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Research article

A variant of Jensen-type inequality and related results for harmonic convex functions

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Abstract: In this article, we present a variant of discrete Jensen-type inequality for harmonic convex functions and establish a Jensen-type inequality for harmonic h-convex functions. Furthermore, we found a variant of Jensen-type inequality for harmonic h-convex functions.

Keywords: convex functions; harmonic convex functions; harmonic h-convex functions; Jensen-type inequality; discrete Hölder inequality; Hermite-Hadamard-type inequality
Mathematics Subject Classification: 26D15, 26A51, 26D10, 26A15

1. Introduction

A real-valued function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex (concave) on I if the inequality

$$f(tx + (1-t)y) \leq (\geq) tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Recently, the generalizations, variants and extensions for the convexity have attracted the attention of many researchers, for example, the harmonic-convexity [1, 2, 3], harmonic (s,m)-convexity [4, 5], harmonic (p,(s,m))-convexity [6], harmonic (s,m)-preinvexity [7], harmonic log-convexity [8], harmonic (p,h,m)-preinvexity [9], exponential-convexity [10, 11], s-convexity [12, 13], $H_{p,q}$-convexity [14], generalized convexity [15], GG-and GA-convexities [16] and quasi-convexity [17, 18]. In particular, many remarkable inequalities can be found in the literature [19–36] via the convexity theory.

Jensen [37] provided a characterization for the convex functions as follows.

Theorem 1.1. *Let f be a convex function defined on the interval $I \subseteq \mathbb{R}$. Then the inequality*

$$f\Big(\sum_{i=1}^{n}\lambda_i x_i\Big) \le \sum_{i=1}^{n}\lambda_i f(x_i) \tag{1.1}$$

holds for all $x_1, x_2, \cdots, x_n \in I$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \ge 0$ with $\sum_{i=1}^{n}\lambda_i = 1$.

If f is a concave function, then the inequality (1.1) is reversed. The inequality (1.1) for convex functions plays a pivotal role in the theory of inequalities due to many other inequalities, for instance, the Hölder inequality, Minkowski inequality and arithmetic mean-geometric mean inequality can be obtained as a particular case of inequality (1.1). Furthermore, it is also important to observe that the inequality (1.1) has a close relation with numerous other prime inequalities like the reverse Minkowski inequality [38], Ostrowski inequality [39, 40], Petrović inequality [41], Hermite-Hadamard inequality [42], Bessel function inequality [43] and Pólya-Szegö inequality [44].

Next, we recall the definition of harmonic convex functions [2, 3, 45].

Definition 1.2. A real-valued function $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonic convex on I if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le tf(y) + (1-t)f(x) \tag{1.2}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If inequality (1.2) is reversed, then f is said to be harmonic concave.

Now, we provide several examples of harmonic convex functions. The function $f(x) = \ln x$ is a harmonic convex function on the interval $(0, \infty)$, but it is not a convex function (See Figure 1).

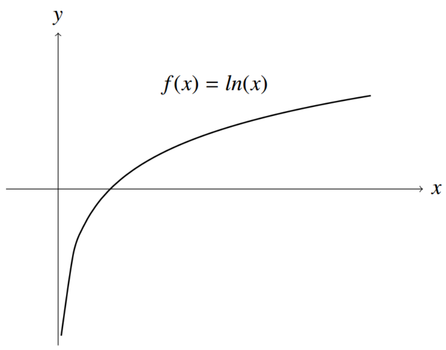

Figure 1. Harmonic convex function but not convex function.

The following three functions are harmonic convex on the interval $(0, \infty)$ (See Figure 2).

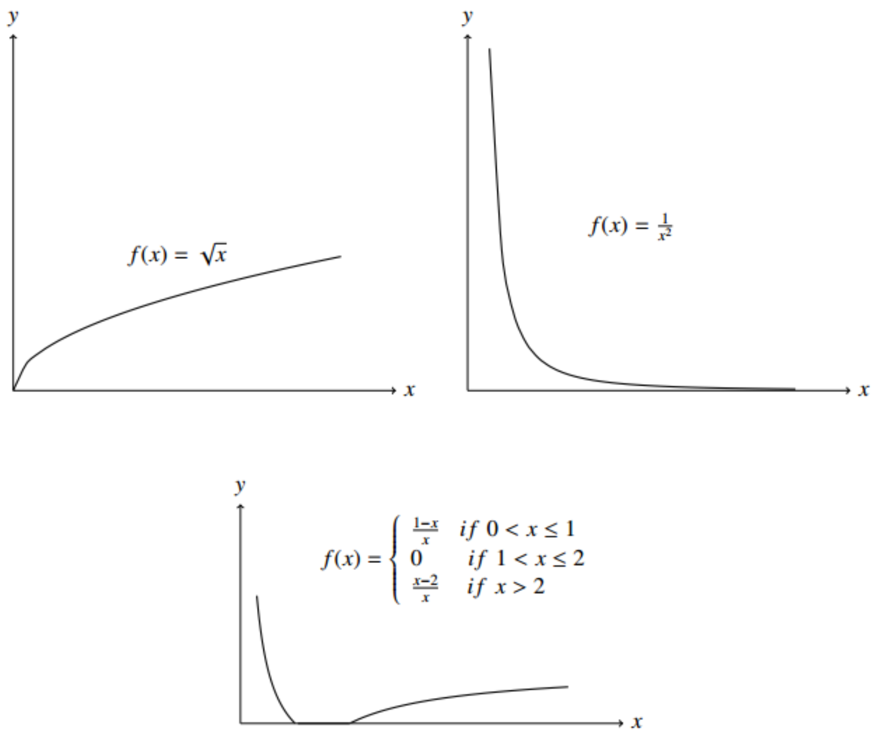

Figure 2. Three harmonic convex functions.

Very recently, Dragomir [46] established a Jensen-type inequality for harmonic convex function as follows.

Theorem 1.3. *Let $I \subseteq (0, \infty)$ be an interval and $f : I \to \mathbb{R}$ be a harmonic convex function. Then the Jensen-type inequality*

$$f\left(\frac{1}{\sum_{i=1}^{n} \frac{t_i}{x_i}}\right) \leq \sum_{i=1}^{n} t_i f(x_i) \tag{1.3}$$

holds for all $x_1, \cdots, x_n \in I$ and $t_1, \cdots, t_n \geq 0$ with $\sum_{i=1}^{n} t_i = 1$.

In [47], Varošanec introduced the concept of h-convex function which is the generalizations of many generalized convex functions, like s-convex function, Godunova-Levin function, s-Godunova-Levin function, P-convex function and so on. In the similar manner, the harmonic h-convexity was introduced to unify various types of harmonic convexities.

Definition 1.4. (See [48]) Let $h : [0, 1] \to \mathbb{R}^+$ be a non-negative function. Then the real-valued function $f : I \subseteq \mathbb{R}\backslash\{0\} \to (0, \infty)$ is said to be harmonic h-convex on I if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq h(t) f(y) + h(1-t) f(x) \tag{1.4}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If inequality (1.4) is reversed, then f is said to be harmonic concave.

Remark 1.5. We provide several examples of harmonic h-convex (concave) functions as follows:

- We clearly see that if $h(t) = t$, then the class of non-negative harmonic convex (concave) functions on I is contained in the class of harmonic h-convex (concave) functions on I.
- Let $t \in (0, 1)$ and $h(t) = t^2$. Then the function $f : [-1, 0) \cup (0, 1] \to \mathbb{R}$ defined by $f(x) = 1$ is neither non-increasing nor non-decreasing h-convex function. Therefore, we know that f is a harmonic h-convex function by the Proposition 2.1 given in [49].
- Let $t \in (0, 1)$ and $h : (0, 1) \to (0, \infty)$ be a real-valued function such that $h(t) \geq t$ on $(0, 1)$. Then the following four functions: $h_1(t) = t$, $h_2(t) = t^s$ $(s \in (0, 1))$, $h_3(t) = \frac{1}{t}$ and $h_4(t) = 1$ satisfy the conditions of the function h mentioned above. Therefore, f is a harmonic h_k-convex function for $k = 1, 2, 3, 4$ if $f : I \subseteq (0, \infty) \to (0, \infty)$ is a a non-decreasing convex function, or harmonic s-convex function, or harmonic Godunova-Levin function or harmonic P-function.
- Let $f : (0, \infty) \to (0, \infty)$ be a non-decreasing continuous function and $h : [0, 1] \to (0, \infty)$ be a continuous self-concave function such that $f(tx + (1 - t)y) \leq h(t)f(x) + (1 - t)f(y)$ for some $t \in (0, 1)$ and all $x, y \in (0, \infty)$. Then f is a h-convex function by Lemma 1 of [50] and hence f is a harmonic h-convex function by proposition 2.1 of [49].

Definition 1.6. A real-valued function $h : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be a submultiplicative function if the inequality

$$h(xy) \leq h(x)h(y) \tag{1.5}$$

for all $x, y \in I$. If inequality (1.5) is reversed, then h is said to be a supermultiplicative function. If just equality holds in the relation (1.5), then h is said to be a multiplicative function.

Definition 1.7. Let $n \geq 2$, $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ be two n-tuples of real numbers, and $[\mathbf{a}] = (a_{[1]}, \cdots, a_{[n]})$ and $[\mathbf{b}] = (b_{[1]}, \cdots, b_{[n]})$ be the descending rearrangements of $\mathbf{a}$ and $\mathbf{b}$, namely

$$a_{[1]} \geq a_{[2]} \geq \cdots \geq a_{[n]}, \quad b_{[1]} \geq b_{[2]} \geq \cdots \geq b_{[n]}.$$

Then we say $\mathbf{a}$ majorizes $\mathbf{b}$ (in symbols $\mathbf{a} \succ \mathbf{b}$) if

- $\sum_{i=1}^{k} a_{[i]} \geq \sum_{i=1}^{k} b_{[i]}$ for $k = 1, 2, \cdots, n - 1$.
- $\sum_{i=1}^{n} a_{[i]} = \sum_{i=1}^{n} b_{[i]}$.

The well-known memorization type theorem can be stated as follows.

Theorem 1.8. *(See [2]) Let $\boldsymbol{a} = (a_1, ..., a_n)$, $\boldsymbol{b} = (b_1, ..., b_n)$ be finite sequences from $I \subseteq \mathbb{R}\backslash\{0\}$ and if $\boldsymbol{a}$ majorizes $\boldsymbol{b}$ (in symbols $\boldsymbol{a} \succ \boldsymbol{b}$). Now, if $f : I \to \mathbb{R}$ is harmonic convex, then following inequality*

$$\sum_{i=1}^{n} a_i f(a_i) \geq \sum_{i=1}^{n} b_i f(b_i) \tag{1.6}$$

holds.

2. Main results

In this section, we will establish a variant of the Jensen-type inequality presented in Theorem 1.3, provide a Jensen-type inequality and its variant for the harmonic h-convex functions. In order to prove our main results, we need a lemma which we present in this section.

Lemma 2.1. *Let $I \subseteq \mathbb{R}\backslash\{0\}$ be an interval, $\{x_k\}_{k=1}^{n} \in I$ be a finite positive increasing sequence and f be a harmonic convex function on the interval I. Then the inequality*

$$f\left(\frac{1}{\frac{1}{x_1}+\frac{1}{x_n}-\frac{1}{x_k}}\right) \leq f(x_1)+f(x_n)-f(x_k) \tag{2.1}$$

holds for all $1 \leq k \leq n$.

Proof. Let $\frac{1}{y_k} = \frac{1}{x_1}+\frac{1}{x_n}-\frac{1}{x_k}$. Then $\frac{1}{y_k}+\frac{1}{x_k} = \frac{1}{x_1}+\frac{1}{x_n}$, so that the pairs x_1, x_n and x_k, y_k possess the same harmonic mean. Therefore, we can find $\mu, \lambda \in [0, 1]$ with $\mu + \lambda = 1$ such that

$$x_k = \frac{x_1 x_n}{\mu x_1 + \lambda x_n}$$

and

$$y_k = \frac{x_1 x_n}{\lambda x_1 + \mu x_n}.$$

It follows from the harmonic convexity of f that

$$\begin{aligned} f(y_k) &= f\left(\frac{x_1 x_n}{\lambda x_1 + \mu x_n}\right) \\ &\leq \mu f(x_1) + \lambda f(x_n) \\ &= (1-\lambda) f(x_1) + (1-\mu) f(x_n) \\ &= f(x_1) + f(x_n) - [\lambda f(x_1) + \mu f(x_n)] \\ &\leq f(x_1) + f(x_n) - f\left(\frac{x_1 x_n}{\mu x_1 + \lambda x_n}\right) \\ &= f(x_1) + f(x_n) - f(x_k). \end{aligned}$$

Therefore, inequality (2.1) follows from $\frac{1}{y_k} = \frac{1}{x_1}+\frac{1}{x_n}-\frac{1}{x_k}$. □

Remark 2.2. Lemma 2.1 lead to the conclusion that

1. Since $f(x) = \ln x$ is harmonic convex on $(0, \infty)$, so using (2.1) we get $\frac{2ab}{a+b} \leq \frac{a+b}{2}$ for all $a, b \in (0, \infty)$.
2. Since $f(x) = \frac{1}{x^2}$ is harmonic convex on $(0, \infty)$, so using (2.1) we get $(\frac{2ab}{a+b})^2 \leq \frac{a^2+b^2}{2}$ for all $a, b \in (0, \infty)$.
3. Since $f(x) = \sqrt{x}$ is harmonic convex on $(0, \infty)$, so using (2.1) we get $\sqrt{\frac{2ab}{a+b}} \leq \frac{\sqrt{a}+\sqrt{b}}{2}$ for all $a, b \in (0, \infty)$.
4. It was proved in [49] that $(\frac{a+b}{2})^2 \leq \frac{1}{3}(a^2+ab+b^2) \leq \frac{a^2+b^2}{2}$ for all $a, b \in (0, \infty)$. Therefore, in the light of previous remarks, we get its improvement as

$$\left(\frac{2ab}{a+b}\right)^2 \leq \left(\frac{a+b}{2}\right)^2 \leq \frac{1}{3}(a^2+ab+b^2) \leq \frac{a^2+b^2}{2}. \tag{2.2}$$

5. Since $f(x) = e^x$ is harmonic convex on $(0, \infty)$, so using (2.1) we get $e^{\frac{2ab}{a+b}} \leq \frac{e^a+e^b}{2}$.

Remark 2.3. (**Discrete form of Hölder's inequality**) Let $x_i, y_i > 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then one has

$$\sum_{i=1}^{n} x_i y_i \le \Big(\sum_{i=1}^{n} x_i^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} y_i^q\Big)^{\frac{1}{q}}. \tag{2.3}$$

Proof. Since $f(x) = \frac{1}{x^p}$ is harmonic convex on the interval $(0, \infty)$, hence by use of the substitutions $t_i = \frac{y_i}{\sum_{i=1}^{n} y_i^q}$ and $s_i = x_i^{-1} y_i^{\frac{q}{p}}$ we get

$$\Big(\frac{1}{\sum_{i=1}^{n} |y_i|^q} \sum_{i=1}^{n} |y_i|^q |x_i| |y_i|^{-\frac{q}{p}}\Big)^p \le \frac{1}{\sum_{i=1}^{n} |y_i|^q} \sum_{i=1}^{n} |y_i|^q (|x_i| |y_i|^{-\frac{q}{p}})^p,$$

which gives the desired inequality (2.3). □

Theorem 2.4. *Let $I \subseteq \mathbb{R} \backslash \{0\}$ be an interval and $f : I \to \mathbb{R}$ be a harmonic convex function. Then the inequality*

$$f\Bigg(\frac{1}{\frac{1}{x_1} + \frac{1}{x_n} - \sum_{k=1}^{n} \frac{t_k}{x_k}}\Bigg) \le f(x_1) + f(x_n) - \sum_{k=1}^{n} t_k f(x_k). \tag{2.4}$$

holds for any finite positive sequence $\{x_k\}_{k=1}^{n} \in I$ and $t_1, \cdots, t_n \ge 0$ with $\sum_{i=1}^{n} t_i = 1$.

Proof. It follows from Theorem 1.3 and Lemma 2.1 together with the harmonic convexity of f on the interval I that

$$\begin{aligned} f\Bigg(\frac{1}{\frac{1}{x_1} + \frac{1}{x_n} - \sum_{k=1}^{n} \frac{t_k}{x_k}}\Bigg) &= f\Bigg(\frac{1}{\sum_{k=1}^{n} t_k(\frac{1}{x_1} + \frac{1}{x_n} - \frac{1}{x_k})}\Bigg) \\ &\le \sum_{k=1}^{n} t_k f\Bigg(\frac{1}{\frac{1}{x_1} + \frac{1}{x_n} - \frac{1}{x_k}}\Bigg) \\ &\le \sum_{k=1}^{n} t_k (f(x_1) + f(x_n) - f(x_k)) \\ &= f(x_1) + f(x_n) - \sum_{k=1}^{n} t_k f(x_k), \end{aligned}$$

which completes the proof of Theorem 2.4. □

Theorem 2.5. *Let $n \ge 2$, $J \subseteq (0, 1)$ be an interval, $\{x_k\}_{k=1}^{n} \in I \subseteq \mathbb{R} \setminus \{0\}$, $p_1, p_2, \cdots, p_n > 0$, $P_n = \sum_{i=1}^{n} p_i$, $h : J \to \mathbb{R}$ be a non-negative supermultiplicative function and f be a non-negative harmonic h-convex function on I. Then one has*

$$f\Bigg(\frac{1}{\frac{1}{P_n} \sum_{i=1}^{n} \frac{p_i}{x_i}}\Bigg) \le \sum_{i=1}^{n} h\Big(\frac{p_i}{P_n}\Big) f(x_i). \tag{2.5}$$

Proof. We use mathematical induction to prove Theorem 2.5. If $n = 2$, then inequality (2.5) is equivalent to inequality (1.4) with $t = \frac{p_1}{P_2}$ and $1 - t = \frac{p_2}{P_2}$.

Suppose that inequality (2.5) holds for $n-1$. Then for the n-tuples $(x_1, \cdots, x_n)$ and $(p_1, \cdots, p_n)$, we have

$$\begin{aligned}
f\Big(\frac{1}{\frac{1}{P_n}\sum_{i=1}^{n}\frac{p_i}{x_i}}\Big) &= f\Big(\frac{1}{\frac{p_n}{P_n x_n}+\sum_{i=1}^{n-1}\frac{p_i}{P_n x_i}}\Big) = f\Big(\frac{1}{\frac{p_n}{P_n x_n}+\frac{P_{n-1}}{P_n}\sum_{i=1}^{n-1}\frac{p_i}{P_{n-1}x_i}}\Big)\\
&\leq h\Big(\frac{p_n}{P_n}\Big)f(x_n) + h\Big(\frac{P_{n-1}}{P_n}\Big)f\Big(\frac{1}{\sum_{i=1}^{n-1}\frac{p_i}{P_{n-1}.x_i}}\Big)\\
&\leq h\Big(\frac{p_n}{P_n}\Big)f(x_n) + h\Big(\frac{P_{n-1}}{P_n}\Big)\sum_{i=1}^{n-1} h\Big(\frac{p_i}{P_{n-1}}\Big)f(x_i)\\
&\leq h\Big(\frac{p_n}{P_n}\Big)f(x_n) + \sum_{i=1}^{n-1} h\Big(\frac{p_i}{P_n}\Big)f(x_i)\\
&= \sum_{i=1}^{n} h\Big(\frac{p_i}{P_n}\Big)f(x_i),
\end{aligned}$$

which completes the proof of Theorem 2.5. □

Remark 2.6. Let $h(\alpha) = \alpha$. Then inequality (2.5) becomes the Jensen-type inequality (1.3) for harmonic convex function.

In order to prove our next result, we need the following Lemma 2.7 which is a generalization of Lemma 2.1.

Lemma 2.7. *Let $h : J \supseteq (0,1) \to \mathbb{R}$ be a non-negative supermutiplicative function on J, $\mu, \lambda \in [0,1]$ such that $\mu + \lambda = 1$ and $h(\mu) + h(\lambda) \leq 1$. If $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is a non-negative harmonic h-convex function, then for finite positive increasing sequence $\{x_k\}_{k=1}^{n} \in I$, we again have the inequality (2.1).*

Proof. Similarly to proof of Lemma 2.1, suppose that $\frac{1}{y_k} = \frac{1}{x_1} + \frac{1}{x_n} - \frac{1}{x_k}$. Then $\frac{1}{y_k} + \frac{1}{x_k} = \frac{1}{x_1} + \frac{1}{x_n}$, so that the pairs x_1, x_n and x_k, y_k possess the same harmonic mean. Therefore, we can find $\mu, \lambda \in [0,1]$ such that

$$x_k = \frac{x_1 x_n}{\mu x_1 + \lambda x_n}$$

and

$$y_k = \frac{x_1 x_n}{\lambda x_1 + \mu x_n},$$

where $\mu + \lambda = 1$ and $1 \leq k \leq n$. Now, by taking into account that f is harmonic h-convex, we get

$$\begin{aligned}
f(y_k) &= f\Big(\frac{x_1 x_n}{\lambda x_1 + \mu x_n}\Big)\\
&\leq h(\mu)f(x_1) + h(\lambda)f(x_n)\\
&\leq (1-h(\lambda))f(x_1) + (1-h(\mu))f(x_n)\\
&= f(x_1) + f(x_n) - [h(\lambda)f(x_1) + h(\mu)f(x_n)]\\
&\leq f(x_1) + f(x_n) - f\Big(\frac{x_1 x_n}{\mu x_1 + \lambda x_n}\Big)\\
&= f(x_1) + f(x_n) - f(x_k),
\end{aligned}$$

which completes the proof of Lemma 2.7. □

Theorem 2.8. *Let $h : J \supseteq (0,1) \to \mathbb{R}$ be a non-negative supermutiplicative function, $p_1, \cdots, p_n$ be positive real numbers ($n \geq 2$) such that $P_n = \sum_{k=1}^{n} p_i$ and $\sum_{k=1}^{n} h\Big(\frac{p_k}{P_n}\Big) \leq 1$. If f is non-negative harmonic h-convex on $I \subseteq \mathbb{R}\backslash\{0\}$, then for any finite positive increasing sequence $\{x_k\}_{k=1}^{n} \in I$, we have*

$$f\Bigg(\frac{1}{\frac{1}{x_1} + \frac{1}{x_n} - \frac{1}{P_n}\sum_{k=1}^{n}\frac{p_i}{x_i}}\Bigg) \leq f(x_1) + f(x_n) - \sum_{k=1}^{n} h\Big(\frac{p_k}{P_n}\Big) f(x_k). \tag{2.6}$$

If h is submultiplicative function, $\sum_{k=1}^{n} h\Big(\frac{p_k}{P_n}\Big) \geq 1$ and f is harmonic h-concave, then the inequality (2.6) is reversed.

Proof. Since $\sum_{k=1}^{n}\frac{p_k}{P_n} = 1$ and f is harmonic h-convex on I, so by taking into account Theorem 2.5 and Lemma 2.7 we have

$$\begin{aligned}
f\Bigg(\frac{1}{\frac{1}{x_1} + \frac{1}{x_n} - \frac{1}{P_n}\sum_{k=1}^{n}\frac{p_k}{x_k}}\Bigg) &= f\Bigg(\frac{1}{\sum_{k=1}^{n}\frac{p_k}{P_n}(\frac{1}{x_1} + \frac{1}{x_n} - \frac{1}{x_k})}\Bigg) \\
&\leq \sum_{k=1}^{n} h\Big(\frac{p_k}{P_n}\Big) f\Bigg(\frac{1}{\frac{1}{x_1} + \frac{1}{x_n} - \frac{1}{x_k}}\Bigg) \\
&\leq \sum_{k=1}^{n} h\Big(\frac{p_k}{P_n}\Big)(f(x_1) + f(x_n) - f(x_k)) \\
&= [f(x_1) + f(x_n)]\sum_{k=1}^{n} h\Big(\frac{p_k}{P_n}\Big) - \sum_{k=1}^{n} h\Big(\frac{p_k}{P_n}\Big) f(x_k) \\
&\leq f(x_1) + f(x_n) - \sum_{k=1}^{n} h\Big(\frac{p_k}{P_n}\Big) f(x_k),
\end{aligned}$$

which completes the proof of Theorem 2.8. □

3. Related results

In this section, we present an extension of inequality (2.2) and some related results.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous harmonic convex function on $[a,b] \subset (0,\infty)$. Suppose that $\boldsymbol{a} = (a_1, \cdots, a_m)$ with $a_j \in [a,b]$ and $\boldsymbol{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in [a,b]$ for all $i = 1,2,3,\cdots,n$ and $j = 1,2,3,\cdots,m$. If $\boldsymbol{a}$ majorizes each row of $\boldsymbol{X}$, that is*

$$\boldsymbol{x}_i = (x_{i1}, x_{i2}, x_{i3}, \cdots, x_{im}) \prec (a_1, \cdots, a_m) = \boldsymbol{a} \quad for\ each\ i = 1,2,3\cdots,n$$

and

$$\sum_{j=1}^{m}\frac{1}{a_j} = \sum_{j=1}^{m}\frac{1}{x_{ij}}, \tag{3.1}$$

then we have the inequality

$$f\Bigg(\frac{1}{\sum_{j=1}^{m}\frac{1}{a_j} - \sum_{j=1}^{m-1}\sum_{i=1}^{n}\frac{w_i}{x_{ij}}}\Bigg) \leq \sum_{j=1}^{m}\frac{a_j f(a_j)}{x_{im}} - \sum_{j=1}^{m-1}\sum_{i=1}^{n}\frac{w_i x_{ij} f(x_{ij})}{x_{im}}, \tag{3.2}$$

where $w_i \geq 0$ with $\sum_{i=1}^{n} w_i = 1$.

Proof. Since f is harmonic convex. Therefore, by taking into account the Jensen-type inequality for harmonic convex function, we have

$$\begin{aligned} f\left(\frac{1}{\sum_{j=1}^{m}\frac{1}{a_j}-\sum_{j=1}^{m-1}\sum_{i=1}^{n}\frac{w_i}{x_{ij}}}\right) &= f\left(\frac{1}{\sum_{j=1}^{m}\sum_{i=1}^{n}w_i\frac{1}{a_j}-\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\frac{1}{x_{ij}}}\right) \\ &= f\left(\frac{1}{\sum_{i=1}^{n}w_i\Big[\sum_{j=1}^{m}\frac{1}{a_j}-\sum_{j=1}^{m-1}\frac{1}{x_{ij}}\Big]}\right) \\ &\leq \sum_{i=1}^{n}w_i f\left(\frac{1}{\sum_{j=1}^{m}\frac{1}{a_j}-\sum_{j=1}^{m-1}\frac{1}{x_{ij}}}\right) \\ &= \sum_{i=1}^{n}\frac{w_i}{x_{im}}\left[x_{im}f\left(\frac{1}{\sum_{j=1}^{m}\frac{1}{a_j}-\sum_{j=1}^{m-1}\frac{1}{x_{ij}}}\right)\right]. \end{aligned} \tag{3.3}$$

From Eq (3.1) and using majorization-type Theorem 1.8 for f, we have

$$\begin{aligned} x_{im}f\left(\frac{1}{\sum_{j=1}^{m}\frac{1}{a_j}-\sum_{j=1}^{m-1}\frac{1}{x_{ij}}}\right) &= x_{im}f(x_{im}) \\ &\leq \sum_{j=1}^{m}a_jf(a_j)-\sum_{j=1}^{m-1}x_{ij}f(x_{ij}). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we get the required result. □

Now, we present the alternative form of above theorem as follows.

Theorem 3.2. *If all conditions of Theorem 3.1 are satisfied, then we have*

$$\begin{aligned} & f\left(\frac{1}{\sum_{j=1}^{m}\frac{1}{a_j}-\sum_{j=1}^{k-1}\sum_{i=1}^{n}\frac{w_i}{x_{ij}}-\sum_{j=k+1}^{m-1}\sum_{i=1}^{n}\frac{w_i}{x_{ij}}}\right) \\ & \leq \sum_{j=1}^{m}\frac{a_jf(a_j)}{x_{im}}-\sum_{j=1}^{k-1}\sum_{i=1}^{n}\frac{w_ix_{ij}f(x_{ij})}{x_{im}}-\sum_{j=k+1}^{m-1}\sum_{i=1}^{n}\frac{w_ix_{ij}f(x_{ij})}{x_{im}}. \end{aligned} \tag{3.5}$$

Proof. Using the technique of Theorem 3.1, the proof is quite obvious. □

Theorem 3.3. *If* $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ *is an harmonic convex function and* $x_1,\cdots,x_n\in[a,b]$, *then*

$$\begin{aligned} & \frac{n-1}{n}\sum_{k=1}^{n}f(x_k)+f(x_1)+f(x_n)-f\left(\frac{1}{\frac{1}{x_1}+\frac{1}{x_n}-\sum_{k=1}^{n}\frac{\frac{1}{n}}{x_k}}\right) \\ & \geq f\left(\frac{2x_1x_2}{x_1+x_2}\right)+\ldots+f\left(\frac{2x_1x_{n-1}}{x_{n-1}+x_n}\right)+f\left(\frac{2x_nx_1}{x_n+x_n}\right). \end{aligned} \tag{3.6}$$

Proof. Using (1.2) with $t = \frac{1}{2}$, we get

$$\begin{aligned} & f\Big(\frac{2x_1x_2}{x_1+x_2}\Big) + ... + f\Big(\frac{2x_1x_{n-1}}{x_{n-1}+x_n}\Big) + f\Big(\frac{2x_nx_1}{x_n+x_n}\Big) \\ \leq & \frac{1}{2}[f(x_1)+f(x_2)] + ... + \frac{1}{2}[f(x_{n-1})+f(x_n)] + \frac{1}{2}[f(x_n)+f(x_1)] \\ = & f(x_1) + ... + f(x_n) = \sum_{k=1}^{n} f(x_k). \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned} \sum_{k=1}^{n} f(x_k) &= \frac{n-1}{n}\sum_{k=1}^{n} f(x_k) + \sum_{k=1}^{n}\frac{1}{n}f(x_k) \\ &= \frac{n-1}{n}\sum_{k=1}^{n} f(x_k) + f(x_1) + f(x_n) - \Big[f(x_1) + f(x_n) - \sum_{k=1}^{n}\frac{1}{n}f(x_k)\Big]. \end{aligned} \tag{3.8}$$

Therefore, the required result follows from (2.4) and (3.7) and (3.8). □

Theorem 3.4. *If $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is a harmonic convex function and $x_1, \cdots, x_n \in [a,b]$, then*

$$\sum_{k=1}^{n} f(y_k) \leq \frac{n-1}{n}\sum_{k=1}^{n} f(x_k) + f(x_1) + f(x_n) - f\Bigg(\frac{1}{\frac{1}{x_1} + \frac{1}{x_n} - \sum_{k=1}^{n}\frac{\frac{1}{n}}{x_k}}\Bigg), \tag{3.9}$$

where $y_k = \frac{n}{(n-1)\alpha^{-1}+x_k^{-1}}$ *and* $\alpha = \frac{n}{x_1^{-1}+...+x_n^{-1}}$.

Proof. Using Jensen-type inequality for harmonic convex function, we have

$$\begin{aligned} \sum_{k=1}^{n} f(y_k) &= f(y_1) + ... + f(y_n) \\ &= f\Big(\frac{n}{(n-1)\alpha^{-1}+x_1^{-1}}\Big) + ... + f\Big(\frac{n}{(n-1)\alpha^{-1}+x_n^{-1}}\Big) \\ &\leq [\frac{1}{n}f(\alpha) + \frac{n-1}{n}f(x_1)] + ... + [\frac{1}{n}f(\alpha) + \frac{n-1}{n}f(x_n)] \\ &= f(\alpha) + \frac{n-1}{n}\sum_{k=1}^{n} f(x_k) \\ &= f\Big(\frac{n}{x_1^{-1}+...+x_n^{-1}}\Big) + \frac{n-1}{n}\sum_{k=1}^{n} f(x_k) \\ &\leq \frac{1}{n}\sum_{k=1}^{n} f(x_k) + \frac{n-1}{n}\sum_{k=1}^{n} f(x_k). \end{aligned}$$

Therefore, the required result follows from (2.4) and (3.8). □

Theorem 3.5. *Let f be a harmonic convex function on $[m, M]$. Then*

$$\begin{aligned} f\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\frac{2xy}{x+y}}\right) &\leq f(m)+f(M)-\int_0^1 f\left(\frac{xy}{tx+(1-t)y}\right)dt \\ &\leq f(m)+f(M)-f\left(\frac{2xy}{x+y}\right). \end{aligned} \tag{3.10}$$

Proof. It follows from the inequality (2.4) that

$$f\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\frac{2ab}{a+b}}\right) \leq f(m)+f(M)-\frac{f(a)+f(b)}{2} \tag{3.11}$$

for all $a, b \in [m, M]$.

Let $t \in [0, 1]$ and $x, y \in [m, M]$. Then Replacing a and b respectively by $\frac{xy}{tx+(1-t)y}$ and $\frac{xy}{ty+(1-t)x}$ in (3.10), we obtain

$$f\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\frac{2xy}{x+y}}\right) \leq f(m)+f(M)-\frac{f\left(\frac{xy}{tx+(1-t)y}\right)+f\left(\frac{xy}{ty+(1-t)x}\right)}{2}. \tag{3.12}$$

Integrating both sides of (3.12) with respect to t on $[0, 1]$, we get

$$\begin{aligned} f\left(\frac{1}{\frac{1}{m}+\frac{1}{M}-\frac{2xy}{x+y}}\right) \leq\ & f(m) + f(M) \\ - & \frac{1}{2}\int_0^1 [f\left(\frac{xy}{tx+(1-t)y}\right)+f\left(\frac{xy}{ty+(1-t)x}\right)]dt. \end{aligned} \tag{3.13}$$

Due to

$$\int_0^1 f\left(\frac{xy}{tx+(1-t)y}\right)dt = \int_0^1 f\left(\frac{xy}{ty+(1-t)x}\right)]dt = \frac{xy}{y-x}\int_x^y \frac{f(t)}{t^2}dt, \tag{3.14}$$

the inequality (3.13) give rise to the first inequality of (3.10). The second inequality of (3.10) follows directly from the Hermite-Hadamard type inequality for harmonic convex functions. □

4. Conclusions

We have found a variant of the discrete Jensen-type inequality for harmonic convex functions, and have provided a Jensen-type inequality and it variant for the harmonic h-convex functions. In Addition, we considered here different examples of harmonic convex functions and give a short proof of **Hölder's inequality** by using Jensen-type inequality. Our obtained results are the improvements and generalization of many previously known results, and our ideas and approach may lead to a lot of follow-up research.

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Conflict of interest

The authors declare no conflict of interest.

Authors' contributions

Imran Abbas Baloch provided the main idea and carried out the proof of Lemma 2.1 and Theorems 2.4, 2.5, 3.1, 3.2, 3.3, 3.4, 3.5 and drafted the manuscript. Aqeel Ahmad Mughal carried out the proof of Lemma 2.7 and Theorem 2.8. Absar ul Haq, Y.-M. Chu and Manuel De La Sen reviewed whole mathematics of article, completed the final revision of the article. All authors read and approved the final manuscript.

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