



Research article

2p-th mean dynamic behaviors for semi-discrete stochastic competitive neural networks with time delays

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Abstract: In this article, a novel semi-discrete model for stochastic competitive neural networks (SCNNs) is proposed. At first, taking advantage of some famous inequalities and fixed point theory, a few conditions are obtained for the existence of $2p$ -th mean almost periodic sequence (MAPS) of the semi-discrete stochastic model. In the next place, $2p$ -th moment global exponential stability (MGES) of the above model is discussed as well. The research findings exhibit the stochastic and delayed effects on the mean dynamics of the semi-discrete stochastic networks. In the end, some numerical illustrations are presented to visually expound the feasibility of the works in this paper. The methods in this article could be applied to investigate other models in the areas of science and engineering.

Keywords: competitive neural networks; semi discreteness; stochastic process; fixed point theory

Mathematics Subject Classification: 39A24, 39A30, 39A50, 92B20

1. Introduction

Competitive neural networks (CNNs) simulate the cortical cognitive maps with unsupervised synaptic modifications. In the structure of CNNs, it has two kinds of states: one depicts the fast neural activity and the other one depicts the slow unsupervised synaptic modifications. Recently, CNNs had been extensively used in artificial intelligence, robotics, adaptive control and combinatorial optimization, etc. In the dynamical behaviours of CNNs, (almost) periodicity and exponential stability are extremely important. Consequently, many literatures [1–8] concerned synchronization, the existence and global exponential stability of equilibrium point, (almost) periodic solutions of

CNNs. Such as, Wang and Huang [6] studied the CNNs below:

$$\begin{cases} \dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n c_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) + B_i(t) \sum_{j=1}^p m_{ij}(t)y_j + I_i(t), \\ \dot{m}_{ij}(t) = -\tilde{a}_i(t)m_{ij}(t) + \tilde{b}_i(t)y_j\tilde{f}_i(x_i(t)), \end{cases} \quad (1.1)$$

where x_i is the i th neuron, m_{ij} is the synaptic efficiency, f_j and \tilde{f}_i are the activation functions, $i, j = 1, 2, \dots, n$. In [6], the authors studied the equilibrium point and exponential asymptotic stability for system (1.1).

In practical applications, stochastic disturbances frequently occur in NNs and the stochastic factors always happen in the uncertainties of the parameter process. Therefore, this paper concerns the SCNNs below:

$$\begin{cases} dx_i(t) = \left[-a_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n c_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) + B_i(t) \sum_{j=1}^p m_{ij}(t)y_j + I_i(t) \right] dt \\ \quad + \sum_{j=1}^n d_{ij}(t)\sigma_j(x_j(t - \eta_{ij}(t)))dw_j(t), \\ dm_{ij}(t) = \left[-\tilde{a}_i(t)m_{ij}(t) + \tilde{b}_i(t)y_j\tilde{f}_i(x_i(t)) \right] dt + \tilde{d}_i(t)m_{ij}(t)d\tilde{w}_i(t), \end{cases} \quad (1.2)$$

where w_j and \tilde{w}_i are the standard Brownian motions, $\tau_{ij}(t)$ and $\eta_{ij}(t)$ are bounded continuous functions with $0 \leq \tau_{ij}(t) \leq \bar{\tau}_{ij}$, $0 \leq \eta_{ij}(t) \leq \bar{\eta}_{ij}$, $\forall t \in \mathbb{R}$, $\bar{\tau}_{ij}$ and $\bar{\eta}_{ij}$ are positive constants, $i, j = 1, 2, \dots, n$.

The discreteness of NNs is a very effective tool to investigate the dynamics of the networks in a digital way. Therefore, lots of research findings were gained for the dynamics of discrete-time determinant or stochastic NNs by Euler scheme [9–13]. In recent years, the semi-discretization [14] of determinant differential equations received wide attentions and the authors [14–18] displayed that semi-discrete models provide more precise approximation of continuous-time models than those by Euler method. For example, the following semi-discrete NNs had been discussed in [16]:

$$x_i(k+1) = e^{-a_i(k)}x_i(k) + \frac{1 - e^{-a_i(k)}}{a_i(k)} \left[\sum_{j=1}^n b_{ij}(k)f_j(x_j(k)) + I_i(k) \right], \quad k \in \mathbb{Z}, i = 1, 2, \dots, n. \quad (1.3)$$

Assuming that all coefficients are almost periodic, the stable almost periodic sequence solution of system (1.3) was studied. Next, Huang et al. [18] studied the semi-discrete analogue of a general neural networks as follows:

$$x_i(k+1) = e^{-a_i(k)}x_i(k) + \frac{1 - e^{-a_i(k)}}{a_i(k)} \left[\sum_{l=1}^m \sum_{j=1}^n b_{ijl}(k)f_j(x_j(k - \tau_{ijl})) + I_i(k) \right], \quad k \in \mathbb{Z}, i = 1, 2, \dots, n.$$

In [18], some sufficient conditions were obtained for the existence of 2^n locally stable almost periodic solutions of the above NNs.

In literatures [14–18], the disquisitive models are deterministic. To the best of authors' knowledge, few results study the semi-discrete stochastic NNs. The major contributions of this paper are as follows:

- Semi-discrete analogue of SCNNs (1.2) is achieved.
- A Volterra integral expression in difference form is got for semi-discrete model of SCNNs (1.2).
- Some new results are gained for $2p$ -th MAPS and MGES to semi-discrete stochastic model.

- The issues discussed in this article can motivate the researches of other discrete stochastic models.

Motivated by the above reasons, the aim of this literature is to achieve the semi-discrete SCNNs and the mean dynamic behaviours (MAPS and MGES) of the model should be discussed. The rest of this article is organized as follows. In Section 2, the semi-discrete model of system (1.2) is obtained and some preparatory works are given. In Section 3, the $2p$ -th MAPS of the semi-discrete model is studied by fixed point theory. In Section 4, $2p$ -th MGES of the model is also discussed. An example and simulations are provided to expound the chief results of this paper in Section 5. Section 6 displays the conclusions and future perspectives of this article.

Denote by \mathbb{R}^n the n -dimensional real vector space and (Ω, \mathcal{F}, P) is a complete probability space. Let $BC(\mathbb{Z}, L^p(\Omega, \mathbb{R}^n))$ be the space of all bounded continuous functions from \mathbb{Z} to $L^p(\Omega, \mathbb{R}^n)$, where $L^p(\Omega, \mathbb{R}^n)$ is the space of all p -th integrable \mathbb{R}^n -valued random variables. So $BC(\mathbb{Z}, L^p(\Omega, \mathbb{R}^n))$ is a Banach space with $\|X\|_p = \sup_{k \in \mathbb{Z}} \|X\|_p$, $\|X\|_p = \max_{1 \leq i \leq n} (E|x_i(k)|^p)^{\frac{1}{p}}$, $\forall X = \{x_i\} := \{x_1, x_2, \dots, x_n\} \in BC(\mathbb{Z}, L^p(\Omega, \mathbb{R}^n))$, where $p > 1$ and $E(\cdot)$ stands for the expectation operator. $[a, b]_{\mathbb{Z}} = [a, b] \cap \mathbb{Z}$, $\forall a, b \in \mathbb{R}$.

2. Model description and preparatory work

2.1. Model description

Letting $S_i(t) = \sum_{j=1}^p m_{ij}(t)y_j = \mathbf{y}^T \mathbf{m}_i(t)$, where $\mathbf{y} = (y_1, y_2, \dots, y_p)^T$, $\mathbf{m}_i = (m_{i1}, m_{i2}, \dots, m_{ip})^T$, then system (1.2) is changed to

$$\begin{cases} dx_i(t) = \left[-a_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n c_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) + B_i(t)S_i(t) + I_i(t) \right] dt \\ \quad + \sum_{j=1}^n d_{ij}(t)\sigma_j(x_j(t - \eta_{ij}(t)))dw_j(t), \\ dS_i(t) = \left[-\tilde{a}_i(t)S_i(t) + \tilde{b}_i(t)\tilde{f}_i(x_i(t)) \right] dt + \tilde{d}_i(t)S_i(t)d\tilde{w}_i(t), \quad i = 1, 2, \dots, n, k \in \mathbb{Z}. \end{cases} \quad (2.1)$$

Similar to the argument as that in [14], the semi-discrete analogue of system (2.1) is obtained as follows:

$$\begin{cases} x_i(k+1) = e^{-a_i(k)}x_i(k) + \frac{1-e^{-a_i(k)}}{a_i(k)} \left[\sum_{j=1}^n b_{ij}(k)f_j(x_j(k)) + \sum_{j=1}^n c_{ij}(k)f_j(x_j(k - \tau_{ij}(k))) \right. \\ \quad \left. + B_i(k)S_i(k) + I_i(k) + \sum_{j=1}^n d_{ij}(k)\sigma_j(x_j(k - \eta_{ij}(k)))\Delta w_j(k) \right], \\ S_i(k+1) = e^{-\tilde{a}_i(k)}S_i(k) + \frac{1-e^{-\tilde{a}_i(k)}}{\tilde{a}_i(k)} \left[\tilde{b}_i(k)\tilde{f}_i(x_i(k)) + \tilde{d}_i(k)S_i(k)\Delta \tilde{w}_i(k) \right], \quad i = 1, 2, \dots, n, k \in \mathbb{Z}. \end{cases} \quad (2.2)$$

2.2. Preparatory work

Definition 2.1. ([19]) For $X \in BC(\mathbb{Z}; L^p(\Omega; \mathbb{R}^n))$ and each $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains an integer ω for which

$$\sup_{k \in \mathbb{Z}} E|X(k + \omega) - X(k)|_p < \epsilon.$$

Then stochastic process X is said to be p -th MAPS.

Lemma 2.1. ([20]) If $X = \{x_i\}$ is a solution of system (2.2), then

$$\left\{ \begin{array}{l} x_i(k) = \prod_{s=k_0}^{k-1} e^{-a_i(s)} x_i(k_0) \\ \quad + \sum_{v=k_0}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} \left[\sum_{j=1}^n b_{ij}(v) f_j(x_j(v)) + \sum_{j=1}^n c_{ij}(v) f_j(x_j(v - \tau_{ij}(v))) \right. \\ \quad \left. + B_i(v) S_i(v) + I_i(v) + \sum_{j=1}^n d_{ij}(v) \sigma_j(x_j(v - \eta_{ij}(v))) \Delta w_j(v) \right], \\ S_i(k) = \prod_{s=k_0}^{k-1} e^{-\tilde{a}_i(s)} S_i(k_0) \\ \quad + \sum_{v=k_0}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-\tilde{a}_i(s)} [1 - e^{-\tilde{a}_i(v)}]}{\tilde{a}_i(v)} \left[\tilde{b}_i(v) \tilde{f}_i(x_i(v)) + \tilde{d}_i(v) S_i(v) \Delta \tilde{w}_i(v) \right], \end{array} \right. \quad (2.3)$$

where $k_0 \in \mathbb{Z}$, $k \in (k_0, +\infty)_{\mathbb{Z}}$, $i = 1, 2, \dots, n$.

Lemma 2.2. ([21]) If $g \in L^2([a, b], \mathbb{R})$, then

$$E \left[\sup_{t \in [a, b]} \left| \int_a^t g(s) d\omega(s) \right|^p \right] \leq C_p E \left[\int_a^b |g(t)|^2 dt \right]^{\frac{p}{2}}, \quad C_p = \begin{cases} (32/p)^{p/2}, & 0 < p < 2, \\ 4, & p = 2, \\ \left[\frac{p^{p+1}}{2(p-1)^{(p-1)}} \right]^{\frac{p}{2}}, & p > 2. \end{cases}$$

Lemma 2.3. ([20]) If $\{x(k) : k \in \mathbb{Z}\}$ is real-valued stochastic process and $w(k)$ is the standard Brownian motion, then

$$E |x(k) \Delta w(k)|^p \leq C_p E |x(k)|^p, \quad k \in \mathbb{Z},$$

where C_p is defined as that in Lemma 2.2, $p > 0$.

Let $f^u = \sup_{k \in \mathbb{Z}} |f(k)|$ and $f^l = \inf_{k \in \mathbb{Z}} |f(k)|$ for bounded function f defined on \mathbb{Z} . In this paper, assume that the assumptions below hold:

(H₁) $a_i^l > 0$ and $\tilde{a}_i^l > 0$, $i = 1, 2, \dots, n$.

(H₂) There are some positive constants L_j^f , L_j^σ and $L_i^{\tilde{f}}$ such that

$$|f_j(u) - f_j(v)| \leq L_j^f |u - v|, \quad |\sigma_j(u) - \sigma_j(v)| \leq L_j^\sigma |u - v|, \quad |\tilde{f}_i(u) - \tilde{f}_i(v)| \leq L_i^{\tilde{f}} |u - v|,$$

for all $u, v \in \mathbb{R}$, $i, j = 1, 2, \dots, n$.

3. 2p-th mean almost periodic sequence

Define

$$a^u := \max_{1 \leq i \leq n} a_i^u, \quad a^l := \min_{1 \leq i \leq n} a_i^l, \quad \tilde{a}^u := \max_{1 \leq i \leq n} \tilde{a}_i^u, \quad \tilde{a}^l := \max_{1 \leq i \leq n} \tilde{a}_i^l,$$

$$D^* := \max_{1 \leq i \leq n} \left[B_i^u + \sum_{j=1}^n (b_{ij}^u + c_{ij}^u) L_j^f \right], \quad \tilde{D}^* := \max_{1 \leq i \leq n} \tilde{b}_i^u L_i^{\tilde{f}}, \quad K^* := \max_{1 \leq i \leq n} \sum_{j=1}^n d_{ij}^u L_j^\sigma, \quad \tilde{K}^* := \max_{1 \leq i \leq n} \tilde{d}_i^u,$$

$$r_{2p} := \max \left\{ \frac{(1 - e^{-a^u})}{a^l(1 - e^{-a^l})} \left(D^* + K^* C_{2p}^{\frac{1}{2p}} \right), \frac{(1 - e^{-\tilde{a}^u})}{\tilde{a}^l(1 - e^{-\tilde{a}^l})} \left(\tilde{D}^* + \tilde{K}^* C_{2p}^{\frac{1}{2p}} \right) \right\}, \quad \beta_{2p} := \frac{\alpha_{2p}}{1 - r_{2p}},$$

where

$$\alpha_{2p} := \max\{e_{2p}, \tilde{e}_{2p}\}, \quad e_{2p} := \frac{(1 - e^{-a^u})}{a^l(1 - e^{-a^l})} \max_{1 \leq i \leq n} \left[\sum_{j=1}^n (b_{ij}^u + c_{ij}^u) |f_j(0)| + I_i^u + \sum_{j=1}^n d_{ij}^u \sigma_j(0) C_{2p}^{\frac{1}{2p}} \right],$$

$$\tilde{e}_{2p} := \frac{(1 - e^{-\tilde{a}^u})}{\tilde{a}^l(1 - e^{-\tilde{a}^l})} \max_{1 \leq i \leq n} \tilde{b}_i^u |\tilde{f}_i(0)|.$$

Lemma 3.1. ([22]) *If \mathbb{X} is a Banach space, $\Lambda \subseteq \mathbb{X}$ is closed and convex. Suppose further that $\mathcal{B}, C: \Lambda \rightarrow \mathbb{X}$ such that*

- (1) $x, y \in \Lambda \Rightarrow \mathcal{B}x + Cy \in \Lambda$;
- (2) \mathcal{B} is continuous and $\mathcal{B}\Lambda$ is contained in a compact set;
- (3) C is contractive.

Then it has $z \in \Lambda$ with $z = \mathcal{B}z + Cz$.

Theorem 3.1. *Assume that (H_1) – (H_2) and the following conditions hold.*

(AP) *All of the coefficients in system (2.2) are almost periodic sequences.*

(H₃) $r_{2p} < 1$.

Then it has a $2p$ -th MAPS X of system (2.2) with $\|X\|_{2p} \leq \beta_{2p}$, $p \geq 1$.

Proof. Let $\Lambda \subseteq BC(\mathbb{Z}; L^{2p}(\Omega; \mathbb{R}^{m+n}))$ denote the set of all $2p$ -th MAPS X satisfying $\|X\|_{2p} \leq \beta_{2p}$.

On the basis of (2.3), define $\Phi X(k) = \mathcal{B}X(k) + CX(k)$, where

$$\Phi X(k) = \{(\Phi X)_i(k); (\Phi X)_{n+i}(k)\} = ((\Phi X)_1(k), \dots, (\Phi X)_n(k), (\Phi X)_{n+1}(k), \dots, (\Phi X)_{2n}(k))^T,$$

$$(\Phi X)_i(k) = (\mathcal{B}X)_i(k) + (CX)_i(k), \quad (\Phi X)_{n+i}(k) = (\mathcal{B}X)_{n+i}(k) + (CX)_{n+i}(k), \quad (3.1)$$

$$\left\{ \begin{array}{l} (\mathcal{B}X)_i(k) = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)[1-e^{-a_i(v)}]}}{a_i(v)} \left[\sum_{j=1}^n b_{ij}(v) f_j(x_j(v)) + \sum_{j=1}^n c_{ij}(v) f_j(x_j(v - \tau_{ij}(v))) \right. \\ \quad \left. + B_i(v) S_i(v) + I_i(v) \right], \\ (\mathcal{B}X)_{n+i}(k) = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-\tilde{a}_i(s)[1-e^{-\tilde{a}_i(v)}]}}{\tilde{a}_i(v)} \tilde{b}_i(v) \tilde{f}_i(x_i(v)), \end{array} \right. \quad (3.2)$$

$$\left\{ \begin{array}{l} (CX)_i(k) = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)[1-e^{-a_i(v)}]}}{a_i(v)} \sum_{j=1}^n d_{ij}(v) \sigma_j(x_j(v - \eta_{ij}(v))) \Delta w_j(v), \\ (CX)_{n+i}(k) = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-\tilde{a}_i(s)[1-e^{-\tilde{a}_i(v)}]}}{\tilde{a}_i(v)} \tilde{d}_i(v) S_i(v) \Delta \tilde{w}_i(v), \quad i = 1, 2, \dots, n, k \in \mathbb{Z}. \end{array} \right. \quad (3.3)$$

Let $X^0 = \{x_i^0; S_i^0\}$ be given by

$$\begin{cases} x_i^0(k) = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \left[\sum_{j=1}^n b_{ij}(v)f_j(0) \right. \\ \quad \left. + \sum_{j=1}^n c_{ij}(v)f_j(0) + \sum_{j=1}^n d_{ij}(v)\sigma_j(0)\Delta w_j(v) + I_i(v) \right], \\ S_i^0(k) = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-\tilde{a}_i(s)}[1-e^{-\tilde{a}_i(v)}]}{\tilde{a}_i(v)} \tilde{b}_i(v)\tilde{f}_i(0), \quad i = 1, 2, \dots, n, k \in \mathbb{Z}. \end{cases}$$

By the above equations, it has

$$\begin{aligned} \|\{x_i^0\}\|_{2p} &\leq \max_{1 \leq i \leq n} \sup_{k \in \mathbb{Z}} \left\{ \left[E \left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \sum_{j=1}^n (b_{ij}(v)f_j(0) + c_{ij}(v)f_j(0)) \right|^{2p} \right]^{\frac{1}{2p}} \right. \\ &\quad + \left[E \left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \sum_{j=1}^n d_{ij}(v)\sigma_j(0)\Delta w_j(v) \right|^{2p} \right]^{\frac{1}{2p}} \\ &\quad \left. + \left[E \left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} I_i(v) \right|^{2p} \right]^{\frac{1}{2p}} \right\}. \end{aligned}$$

Then

$$\begin{aligned} \|\{x_i^0\}\|_{2p} &\leq \max_{1 \leq i \leq n} \sup_{k \in \mathbb{Z}} \left\{ \frac{(1-e^{-a^u})}{a^l(1-e^{-a^l})} \left[\sum_{j=1}^n (b_{ij}^u|f_j(0)| + c_{ij}^u|f_j(0)|) + I_i^u \right] \right. \\ &\quad + \sum_{j=1}^n d_{ij}^u \sigma_j(0) \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} \right]^{1-\frac{1}{2p}} \\ &\quad \left. \times \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}[1-e^{-a_i(v)}]}{a_i(v)} E|\Delta w_j(v)|^{2p} \right]^{\frac{1}{2p}} \right\} \\ &\leq \frac{(1-e^{-a^u})}{a^l(1-e^{-a^l})} \max_{1 \leq i \leq n} \left[\sum_{j=1}^n (b_{ij}^u + c_{ij}^u)|f_j(0)| + I_i^u + \sum_{j=1}^n d_{ij}^u \sigma_j(0) C_{2p}^{\frac{1}{2p}} \right] := e_{2p}. \end{aligned}$$

Homoplastically,

$$\|\{S_i^0\}\|_{2p} \leq \frac{(1-e^{-\tilde{a}^u})}{\tilde{a}^l(1-e^{-\tilde{a}^l})} \max_{1 \leq i \leq n} \tilde{b}_i^u |\tilde{f}_i(0)| := \tilde{e}_{2p}.$$

Thus

$$\|X^0\|_{2p} \leq \max\{e_{2p}, \tilde{e}_{2p}\} := \alpha_{2p}.$$

From (3.1–3.3), one has

$$\begin{aligned}
& \| \{ (\Phi X)_i(k) - x_i^0(k) \}_{i=1}^n \|_{2p} \\
& \leq \max_{1 \leq i \leq n} \sup_{k \in \mathbb{Z}} \sum_{j=1}^n b_{ij}^u L_j^f \left\{ E \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} |x_j(v)| \right]^{2p} \right\}^{\frac{1}{2p}} \\
& \quad + \max_{1 \leq i \leq n} \sup_{k \in \mathbb{Z}} B_i^u \left\{ E \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} |S_i(v)| \right]^{2p} \right\}^{\frac{1}{2p}} \\
& \quad + \max_{1 \leq i, j \leq n} \sup_{k \in \mathbb{Z}} D_i^{**} \left\{ E \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} |x_j(v - \tau_{ij}(v))| \right]^{2p} \right\}^{\frac{1}{2p}} \\
& \quad + \max_{1 \leq i, j \leq n} \sup_{k \in \mathbb{Z}} K^* \left\{ E \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} |x_j(v - \eta_{ij}(v)) \Delta w_j(v)| \right]^{2p} \right\}^{\frac{1}{2p}},
\end{aligned}$$

which implies

$$\begin{aligned}
\| \{ (\Phi X)_i(k) - x_i^0(k) \}_{i=1}^n \|_{2p} & \leq \max_{1 \leq i \leq n} \sup_{k \in \mathbb{Z}} \sum_{j=1}^n b_{ij}^u L_j^f \left\{ \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} \right]^{p-1} \right. \\
& \quad \times \left. \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} E |x_j(v)|^{2p} \right\}^{\frac{1}{2p}} \\
& \quad + \max_{1 \leq i \leq n} \sup_{k \in \mathbb{Z}} B_i^u \left\{ \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} \right]^{p-1} \right. \\
& \quad \times \left. \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} E |S_i(v)|^{2p} \right\}^{\frac{1}{2p}} \\
& \quad + \max_{1 \leq i, j \leq n} \sup_{k \in \mathbb{Z}} D_i^{**} \left\{ \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} \right]^{p-1} \right. \\
& \quad \times \left. \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} E |x_j(v - \tau_{ij}(v))|^{2p} \right\}^{\frac{1}{2p}} \\
& \quad + \max_{1 \leq i, j \leq n} \sup_{k \in \mathbb{Z}} K^* \left\{ \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} \right]^{p-1} \right. \\
& \quad \times \left. \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)} [1 - e^{-a_i(v)}]}{a_i(v)} E |x_j(v - \eta_{ij}(v)) \Delta w_j(v)|^{2p} \right\}^{\frac{1}{2p}},
\end{aligned}$$

where $D_i^{**} = D^* - B_i^u - \sum_{j=1}^n b_{ij}^u L_j^f$, $i = 1, 2, \dots, n$. It gains from Lemma 2.3 that

$$\| \{ (\Phi X)_i(k) - x_i^0(k) \}_{i=1}^n \|_{2p} \leq \frac{(1 - e^{-a^l})}{a^l(1 - e^{-a^l})} \left\{ D^* + K^* C_{2p}^{\frac{1}{2p}} \right\} \|X\|_{2p} \leq r_{2p} \|X\|_{2p} \leq \frac{r_{2p} \alpha_{2p}}{1 - r_{2p}}.$$

Homoplastically,

$$\| \{ (\Phi X)_{n+i}(k) - S_i^0(k) \}_{i=1}^n \|_{2p} \leq \frac{(1 - e^{-\tilde{a}^l})}{\tilde{a}^l(1 - e^{-\tilde{a}^l})} \left\{ \tilde{D}^* + \tilde{K}^* C_{2p}^{\frac{1}{2p}} \right\} \|X\|_{2p} \leq r_{2p} \|X\|_{2p} \leq \frac{r_{2p} \alpha_{2p}}{1 - r_{2p}}.$$

Consequently,

$$\|\Phi X - X^0\|_{2p} \leq \frac{r_{2p}\alpha_{2p}}{1 - r_{2p}}.$$

So

$$\|\Phi X\|_{2p} \leq \|X^0\|_{2p} + \|\Phi X - X^0\|_{2p} \leq \alpha_{2p} + \frac{r_{2p}\alpha_{2p}}{1 - r_{2p}} = \frac{\alpha_{2p}}{1 - r_{2p}} := \beta_{2p}. \quad (3.4)$$

By (3.4) and similar to the argument as that in [20], (1)–(2) of Lemma 3.1 hold.

At last, $\forall X = \{x_i; S_i\}, X^* = \{x_i^*; S_i^*\} \in \Lambda$, one has

$$\begin{aligned} \|\{(CX)_i(k) - (CX^*)_i(k)\}_{i=1}^n\|_{2p} &\leq \frac{[1 - e^{-a^l}]}{a^l} \max_{1 \leq i \leq n} \sup_{k \in \mathbb{Z}} \left\{ E \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a^l} \right. \right. \\ &\quad \left. \left. \times \sum_{j=1}^n d_{ij}(v) (\sigma_j(x_j(v - \eta_{ij}(v))) - \sigma_j(y_j(v - \eta_{ij}(v)))) \Delta w_j(v) \right]^{2p} \right\}^{\frac{1}{2p}} \\ &\leq \frac{[1 - e^{-a^l}]}{a^l} \max_{1 \leq i, j \leq n} \sup_{k \in \mathbb{Z}} K^* \left\{ \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a^l} \right]^{2p-1} \right. \\ &\quad \left. \times \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a^l} E \left[|x_j(v - \eta_{ij}(v)) - y_j(v - \eta_{ij}(v))| \Delta w_j(v) \right]^{2p} \right\}^{\frac{1}{2p}} \\ &\leq \frac{K^* C_{2p}^{\frac{1}{2p}} (1 - e^{-a^l})}{a^l (1 - e^{-a^l})} \|X - Y\|_{2p} \leq r_{2p} \|X - Y\|_{2p}, \end{aligned}$$

homoplastically,

$$\|\{(CX)_{n+i}(k) - (CX^*)_{n+i}(k)\}_{i=1}^n\|_{2p} \leq \frac{\tilde{K}^* C_{2p}^{\frac{1}{2p}} (1 - e^{-\tilde{a}^l})}{\tilde{a}^l (1 - e^{-\tilde{a}^l})} \|X - Y\|_{2p} \leq r_{2p} \|X - Y\|_{2p}.$$

Then

$$\|CX - CY\|_{2p} \leq r_{2p} \|X - Y\|_{2p}.$$

By (H_3) , C is contractive. So (3) of Lemma 3.1 holds. By Lemma 3.1, system (2.2) has a $2p$ -th MAPS. The proof is finished. \square

Remark 3.1. In view of Theorem 3.1, (H_3) is hard to be verified. For the sake of enabling (H_3) to be met, the following items should be followed in applications:

- (1) The coefficients a_i and \tilde{a}_i in NNs (2.2), $i = 1, 2, \dots, n$, should be selected to be large enough.
- (2) The coefficients b_{ij} , c_{ij} , B_i , d_{ij} , \tilde{b}_i and \tilde{d}_i of NNs (2.2) should be selected to meet (H_3) with large enough numbers, $i = 1, 2, \dots, n$.
- (3) If the activation functions f_j , σ_j and \tilde{f}_i in NNs (2.2) are selected by some small enough constants L_j^f , L_j^σ and $L_i^{\tilde{f}}$, then (H_2) is easier to be valid, $i, j = 1, 2, \dots, n$.

4. $2p$ -th moment global exponential stability

Theorem 4.1. *If (H_1) – (H_3) hold, then system (2.2) has the property of $2p$ -th MGES.*

Proof. Assume that $X = \{x_i; S_i\}$ with initial condition $\varphi = \{\varphi_i; \psi_i\}$ and $X^* = \{x_i^*; S_i^*\}$ with initial condition $\varphi^* = \{\varphi_i^*; \psi_i^*\}$ are any two solutions of system (2.2). So

$$\begin{aligned}
 |x_i(k) - x_i^*(k)| &\leq \prod_{s=0}^{k-1} e^{-a_i(s)} |\varphi_i(0) - \varphi_i^*(0)| + \frac{(1 - e^{-a^u})}{a^l} \sum_{v=0}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_i(s)} \left\{ \sum_{j=1}^n \left[b_{ij}^u L_j^f |x_j(v) - x_j^*(v)| \right. \right. \\
 &\quad \left. \left. + c_{ij}^u L_j^f |x_j(v - \tau_{ij}(v)) - x_j^*(v - \tau_{ij}(v))| \right. \right. \\
 &\quad \left. \left. + d_{ij}^u L_j^\sigma |x_j(v - \eta_{ij}(v)) - x_j^*(v - \eta_{ij}(v))| \|\Delta w_j(v)\| \right] + B_i^u |S_i(v) - S_i^*(v)| \right\} \\
 &\leq e^{-a^l k} |\varphi_i(0) - \varphi_i^*(0)| + \frac{(1 - e^{-a^u})}{a^l} \sum_{v=0}^{k-1} e^{-a^l(k-v-1)} \sum_{j=1}^n \left\{ \sum_{j=1}^n \left[b_{ij}^u L_j^f |x_j(v) - x_j^*(v)| \right. \right. \\
 &\quad \left. \left. + c_{ij}^u L_j^f |x_j(v - \tau_{ij}(v)) - x_j^*(v - \tau_{ij}(v))| \right. \right. \\
 &\quad \left. \left. + d_{ij}^u L_j^\sigma |x_j(v - \eta_{ij}(v)) - x_j^*(v - \eta_{ij}(v))| \|\Delta w_j(v)\| \right] + B_i^u |S_i(v) - S_i^*(v)| \right\}, \quad (4.1)
 \end{aligned}$$

$$\begin{aligned}
 &|S_i(k) - S_i^*(k)| \\
 &\leq \prod_{s=0}^{k-1} e^{-\tilde{a}_i(s)} |\psi_i(0) - \psi_i^*(0)| \\
 &\quad + \frac{(1 - e^{-\tilde{a}^u})}{\tilde{a}^l} \sum_{v=0}^{k-1} \prod_{s=v+1}^{k-1} e^{-\tilde{a}_i(s)} \left\{ \tilde{b}_i^u L_i^{\tilde{f}} |x_i(v) - x_i^*(v)| \tilde{d}_i^u |S_i(v) - S_i^*(v)| \|\Delta \tilde{w}_i(v)\| \right\} \\
 &\leq e^{-\tilde{a}^l k} |\psi_i(0) - \psi_i^*(0)| \\
 &\quad + \frac{(1 - e^{-\tilde{a}^u})}{\tilde{a}^l} \sum_{v=0}^{k-1} e^{-\tilde{a}^l(k-v-1)} \left\{ \tilde{b}_i^u L_i^{\tilde{f}} |x_i(v) - x_i^*(v)| + \tilde{d}_i^u |S_i(v) - S_i^*(v)| \|\Delta \tilde{w}_i(v)\| \right\}, \quad (4.2)
 \end{aligned}$$

where $i = 1, 2, \dots, n$, $k \in [-\mu_0, +\infty)_{\mathbb{Z}}$, $\mu_0 = \max_{(i,j)} \{\tau_{ij}^u, \eta_{ij}^u\}$.

To facilitate, set

$$a_0 = \frac{1 - e^{-a^u}}{a^l}, \quad \tilde{a}_0 = \frac{1 - e^{-\tilde{a}^u}}{\tilde{a}^l},$$

$$\gamma_{2p} = \max_{1 \leq i \leq n} \sup_{s \in [-\mu_0, 0]_{\mathbb{Z}}} \{(E|\varphi_i(s) - \varphi_i^*(s)|^{2p})^{\frac{1}{2p}}, (E|\psi_i(s) - \psi_i^*(s)|^{2p})^{\frac{1}{2p}}\},$$

$Z = \{z_i\}$, $W = \{w_i\}$, $z_i(k) = x_i(k) - x_i^*(k)$, $w_i(k) = S_i(k) - S_i^*(k)$, $i = 1, 2, \dots, n$, $k \in \mathbb{Z}$.

By (4.1), it has

$$\begin{aligned}
|Z(k)|_{2p} &\leq e^{-d'k} \gamma_{2p} + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 b_{ij}^u L_j^f \left\{ \left[\sum_{s=0}^{k-1} e^{-d'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-d'(k-s-1)} E |x_j(s) - x_j^*(s)|^{2p} \right\}^{\frac{1}{2p}} \\
&\quad + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 c_{ij}^u L_j^f \left\{ \left[\sum_{s=0}^{k-1} e^{-d'(k-s-1)} \right]^{2p-1} \right. \\
&\quad \times \left. \sum_{s=0}^{k-1} e^{-d'(k-s-1)} E |x_j(s - \tau_{ij}(s)) - x_j^*(s - \tau_{ij}(s))|^{2p} \right\}^{\frac{1}{2p}} \\
&\quad + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 d_{ij}^u L_j^\sigma \left\{ \left[\sum_{s=0}^{k-1} e^{-d'(k-s-1)} \right]^{2p-1} \right. \\
&\quad \times \left. \sum_{s=0}^{k-1} e^{-d'(k-s-1)} E |x_j(s - \eta_{ij}(s)) - x_j^*(s - \eta_{ij}(s))| \Delta w_j(s) \right\}^{2p} \\
&\quad + \max_{1 \leq i \leq n} B_i^u \left\{ \left[\sum_{s=0}^{k-1} e^{-d'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-d'(k-s-1)} E |S_i(s) - S_i^*(s)|^{2p} \right\}^{\frac{1}{2p}} \\
&\leq e^{-d'k} \gamma_{2p} + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 b_{ij}^u L_j^f \left\{ \left[\sum_{s=0}^{k-1} e^{-d'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-d'(k-s-1)} |Z(s)|_{2p}^{2p} \right\}^{\frac{1}{2p}} \\
&\quad + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 c_{ij}^u L_j^f \left\{ \left[\sum_{s=0}^{k-1} e^{-d'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-d'(k-s-1)} |Z(s - \tau_{ij}(s))|_{2p}^{2p} \right\}^{\frac{1}{2p}} \\
&\quad + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 C_{2p}^{\frac{1}{2p}} d_{ij}^u L_j^\sigma \left\{ \left[\sum_{s=0}^{k-1} e^{-d'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-d'(k-s-1)} |Z(s - \eta_{ij}(s))|_{2p}^{2p} \right\}^{\frac{1}{2p}} \\
&\quad + \max_{1 \leq i \leq n} B_i^u \left\{ \left[\sum_{s=0}^{k-1} e^{-d'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-d'(k-s-1)} |W(s)|_{2p}^{2p} \right\}^{\frac{1}{2p}}. \tag{4.3}
\end{aligned}$$

From (4.2), it induces

$$\begin{aligned}
|W(k)|_{2p} &\leq e^{-\tilde{d}'k} \gamma_{2p} + \max_{1 \leq i \leq n} \tilde{a}_0 \tilde{b}_i^u L_i^{\tilde{f}} \left\{ \left[\sum_{s=0}^{k-1} e^{-\tilde{d}'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-\tilde{d}'(k-s-1)} |Z(s)|_{2p}^{2p} \right\}^{\frac{1}{2p}} \\
&\quad + \max_{1 \leq i \leq n} \tilde{a}_0 C_{2p}^{\frac{1}{2p}} \tilde{d}_i^u \left\{ \left[\sum_{s=0}^{k-1} e^{-\tilde{d}'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-\tilde{d}'(k-s-1)} |W(s)|_{2p}^{2p} \right\}^{\frac{1}{2p}}. \tag{4.4}
\end{aligned}$$

By (H_3) , it has $\lambda > 0$ small enough ensuring that

$$\max_{1 \leq i \leq n} \frac{e^\lambda a_0}{1 - e^{-(d'-2p)\lambda}} \left[B_i^u + \sum_{j=1}^n (b_{ij}^u L_j^f + e^{\mu_0 \lambda} c_{ij}^u L_j^f + e^{\mu_0 \lambda} C_{2p}^{\frac{1}{2p}} d_{ij}^u L_j^\sigma) \right] \stackrel{\text{def}}{=} \rho \leq 1,$$

$$\max_{1 \leq i \leq n} \frac{e^\lambda \tilde{a}_0}{1 - e^{-(\tilde{d}'-2p)\lambda}} \left[\tilde{b}_i^u L_i^{\tilde{f}} + e^{\mu_0 \lambda} C_{2p}^{\frac{1}{2p}} \tilde{d}_i^u \right] \stackrel{\text{def}}{=} \tilde{\rho} \leq 1.$$

Define $V(k) = \max\{|Z(k)|_{2p}, |W(k)|_{2p}\}$, $k \in \mathbb{Z}$.

Next, assume that it has $M_0 > 1$ ensuring that

$$V(k) \leq M_0 \gamma_{2p} e^{-\lambda k}, \quad \forall k \in [-\mu_0, +\infty)_{\mathbb{Z}}. \quad (4.5)$$

If (4.5) fails, then it has $k_0 \in (0, +\infty)_{\mathbb{Z}}$ such that one of the following two cases is valid:

$$(C_1) \quad |Z(k_0)|_{2p} > M_0 \gamma_{2p} e^{-\lambda k_0} \text{ and } V(k) \leq M_0 \gamma_{2p} e^{-\lambda k}, \quad \forall k \in [-\mu_0, k_0)_{\mathbb{Z}}.$$

$$(C_2) \quad |W(k_0)|_{2p} > M_0 \gamma_{2p} e^{-\lambda k_0} \text{ and } V(k) \leq M_0 \gamma_{2p} e^{-\lambda k}, \quad \forall k \in [-\mu_0, k_0)_{\mathbb{Z}}.$$

If (C_1) holds, then

$$\begin{aligned} |Z(k_0)|_{2p} &\leq e^{-a'k_0} \gamma_{2p} + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 b_{ij}^u L_j^f M_0 \gamma_{2p} \left\{ \left[\sum_{s=0}^{k-1} e^{-a'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-a'(k-s-1)} e^{-2p\lambda s} \right\}^{\frac{1}{2p}} \\ &\quad + \max_{1 \leq i \leq n} a_0 B_i^u M_0 \gamma_{2p} \left\{ \left[\sum_{s=0}^{k-1} e^{-a'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-a'(k-s-1)} e^{-2p\lambda s} \right\}^{\frac{1}{2p}} \\ &\quad + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 M_0 \gamma_{2p} \left[c_{ij}^u L_j^f + C_{2p}^{\frac{1}{2p}} d_{ij}^u L_j^\sigma \right] \\ &\quad \times \left\{ \left[\sum_{s=0}^{k-1} e^{-a'(k-s-1)} \right]^{2p-1} \sum_{s=0}^{k-1} e^{-a'(k-s-1)} e^{-2p\lambda(s-\mu_0)} \right\}^{\frac{1}{2p}} \\ &\leq e^{-a'k_0} \gamma_{2p} + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 b_{ij}^u L_j^f M_0 \gamma_{2p} e^{-\lambda k_0} e^{\lambda} \left[\frac{1 - e^{-a'k_0}}{1 - e^{-a'}} \right]^{1 - \frac{1}{2p}} \left[\sum_{s=0}^{k_0-1} e^{-(a'-p\lambda)(k_0-s-1)} \right]^{\frac{1}{2p}} \\ &\quad + \max_{1 \leq i \leq n} a_0 B_i^u M_0 \gamma_{2p} e^{-\lambda k_0} e^{\lambda} \left[\frac{1 - e^{-a'k_0}}{1 - e^{-a'}} \right]^{1 - \frac{1}{2p}} \left[\sum_{s=0}^{k_0-1} e^{-(a'-p\lambda)(k_0-s-1)} \right]^{\frac{1}{2p}} \\ &\quad + \max_{1 \leq i \leq n} \sum_{j=1}^n a_0 M_0 \gamma_{2p} \left[c_{ij}^u L_j^f + C_{2p}^{\frac{1}{2p}} d_{ij}^u L_j^\sigma \right] \\ &\quad \times e^{-\lambda k_0} e^{(\mu_0+1)\lambda} \left[\frac{1 - e^{-a'k_0}}{1 - e^{-a'}} \right]^{1 - \frac{1}{2p}} \left[\sum_{s=0}^{k_0-1} e^{-(a'-p\lambda)(k_0-s-1)} \right]^{\frac{1}{2p}} \\ &\leq e^{-a'k_0} \gamma_{2p} + \max_{1 \leq i \leq n} a_0 M_0 \gamma_{2p} e^{-\lambda k_0} \left[B_i^u + \sum_{j=1}^n (b_{ij}^u L_j^f + e^{\mu_0 \lambda} c_{ij}^u L_j^f + e^{\mu_0 \lambda} C_{2p}^{\frac{1}{2p}} d_{ij}^u L_j^\sigma) \right] \\ &\quad \times e^{\lambda} \left[\frac{1 - e^{-a'k_0}}{1 - e^{-a'}} \right]^{1 - \frac{1}{2p}} \left[\frac{1 - e^{-(a'-2p\lambda)k_0}}{1 - e^{-(a'-2p\lambda)}} \right]^{\frac{1}{2p}} \\ &\leq M_0 \gamma_{2p} e^{-\lambda k_0} \left\{ \frac{1}{M_0} e^{-(a'-\lambda)k_0} \right. \\ &\quad \left. + \max_{1 \leq i \leq n} a_0 \left[B_i^u + \sum_{j=1}^n (b_{ij}^u L_j^f + e^{\mu_0 \lambda} c_{ij}^u L_j^f + e^{\mu_0 \lambda} C_{2p}^{\frac{1}{2p}} d_{ij}^u L_j^\sigma) \right] \frac{e^{\lambda} [1 - e^{-(a'-\lambda)k_0}]}{1 - e^{-(a'-2p\lambda)}} \right\} \\ &\leq M_0 \gamma_{2p} e^{-\lambda k_0} \left\{ e^{-(a'-\lambda)k_0} + \rho [1 - e^{-(a'-\lambda)k_0}] \right\} \\ &\leq M_0 \gamma_{2p} e^{-\lambda k_0}. \end{aligned} \quad (4.6)$$

Thus (C_1) fails. Homoplastically, (C_2) is not satisfied. So (4.5) hold. Then system (2.2) has the property of $2p$ -th MGES. The proof is finished.

By Theorem 3.1, it obtains

Theorem 4.2. *Assume that all conditions in Theorem 3.1 hold, then system (2.2) admits a $2p$ -th MAPS, which has the property of $2p$ -th MGES.*

Remark 4.1. *In literature [17], Ji discussed a delayed semi-discrete neural networks and the existence and global attractivity of a unique almost periodic sequence solution were investigated. In literatures [14–16, 18], the authors discussed the almost periodic dynamics of various NNs. But neither of them considered the stochastic disturbances. Therefore, the work of this paper complements the works in [14–18].*

5. Numerical illustrations

Example 5.1. *Regarding the semi-discrete stochastic CNNs below:*

$$\begin{cases} x(k+1) = 0.4x(k) - 0.6 \left[0.01 \sin(\sqrt{5}k) \sin(x(k)) + 0.05 \sin(\sqrt{7}k)S(k) \right. \\ \quad \left. + 0.01 \Delta w_1(k) + 0.01 \cos^2(\sqrt{17}k) \right], \\ S(k+1) = 0.4S(k) - 0.6 \left[0.03 \cos(\sqrt{2}k) \sin(x(k-1)) + 0.02S(k) \Delta w_2(k) \right], \quad k \in \mathbb{Z}. \end{cases} \quad (5.1)$$

Referring to system (2.2), it has $\tilde{a}^u = \tilde{a}^l = a^u = a^l = 1$, $L^f = L^{\tilde{f}} = 1$, $L^\sigma = 0$, $b^u = 0.01$, $\tilde{b}^u = 0.03$, $B^u = 0.05$, $d^u = 0.01$, $\tilde{d}^u = 0.02$. Taking $p = 2$, then

$$C_4^{1/4} = 4, \quad D^* \approx 0.06, \quad \tilde{D}^* \approx 0.03, \quad K^* = 0, \quad \tilde{K}^* \approx 0.01, \quad r_4 \approx 0.68 < 1.$$

By Theorem 4.2, system (5.1) admits a 4-th MAPS, which has the property of 4-th MGES, see Figures 1–3.

Figure 1 plots the almost periodic trajectories of state variables x and S in system (5.1). Obviously, the amplitude of state variable x is larger than that of state variable S . Figures 2,3 plot the trajectories of state variables x and S in system (5.1) with different initial values, respectively. They demonstrate that the trajectories of state variables in system (5.1) with different initial values will tend to the same trajectory since it has the property of 4-th MGES.

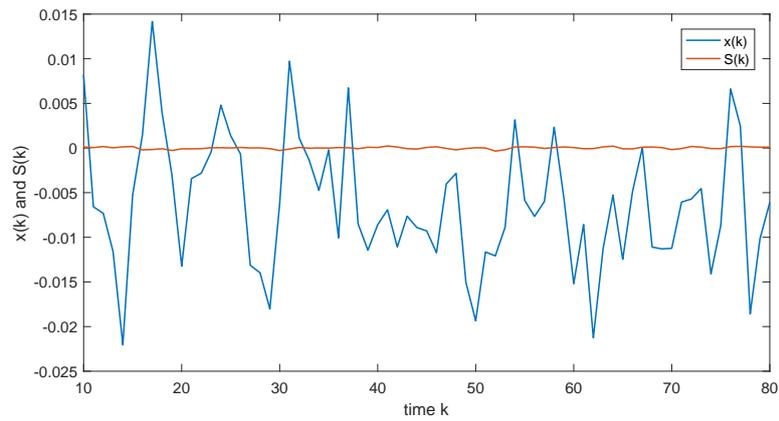


Figure 1. Trajectories of 4-th MAPS of model (5.1).

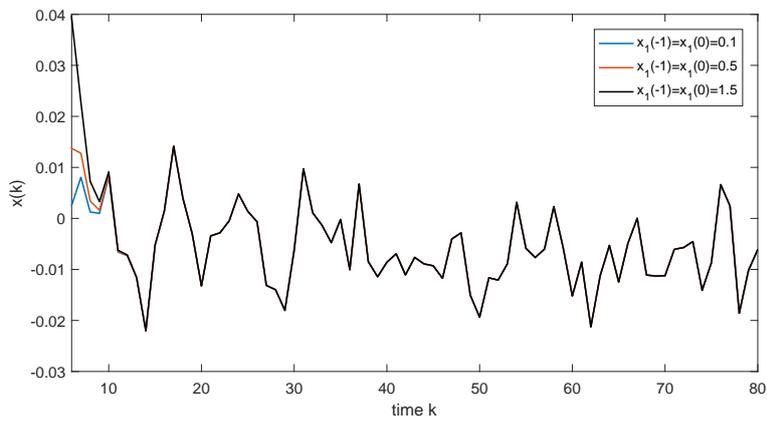


Figure 2. 4-th MGES of state variable x of model (5.1).

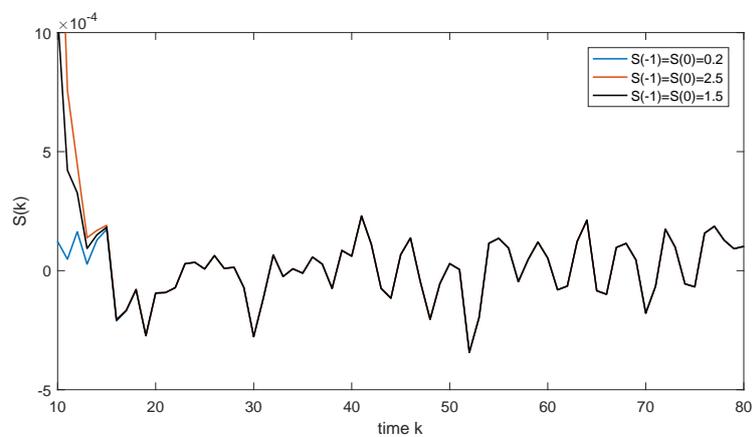


Figure 3. 4-th MGES of state variable S of model (5.1).

Considering the corresponding determinant model of system (5.1) below:

$$\begin{cases} x(k+1) = 0.4x(k) - 0.6 \left[0.01 \sin(\sqrt{5}k) \sin(x(k)) \right. \\ \quad \left. + 0.05 \sin(\sqrt{7}k)S(k) + 0.01 + 0.01 \cos^2(\sqrt{17}k) \right], \\ S(k+1) = 0.4S(k) - 0.6 \left[0.03 \cos(\sqrt{2}k) \sin(x(k-1)) + 0.02S(k) \right], \quad k \in \mathbb{Z}. \end{cases} \quad (5.2)$$

Figures 4 and 5 depict the comparison result between model (5.1) and model (5.2). Figures 4 and 5 imply that the influences of stochastic disturbances on x and S are pretty obvious. Observing Figures 4 and 5, it easily discovers that the effect of stochastic disturbance on x is larger than that on S .

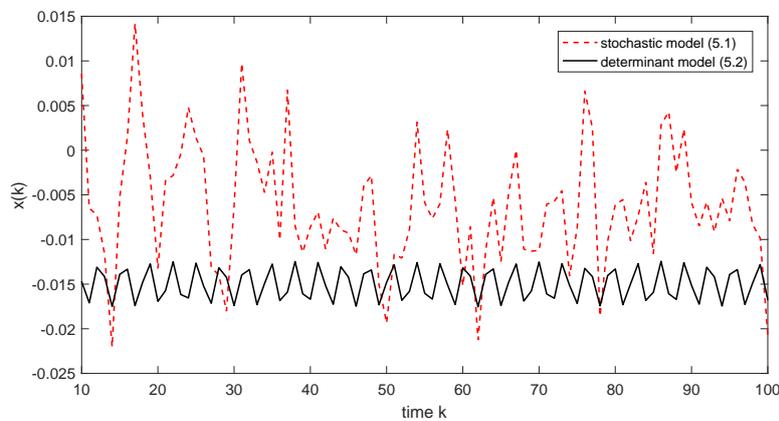


Figure 4. Comparisons of state variable x between stochastic model (5.1) and determinant model (5.2).

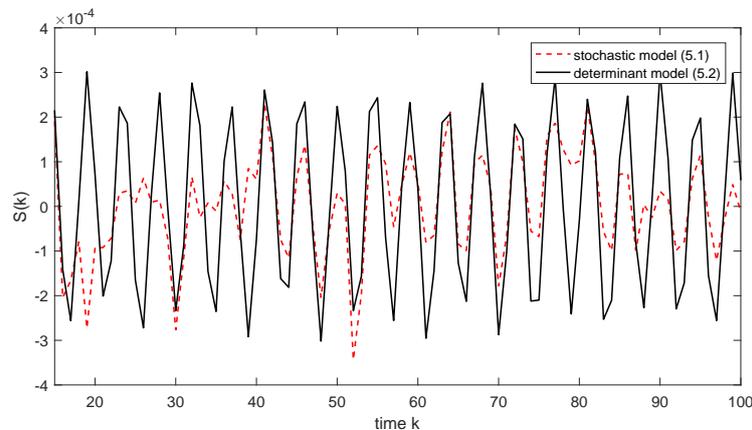


Figure 5. Comparisons of state variable S between stochastic model (5.1) and determinant model (5.2).

6. Conclusions and future perspectives

To date, few people consider the problems of MAPS and MGES of semi-discrete SFCNNs. Hence, this article concerns the issues of $2p$ -th MAPS and $2p$ -th MGES of semi-discrete SFCNNs. The techniques of this literatures give an effective approach to investigate $2p$ -th MAPS and $2p$ -th MGES of semi-discrete stochastic fuzzy models. By (H_3) , $r_{2p} < 1$ is important to the studies of $2p$ -th MAPS and $2p$ -th MGES. There is no influence of time delay on $2p$ -th MAPS and time delay has negative influence in $2p$ -th MGES.

In the future, the following aspects could be studied further:

- (1) $p \in (0, 1]$ in NNs (2.2) could be further investigated.
- (2) Other dynamics in NNs (2.2) should be further considered, e.g., hopf bifurcation, chaos, automatic control, etc.
- (3) Using the idea and method of this paper, other types of dynamic systems could be studied, such as impulsive dynamic systems [23, 24], biomathematical systems [25–27], etc.
- (4) Applications of NNs (2.2) in intelligent control, machine learning, image processing, etc, should be further investigated.

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Conflict of interest

All authors declare no conflict of interest.

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