frac{1}{2})^5)(\frac{1}{2})^{\frac{1}{2}} \\ >& 5 - \frac{1}{500} - \frac{6}{500} \doteq 4.986 \neq 0. \end{aligned}$$

$$L_1 = \sup_{t\in[0,1]} \frac{1}{\Gamma_q(\frac{3}{2})} \int_0^t (t - qs)^{(\frac{3}{2}-1)} \frac{s^2}{2} d_q s = \frac{1}{2} \frac{\Gamma_q(3)}{\Gamma_q(4.5)}.$$

$$L_2 = \sup_{t\in[0,1]} \frac{1}{\Gamma_q(\frac{3}{2} - 1)} \int_0^t (t - qs)^{(\frac{3}{2}-2)} \frac{s^2}{2} d_q s = \frac{1}{2} \frac{\Gamma_q(3)}{\Gamma_q(3.5)}.$$

$$L_3 = \sup_{t\in[0,1]} \frac{1}{\Gamma_q(\frac{3}{2} - \frac{1}{4})} \int_0^t (t - qs)^{(\frac{3}{2}-\frac{1}{4}-1)} \frac{s^2}{2} d_q s = \frac{1}{2} \frac{\Gamma_q(3)}{\Gamma_q(4.25)}.$$

$$\begin{aligned} & L_1 + L_3 + \frac{T^{\alpha-1}}{|M|}\Big(1 + \frac{T^{-\rho}\Gamma_q(\alpha)}{\Gamma_q(\alpha - \rho)}\Big)\Big(|a|L_2 + \sum_{i=0}^{n} |\lambda_i|\beta_i[\frac{1}{\beta_i}]_q \frac{L_1\Gamma_q(\eta_i + 1)}{\Gamma_q(\mu_i + \eta_i + 1)}\Big) \\ & \leq \frac{1}{2}\frac{\Gamma_q(3)}{\Gamma_q(4.5)} + \frac{1}{2}\frac{\Gamma_q(3)}{\Gamma_q(4.25)} + \frac{1}{4.986}\Big(\frac{5}{2}\frac{\Gamma_q(3)}{\Gamma_q(3.5)} + \frac{1}{100} \times \frac{1}{10} \times \frac{1}{2}[10]_q \frac{\Gamma_q(3)}{\Gamma_q(4.5)} + \\ & \frac{3}{100} \times \frac{1}{5} \times \frac{1}{2}\frac{\Gamma_q(3)}{\Gamma_q(4.5)} + \frac{5}{2}\frac{\Gamma_q(\frac{3}{2})\Gamma_q(3)}{\Gamma_q(\frac{5}{4})\Gamma_q(3.5)} + \frac{1}{100} \times \frac{1}{10} \times \frac{1}{2}[10]_q \frac{\Gamma_q(\frac{3}{2})\Gamma_q(3)}{\Gamma_q(\frac{5}{4})\Gamma_q(3.5)} + \\ & \frac{3}{100} \times \frac{1}{5} \times \frac{1}{2}[5]_q \frac{\Gamma_q(\frac{3}{2})\Gamma_q(3)}{\Gamma_q(\frac{5}{4})\Gamma_q(3.5)}\Big) \approx 0.894875 < 1. \end{aligned}$$

Hence, by Theorem 1, the boundary value problem (1.1) has an unique solution on [0, 1].

Example 2 Consider the following fractional-order boundary value problem involving nonlocal Erdélyi-Kober fractional q-integral conditions:

$$\begin{cases} D_q^{\frac{7}{5}}x(t) + \frac{1}{2\pi}\sin(\pi|x|)\frac{(|x|+|D_q^{\delta}x|)^{\frac{1}{2}}}{|x|+|D_q^{\delta}x|+1} + 1 = 0, \ t \in (0,1) \\ x(0) = 0, \\ 5x(1) = \frac{1}{7}I_q^{\frac{1}{3},\frac{5}{4},\frac{1}{10}}x(\frac{1}{4}) + \frac{3}{10}I_q^{\frac{1}{8},\frac{7}{4},\frac{1}{5}}x(\frac{1}{2}). \end{cases}$$

Here, $q = \frac{1}{2}, \delta = \frac{1}{4}$. $f(t, x(t), D_q^{\rho}x(t)) = \frac{1}{2\pi}\sin(\pi|x|)\frac{(|x|+|D_q^{\delta}x|)^{\frac{1}{2}}}{|x|+|D_q^{\delta}x|+1} + 1,$

$$\frac{|f(t, x(t), D_q^{\delta}x(t))|}{\| x \|} = \frac{\left|\frac{1}{2\pi}\sin(\pi|x|)\frac{(|x|+|D_q^{\delta}x|)^{\frac{1}{2}}}{|x|+|D_q^{\delta}x|+1} + 1\right|}{\| x \|} \leq \frac{\frac{1}{2}(|x| + |D_q^{\delta}x|)^{\frac{1}{2}} + 1}{\| x \|} \to 0, \text{ as } \| x \| \to \infty.$$

Therefore, the conclusion of Theorem 4 implies that problem (1.1) has at least one solution on [0, 1].

5. Conclusions

In this work, we utilize Banach contraction principle and Schauder's fixed point theorem to research the existence, uniqueness of solutions for a q-fractional differential equation with nonlocal Erdélyi-Kober q-fractional integral condition and in which the nonlinear term contains a fractional q-derivative of Rieman-Liouville type. Some existence and uniqueness results of solutions are obtained, we also provide an easily verifiable condition for the existence of nontrivial solution for the problem (1.1).

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Conflict of interest

The authors declare no conflict of interest.

References

1. D. O. Jackson, *On q-definite integrals*, Quart. J. Pure Appl. Math., **41** (1910), 193–203.

2. V. Kac, P. Cheung, *Quantum calculus*, Springer, New York, 2001.

3. W. A. Al-Salam, *Some fractional q-integral and q-derivatives*, Proc. Edinburgh Math. Soc., **15** (1966), 135–140.

4. R. P. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc. Camb. Philos. Soc., **66** (1969), 365–370.

5. P. Rajkovic, S. Marinkovic, M. Stankovic, *On q-analogues of Caputo derivative and Mittag-Leffler function*, Fract. Calc. Appl. Anal., **10** (2007), 359–373.

6. P. M. Rajković, S. D. Marinkovic, M. S. Stankovic, *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math., **1** (2007), 311–323.

7. M. H. Annaby, Z. S. I. Mansour, *q-fractional calculus and equations*, Springer, Berlin, 2012.

8. R. Ferreira, *Positive solutions for a class of boundary value problems with fractional q-difference*, Comput. Math. Appl. , **61** (2011), 367–373.

9. Y. Zhao, H. Chen, Q. Zhang, *Existence results for fractional q-difference equations with nonlocal q-integral boundary conditions*, Adv. Differ. Equ., **2013** (2013), 1–15.

10. B. Ahmad, J. J. Nieto, A. Alsaedi, et al. *Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions*, J. Franklin Inst., **351** (2014), 2890–2909.

11. M. Jiang, S. M. Zhong, *Existence of solutions for nonlinear fractional q-difference equations with Riemann-Liouville type q-derivatives*, J. Appl. Math. Comput., **47** (2015), 429–459.

12. B. Ahmad, S. K. Ntouyas, J. Tariboon, *Impulsive fractional q-difference equations with separated boundary conditions*, Appl. Math. Comput., **281** (2016), 199–213.

13. J. Ren, C. B. Zhai, *A fractional q-difference equation with integral boundary conditions and comparison theorem*, Int. J. Nonlinear Sci. Numer. Simul., **18** (2017), 575–583.

14. Y. Cui, S. Kang, H. Chen, *Uniqueness of solutions for an integral boundary value problem with fractional q-differences*, J. Appl. Anal. Comput., **8** (2018), 524–531.

15. S. Etemad, S. K. Ntouyas, B. Ahmad, *Existence theory for a fractional q-integro-difference equation with q-integral boundary conditions of different orders*, Mathematics, **7** (2019), 1–15.

16. T. Zhang, Y. Tang, *A difference method for solving the q-fractional differential equations*, Appl. Math. Lett., **98** (2019), 292–299.

17. K. Ma, X. H. Li, S. R. Sun, *Boundary value problems of fractional q-difference equations on the half-line*, Bound. Value Probl., **2019** (2019), 1–16.

18. G. T. Wang, Z. B. Bai, L. Zhang, *Successive iterations for unique positive solution of a nonlinear fractional q-integral boundary value problem*, J. Appl. Anal. Comput., **9** (2019), 1204–1215.

19. J. Ren, C. Zhai, *Nonlocal q-fractional boundary value problem with Stieltjes integral conditions*, Nonlinear Anal. Model. Control, **24**, (2019), 582–602.

20. X. H. Li, Z. L. Han, X. Li, *Boundary value problems of fractional q-difference Schröinger equations*, Appl. Math. Lett., **46** (2015), 100–105.

21. T. Zhang, Q. Guo, *The solution theory of the nonlinear q-fractional differential equations*, Appl. Math. Lett., **104** (2020), 106282.

22. P. Lyu, S.Vong, *An efficient numerical method for q-fractional differential equations*, Appl. Math. Lett., **103** (2020), 106156.

23. N. Thongsalee, S. Ntouyas, J. Tariboon, *Nonlinear Riemann-Liouville fractional differential equations with nonlocal Eedélyi-Kober fractional integral conditions*, Fract. Calc. Appl. Anal., **19** (2016), 480–497.

24. L. Gaulue, *Some results involving generalized Eedélyi-Kober fractional q-integral operators*, Revista Tecno-Cientfica URU., **6** (2014), 77–89.