Mathematics

## Research article

# An effective method for division of rectangular intervals 

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#### Abstract

This paper focuses on the division of intervals in rectangular form. The particular case where the intervals are in the complex plane is considered. For two rectangular complex intervals $Z_{1}$ and $Z_{2}$ finding the smallest rectangle containing the exact set $\left\{z_{1} * z_{2}: z_{1} \in Z_{1}, z_{2} \in Z_{2}\right\}$ of the operation $* \in\{+,-, \cdot, /\}$ is the main objective of complex interval arithmetic. For the operations addition, subtraction and multiplication, the optimal solution can be easily found. In the case of division the solution requires rather complicated calculations. This is due to the fact that space of rectangular intervals is not closed under division. The quotient of two rectangular intervals is an irregular shape in general. This work introduces a new method for the determination of the smallest rectangle containing the result in the case of division. The method obtains the optimal solution with less computational cost compared to the algorithms currently available.


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## 1. Introduction

Analysis and solutions of various problems in the complex plane which either involve "inexact" data, or require some information on upper error bound of the obtained result or solution, dictate the need for a structure which is referred to as complex interval arithmetic [1]. There are three different forms to represent complex intervals: the rectangular form [2,3], the circular form [4], and the polar form (or sector) [5].

Interval operations should deliver the closest inclusion of the set of all possible values (e.g., [6]), i.e.

$$
\left\{z_{1} * z_{2}: z_{1} \in Z_{1}, z_{2} \in Z_{2}\right\} \subseteq Z_{1} * Z_{2}
$$

for any two intervals $Z_{1}, Z_{2}$ and $* \in\{+,-, \cdot, /\}$.

For rectangular complex arithmetic addition, subtraction and multiplication are optimal, whereas division is not (see, e.g., [7,8]. By optimality it is meant that the computed interval is the least interval that includes the set $\left\{z_{1} * z_{2}: z_{1} \in Z_{1}, z_{2} \in Z_{2}\right\}$. Thus, special algorithms must be built to perform division operation on rectangular intervals. Methods to improve the division operation of rectangular intervals are addressed in e.g. [7,8].

The method presented in [7] is based on the following approach

$$
\frac{Z_{1}}{Z_{2}}=Z_{1} \cdot \frac{1}{Z_{2}}
$$

where $\frac{1}{Z_{2}}:=\inf \left\{X:\left\{\frac{1}{d}: d \in Z_{2}\right\} \subseteq X\right\}$. However, the computed rectangle by this approach is not optimal in general (see, e.g., [8]).

The method presented in [8] is based on an algorithm which calculates the maximum and the minimum of the real and imaginary parts of the division. But, in general, this algorithm requires a significant amount of computations in order to get the optimal rectangle. This paper focuses on the division of rectangular complex intervals. We introduce a highly efficient and low cost algorithm compared to the one in [8]. A crucial ingredient to the efficiency of our algorithm is the fact that it can be easily implemented even without using a computer program.

The rest of the paper is organized as follows. Section 2 presents the definition and arithmetic of rectangular intervals. The proposed algorithm is introduced in Section 3. Section 4 contains the implementation of the proposed algorithm and its comparison with the available algorithms in the literature. Finally, Section 5 concludes the work.

## 2. Rectangular complex intervals

This section provides a brief review of complex interval arithmetic. Details of real interval arithmetic can be found in ([3,9-13]).
Definition 1. Let $[x]=\left[x^{-}, x^{+}\right] \in \mathbb{R} \mathbb{R}$ and $[y]=\left[y^{-}, y^{+}\right] \in \mathbb{R}$ be two closed real intervals. A rectangular complex interval $Z$ is defined by a pair of two real intervals $[x]$ and $[y]$ :

$$
Z=[x]+i[y], Z=\{z=x+i y: x \in[x], y \in[y]\},
$$

where $i=\sqrt{-1}$.
The set of all rectangular complex intervals is denoted by

$$
R(\mathbb{C})=\{Z=[x]+i[y]:[x],[y] \in \mathbb{R}\} .
$$

### 2.1. Arithmetic of rectangular intervals

Definition 2. Let $Z_{1}, Z_{2} \in R(\mathbb{C})$ and $*$ one of the basic operations $* \in\{+,-, \cdot, /\}$. We define the corresponding operations for $Z_{1}$ and $Z_{2}$ by,

$$
Z_{1} * Z_{2}:=\square\left\{Z_{1} \circledast Z_{2}\right\},
$$

where $Z_{1} \circledast Z_{2}:=\left\{z_{1} * z_{2}: z_{1} \in Z_{1}, z_{2} \in Z_{2}\right\}$ and $\square\left\{Z_{1} \circledast Z_{2}\right\}$ is the smallest rectangle in $R(\mathbb{C})$ enclosing $Z_{1} \circledast Z_{2}$.

The set $Z_{1} \circledast Z_{2}$ with $* \in\{\cdot, /\}$ is not necessarily a complex interval. That is, $Z_{1} \circledast Z_{2}$ may not be a rectangle with sides parallel to the axes. Consider the following example.

Let $Z_{1}=[1,2]+i[1,2]$ and $Z_{2}=[1,2]+i[1,2]$. Then $Z_{1} \oplus Z_{2}$ and $Z_{1} \ominus Z_{2}$ produce rectangles in the complex plane with sides parallel to the axes, while the resulting sets from $Z_{1} \odot Z_{2}$ and $Z_{1} \oslash Z_{2}$ have complicated shapes (not rectangles)(See Figure 1).


Figure 1. Arithmetic operations on complex intervals.

For two given intervals, $Z_{1}=\left[x_{1}\right]+i\left[y_{1}\right]$ and $Z_{2}=\left[x_{2}\right]+i\left[y_{2}\right]$, the basic arithmetic operations are defined as follows (see, e.g., $[2,3]$ ):

## Addition and subtraction

The sum (difference) of $Z_{1}$ and $Z_{2}$ is given by

$$
Z_{1} \pm Z_{2}:=\left[x_{1}\right] \pm\left[x_{2}\right] \pm i\left(\left[y_{1}\right] \pm\left[y_{2}\right]\right) .
$$

It is easy to prove that the following is valid:

$$
\begin{aligned}
Z_{1} * Z_{2} & :=\square\left\{Z_{1} \circledast Z_{2}\right\} \\
& =Z_{1} \circledast Z_{2} \text { for } * \in\{+,-\} .
\end{aligned}
$$

## Multiplication

The multiplication of $Z_{1}$ and $Z_{2}$ is given by the formula,

$$
\begin{aligned}
Z_{1} \cdot Z_{2} & :=\left[x_{1}\right]\left[x_{2}\right]-\left[y_{1}\right]\left[y_{2}\right]+i\left(\left[x_{1}\right]\left[y_{2}\right]+\left[x_{2}\right]\left[y_{1}\right]\right) \\
& =\left[x^{-}, x^{+}\right]+i\left[y^{-}, y^{+}\right],
\end{aligned}
$$

where,

$$
\begin{aligned}
x^{-}= & \min \left\{x_{1}^{-} x_{2}^{-}, x_{1}^{-} x_{2}^{+}, x_{1}^{+} x_{2}^{-}, x_{1}^{+} x_{2}^{+}\right\} \\
& +\min \left\{-y_{1}^{+} y_{2}^{-},-y_{1}^{+} y_{2}^{+},-y_{1}^{-} y_{2}^{-},-y_{1}^{-} y_{2}^{+}\right\}, \\
x^{+}= & \max \left\{x_{1}^{-} x_{2}^{-}, x_{1}^{-} x_{2}^{+}, x_{1}^{+} x_{2}^{-}, x_{1}^{+} x_{2}^{+}\right\} \\
& +\max \left\{-y_{1}^{+} y_{2}^{-},-y_{1}^{+} y_{2}^{+},-y_{1}^{-} y_{2}^{-},-y_{1}^{-} y_{2}^{+}\right\}, \\
y^{-} & =\min \left\{x_{1}^{-} y_{2}^{-}, x_{1}^{-} y_{2}^{+}, x_{1}^{+} y_{2}^{-}, x_{1}^{+} y_{2}^{+}\right\}+\min \left\{x_{2}^{-} y_{1}^{-}, x_{2}^{-} y_{1}^{+}, x_{2}^{+} y_{1}^{-}, x_{2}^{+} y_{y}^{+}\right\}, \\
y^{+} & =\max \left\{x_{1}^{-} y_{2}^{-}, x_{1}^{-} y_{2}^{+}, x_{1}^{+} y_{2}^{-}, x_{1}^{+} y_{2}^{+}\right\}+\max \left\{x_{2}^{-} y_{1}^{-}, x_{2}^{-} y_{1}^{+}, x_{2}^{+} y_{1}^{-}, x_{2}^{+} y_{1}^{+}\right\} .
\end{aligned}
$$

The multiplication $Z_{1} \cdot Z_{2}$ as defined above gives a rectangle in the complex plane such that,

$$
Z_{1} \cdot Z_{2}:=\square\left\{Z_{1} \odot Z_{2}\right\} \supseteq Z_{1} \odot Z_{2} .
$$

Consider the previous example $Z_{1}=[1,2]+i[1,2]$ and $Z_{2}=[1,2]+i[1,2]$. Then we get,

$$
Z_{1} \cdot Z_{2}=[-3,3]+i[2,8]
$$

which is the smallest rectangle containing the set $Z_{1} \odot Z_{2}$ (see Figure 2).


Figure 2. The rectangular hull of multiplication.

## Division

The division is defined by,

$$
\frac{Z_{1}}{Z_{2}}:=\frac{\left[x_{1}\right]\left[x_{2}\right]+\left[y_{1}\right]\left[y_{2}\right]}{\left[x_{2}\right]^{2}+\left[y_{2}\right]^{2}}+i \frac{\left[y_{1}\right]\left[x_{2}\right]-\left[x_{1}\right]\left[y_{2}\right]}{\left[x_{2}\right]^{2}+\left[y_{2}\right]^{2}}, 0 \notin\left[x_{2}\right]^{2}+\left[y_{2}\right]^{2} .
$$

The division defined above produces a rectangle in the complex plane that is generally far too pessimistic. In general, we have,

$$
\frac{Z_{1}}{Z_{2}} \supset \square\left\{Z_{1} \oslash Z_{2}\right\} .
$$

Consider again the previous example. Then,

$$
\frac{Z_{1}}{Z_{2}}=[0.25,4]+i[-1.5,1.5]
$$

However, the optimal rectangle is (see [8] and Figure 3),

$$
\square\left\{Z_{1} \oslash Z_{2}\right\}=[0.5,2]+i[-0.618028,0.618028] .
$$



Figure 3. Optimal and non-optimal division.

Therefore, the result of the division $Z_{1} \oslash Z_{2}$ has to be approximated (in the sense of covering) by a smallest rectangle.

## 3. Optimal covering rectangle of division

Let $Z_{1}=\left[x_{1}\right]+i\left[y_{1}\right]$ and $Z_{2}=\left[x_{2}\right]+i\left[y_{2}\right]$ be two rectangular intervals. It is known that $Z_{1} \oslash Z_{2}$ is not a rectangle in general but has a complex shape. This section presents a simple and efficient algorithm to calculate the optimal covering rectangle $\square\left\{Z_{1} \oslash Z_{2}\right\}$. The rectangular hull $\square\left\{Z_{1} \oslash Z_{2}\right\}$ contains two parts, namely the imaginary and real parts. For this reason, it is a good idea to split the optimization problem into optimizations of two functions which represent the imaginary and real parts, solve the problems separately and then combine the results.

The procedure to compute $\square\left\{Z_{1} \oslash Z_{2}\right\}$ will be defined in the following fashion. Let $f, g: B \rightarrow \mathbb{R}$ be two real functions such that,

$$
f=\frac{x_{1} x_{2}+y_{1} y_{2}}{x_{2}^{2}+y_{2}^{2}}, g=\frac{y_{1} x_{2}-x_{1} y_{2}}{x_{2}^{2}+y_{2}^{2}}, x_{2}^{2}+y_{2}^{2}>0
$$

where $B=\left[x_{1}\right] \times\left[x_{2}\right] \times\left[y_{1}\right] \times\left[y_{2}\right]$.Thus, $\square\left\{Z_{1} \oslash Z_{2}\right\}$ is given by,

$$
\square\left\{Z_{1} \oslash Z_{2}\right\}=[\min f, \max f]+i[\min g, \max g] .
$$

Before continuing, the following should be pointed out: $f$ and $g$ are continuous on $B$, and $B$ is closed and bounded (compact). Therefore, by the Extreme Value Theorem, it is known that $f$ and $g$ are bounded and attain their maximum and minimum values on $B$. The extreme values (maximum and minimum) can occur either

- on the interior points of $B$ (critical points) or
- on the boundary points of $B$.

For critical points of $f$ the following equations hold,

$$
\begin{align*}
& \frac{\partial f}{\partial x_{1}}=\frac{\left(x_{2}^{2}+y_{2}^{2}\right) x_{2}}{\left(x_{2}^{2}+y_{2}^{2}\right)^{2}}=\frac{x_{2}}{x_{2}^{2}+y_{2}^{2}}=0,  \tag{3.1}\\
& \frac{\partial f}{\partial x_{2}}=\frac{\left(x_{2}^{2}+y_{2}^{2}\right) x_{1}-\left(x_{1} x_{2}+y_{1} y_{2}\right)\left(2 x_{2}\right)}{\left(x_{2}^{2}+y_{2}^{2}\right)^{2}}=0,  \tag{3.2}\\
& \frac{\partial f}{\partial y_{1}}=\frac{\left(x_{2}^{2}+y_{2}^{2}\right) y_{2}}{\left(x_{2}^{2}+y_{2}^{2}\right)^{2}}=\frac{y_{2}}{x_{2}^{2}+y_{2}^{2}}=0,  \tag{3.3}\\
& \frac{\partial f}{\partial y_{2}}=\frac{\left(x_{2}^{2}+y_{2}^{2}\right) y_{1}-\left(x_{1} x_{2}+y_{1} y_{2}\right)\left(2 y_{2}\right)}{\left(x_{2}^{2}+y_{2}^{2}\right)^{2}}=0 . \tag{3.4}
\end{align*}
$$

The Eqs (3.1)-(3.4) yield to $x_{2}=0$ and $y_{2}=0$. However, points of the form $\left(x_{1}, x_{2}, y_{1}, y_{2}\right)=$ $\left(x_{1}, 0, y_{1}, 0\right)$ are not in the domain of $f$. Thus $f$ has no critical points. In the same way it can be verified that the function $g$ has no critical points either.

Boundary points of $B$ : Since $f$ (and also $g$ ) is linear in $x_{1}$ and $y_{1}$, the candidates for the location of the global extreme values occur among the following types of points:

$$
\begin{aligned}
\bullet & P_{1}=\left\{\left(x_{1}, x_{2}, y_{1}, y_{2}\right):\right. \\
x_{1} & \left.\in\left\{x_{1}^{+}, x_{1}^{-}\right\}, x_{2} \in\right] x_{2}[, \\
y_{1} & \left.\in\left\{y_{1}^{+}, y_{1}^{-}\right\}, y_{2} \in\left\{y_{2}^{+}, y_{2}^{-}\right\}\right\} . \\
\bullet P_{2}=\left\{\left(x_{1}, x_{2}, y_{1}, y_{2}\right): x_{1}\right. & \in\left\{x_{1}^{+}, x_{1}^{-}\right\}, x_{2} \in\left\{x_{2}^{+}, x_{2}^{-}\right\}, \\
y_{1} & \left.\in\left\{y_{1}^{+}, y_{1}^{-}\right\}, y_{2} \in\right] y_{2}[ \} . \\
\bullet P_{3}=\left\{\left(x_{1}, x_{2}, y_{1}, y_{2}\right): x_{1}\right. & \in\left\{x_{1}^{+}, x_{1}^{-}\right\}, x_{2} \in\left\{x_{2}^{+}, x_{2}^{-}\right\}, \\
y_{1} & \left.\in\left\{y_{1}^{+}, y_{1}^{-}\right\}, y_{2} \in\left\{y_{2}^{+}, y_{2}^{-}\right\}\right\} .
\end{aligned}
$$

For $i=1,2,3$, let $p_{i \max } \in P_{i}$ and $p_{i \min } \in P_{i}$ denote the candidates for the locations of the global maximum and minimum, respectively. The aim is to determine these candidates in an efficient way.

### 3.1. Fast computation of $\min f$

The value of $\min f$ will be determined by solving the problem,

$$
\min _{B} f\left(x_{1}, x_{2}, y_{1}, y_{2}\right)
$$

with

$$
f=\frac{x_{1} x_{2}+y_{1} y_{2}}{x_{2}^{2}+y_{2}^{2}}
$$

and

$$
B=\left[x_{1}\right] \times\left[x_{2}\right] \times\left[y_{1}\right] \times\left[y_{2}\right] .
$$

Following propositions will be used when determining $p_{1 \text { min }}$ or $p_{2 \text { min }}$ (for the proof see Theorem 1 below):

- If $0 \in\left[y_{2}\right]$, the global minimum of $f$ can not occur at $p_{1 \text { min }} \in P_{1}$.
- If $0 \in\left[x_{2}\right]$, the global minimum of $f$ can not occur at $p_{2 \text { min }} \in P_{2}$.

Determining $p_{1 \text { min }} \in P_{1}$
Solving Eq (3.2) for $x_{2}$ we get

$$
x_{2}=\frac{-y_{1} y_{2} \pm y_{2} \sqrt{x_{1}^{2}+y_{1}^{2}}}{x_{1}}
$$

Note that if $\left.\frac{-y_{1} y_{2} \pm y_{2} \sqrt{x_{1}^{2}+y_{1}^{2}}}{x_{1}} \notin\right] x_{2}\left[\right.$, then $P_{1}=\emptyset$. Suppose that $P_{1} \neq \emptyset, 0 \notin\left[y_{2}\right]$ and $p_{1 \text { min }}=\left(x_{1}, x_{2 \text { min }}, y_{1}, y_{2}\right)$, where ,

$$
x_{2 \min }=\frac{-y_{1} y_{2} \pm y_{2} \sqrt{x_{1}^{2}+y_{1}^{2}}}{x_{1}}
$$

Since,

$$
f\left(p_{1 \min }\right)=\frac{x_{1} x_{2 \min }+y_{1} y_{2}}{x_{2 \min }^{2}+y_{2}^{2}}=\frac{ \pm y_{2} \sqrt{x_{1}^{2}+y_{1}^{2}}}{x_{2 \min }^{2}+y_{2}^{2}}
$$

one has,

$$
\begin{align*}
& x_{2 \text { min }}=\frac{-y_{1}^{-} y_{2}^{-}-y_{2}^{-} \sqrt{x_{1}^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}} \quad \text { if } y_{2}^{-}>0,  \tag{3.5}\\
& x_{2 \text { min }}=\frac{-y_{1}^{+} y_{2}^{+}+y_{2}^{+} \sqrt{x_{1}^{2}+\left(y_{1}^{+}\right)^{2}}}{x_{1}} \quad \text { if } y_{2}^{+}<0 . \tag{3.6}
\end{align*}
$$

The reason of using $y_{1}=y_{1}^{-}$when $y_{2}^{-}>0$ and $y_{1}=y_{1}^{+}$when $y_{2}^{+}<0$ is clear; while the reason of using $y_{2}=y_{2}^{-}$in Eq (3.5) and $y_{2}=y_{2}^{+}$in Eq (3.6) can be explained as follows.

Consider Eq (3.5), i.e., $y_{2}^{-}>0$. If $y_{2}=y_{2}^{+}$is used instead of $y_{2}=y_{2}^{-}$, one gets,

$$
\begin{aligned}
f\left(x_{1}, x_{2 \min }, y_{1}^{-}, y_{2}^{+}\right) & =\frac{-y_{2}^{+} \sqrt{x_{1}^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{2 \min }^{2}+\left(y_{2}^{+}\right)^{2}} \\
& =\frac{-x_{1}^{2} \sqrt{x_{1}^{2}+\left(y_{1}^{-}\right)^{2}}}{2 y_{2}^{+}\left\{x_{1}^{2}+\left(y_{1}^{-}\right)^{2}+y_{1}^{-} \sqrt{x_{1}^{2}+\left(y_{1}^{-}\right)^{2}}\right\}} \\
& >\frac{-x_{1}^{2} \sqrt{x_{1}^{2}+\left(y_{1}^{-}\right)^{2}}}{2 y_{2}^{-}\left\{x_{1}^{2}+\left(y_{1}^{-}\right)^{2}+y_{1}^{-} \sqrt{x_{1}^{2}+\left(y_{1}^{-}\right)^{2}}\right\}}
\end{aligned}
$$

$$
=f\left(x_{1}, x_{2 \min }, y_{1}^{-}, y_{2}^{-}\right)
$$

Consider now Eq (3.6) with $y_{2}=y_{2}^{-}$. Then,

$$
\begin{aligned}
f\left(x_{1}, x_{2 \min }, y_{1}^{+}, y_{2}^{-}\right) & =\frac{y_{2}^{-} \sqrt{x_{1}^{2}+\left(y_{1}^{+}\right)^{2}}}{x_{2 \text { min }}^{2}+\left(y_{2}^{-}\right)^{2}} \\
& =\frac{x_{1}^{2} \sqrt{x_{1}^{2}+\left(y_{1}^{+}\right)^{2}}}{2 y_{2}^{-}\left\{x_{1}^{2}+\left(y_{1}^{+}\right)^{2}-y_{1}^{+} \sqrt{x_{1}^{2}+\left(y_{1}^{+}\right)^{2}}\right\}} \\
& >\frac{x_{1}^{2} \sqrt{x_{1}^{2}+\left(y_{1}^{+}\right)^{2}}}{2 y_{2}^{+}\left\{x_{1}^{2}+\left(y_{1}^{+}\right)^{2}-y_{1}^{+} \sqrt{x_{1}^{2}+\left(y_{1}^{+}\right)^{2}}\right\}} \\
& =f\left(x_{1}, x_{2 \min }, y_{1}^{+}, y_{2}^{+}\right) .
\end{aligned}
$$

Hence, to determine $x_{2 \text { min }}$ by $\operatorname{Eq}(3.5)(\operatorname{Eq}(3.6))$, it has to be used $y_{2}=y_{2}^{-}\left(y_{2}=y_{2}^{+}\right)$.
For the point $p_{1 \text { min }}=\left(x_{1}, x_{2 \min }, y_{1}, y_{2}\right) \in P_{1}$, there are two cases to consider.
Case 1. $y_{2}^{-}>0$.
Using Eq (3.5) one gets,

$$
x_{2 \min }=\frac{-y_{1}^{-} y_{2}^{-}-y_{2}^{-} \sqrt{\left(x_{1}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}}
$$

where $x_{1}$ is chosen as follows:

- If $x_{2}^{-} \geq 0$, we have to use $x_{1}=x_{1}^{-}$. If $\left.x_{2 \text { min }} \in\right] x_{2}\left[\right.$, then $p_{1 \text { min }}=\left(x_{1}^{-}, x_{2 \text { min }}, y_{1}^{-}, y_{2}^{-}\right)$.
- If $x_{2}^{+} \leq 0$, we have to use $x_{1}=x_{1}^{+}$. If $\left.x_{2 \text { min }} \in\right] x_{2}\left[\right.$, then $p_{1 \text { min }}=\left(x_{1}^{+}, x_{2 \text { min }}, y_{1}^{-}, y_{2}^{-}\right)$.
- If $0 \in] x_{2}$ [ there are three cases to distinguish.

1. $x_{1}^{-} \geq 0$. In this case we have to use $x_{1}=x_{1}^{+}$. If $\left.x_{2 \text { min }} \in\right] x_{2}\left[\right.$, then $p_{1 \text { min }}=\left(x_{1}^{+}, x_{2 \text { min }}, y_{1}^{-}, y_{2}^{-}\right)$.
2. $x_{1}^{+} \leq 0$. In this case we have to use $x_{1}=x_{1}^{-}$. If $\left.x_{2 \text { min }} \in\right] x_{2}\left[\right.$, then $p_{1 \text { min }}=\left(x_{1}^{-}, x_{2 \text { min }}, y_{1}^{-}, y_{2}^{-}\right)$.
3. $0 \in] x_{1}\left[\right.$. In this case we have two possibilities, $x_{1}=x_{1}^{-}$and $x_{1}=x_{1}^{+}$. Suppose that,

$$
x_{2 \min 1}=\frac{-y_{1}^{-} y_{2}^{-}-y_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}^{-}}
$$

and

$$
x_{2 \min 2}=\frac{-y_{1}^{-} y_{2}^{-}-y_{2}^{-} \sqrt{\left(x_{1}^{+}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}^{+}}
$$

If both $\left.x_{2 \text { min } 1} \in\right] x_{2}\left[\right.$ and $\left.x_{2 \text { min } 2} \in\right] x_{2}[$, then
$p_{1 \text { min }} \in\left\{\left(x_{1}^{-}, x_{2 \text { min }}, y_{1}^{-}, y_{2}^{-}\right),\left(x_{1}^{+}, x_{2 \text { min }}, y_{1}^{-}, y_{2}^{-}\right)\right\}$such that
$f\left(p_{1 \text { min }}\right)=\min \left(f\left(x_{1}^{-}, x_{2 \min 1}, y_{1}^{-}, y_{2}^{-}\right), f\left(x_{1}^{-}, x_{2 \min 2}, y_{1}^{-}, y_{2}^{-}\right)\right)$.
If $\left.x_{2 \text { min } 1} \in\right] x_{2}\left[\quad\right.$ and $\left.\quad x_{2 \text { min } 2} \notin\right] x_{2}\left[\quad\left(x_{2 \min 1} \notin\right] x_{2}\left[\quad\right.\right.$ and $\left.\quad x_{2 \min 2} \in\right] x_{2}[)$, then $p_{1 \text { min }}=\left(x_{1}^{-}, x_{2 \text { min } 1}, y_{1}^{-}, y_{2}^{-}\right)\left(p_{1 \text { min }}=\left(x_{1}^{+}, x_{2 \text { min } 2}, y_{1}^{-}, y_{2}^{-}\right)\right)$.

Case 2. $y_{2}^{+}<0$.
From Eq (3.6),

$$
x_{2 \min }=\frac{-y_{1}^{+} y_{2}^{+}+y_{2}^{+} \sqrt{\left(x_{1}\right)^{2}+\left(y_{1}^{+}\right)^{2}}}{x_{1}},
$$

where $x_{1}$ is chosen as in Case 1.
Determining $p_{2 \text { min }} \in P_{2}$
From Eq (3.4), one gets

$$
y_{2}=\frac{-x_{1} x_{2} \pm x_{2} \sqrt{x_{1}^{2}+y_{1}^{2}}}{y_{1}}
$$

Suppose that $P_{2} \neq \emptyset, 0 \notin\left[x_{2}\right]$, and $p_{2 \text { min }}=\left(x_{1}, x_{2}, y_{1}, y_{2 \text { min }}\right)$, where,

$$
y_{2 \min }=\frac{-x_{1} x_{2} \pm x_{2} \sqrt{x_{1}^{2}+y_{1}^{2}}}{y_{1}} .
$$

Since,

$$
f\left(p_{2 \text { min }}\right)=\frac{ \pm x_{2} \sqrt{x_{1}^{2}+y_{1}^{2}}}{x_{2}^{2}+y_{2 \text { min }}^{2}}
$$

it can be obtained that,

$$
\begin{align*}
& y_{2 \min }=\frac{-x_{1}^{-} x_{2}^{-}-x_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+y_{1}^{2}}}{y_{1}} \quad \text { if } x_{2}^{-}>0  \tag{3.7}\\
& y_{2 \text { min }}=\frac{-x_{1}^{+} x_{2}^{+}+x_{2}^{+} \sqrt{\left(x_{1}^{+}\right)^{2}+y_{1}^{2}}}{y_{1}} \quad \text { if } x_{2}^{+}<0, \tag{3.8}
\end{align*}
$$

where $y_{1}$ is chosen as $x_{1}$ in computing $x_{2 \text { min }}$.
Determining $p_{3 \text { min }} \in P_{3}$
The set $P_{3}$ consists of all extreme (corner) points of $B$, and there are 16 of such points, in general. Since $f$ is linear in $x_{1}$ and $y_{1}$, we can find the point $p_{3 \text { min }}$ by considering only four points of $P_{3}$. These points are:

$$
\left\{\left(x_{1}, x_{2}^{-}, y_{1}, y_{2}^{-}\right),\left(x_{1}, x_{2}^{-}, y_{1}, y_{2}^{+}\right),\left(x_{1}, x_{2}^{+}, y_{1}, y_{2}^{-}\right),\left(x_{1}, x_{2}^{+}, y_{1}, y_{2}^{+}\right)\right\},
$$

where the choice of $x_{1}\left(y_{1}\right)$ depends on sign of $x_{2}\left(y_{2}\right)$. That is, $x_{1}=x_{1}^{-}$if $x_{2} \geq 0$ and $x_{1}=x_{1}^{+}$otherwise.
All findings are summarized in the following theorem.
Theorem 1. 1. If $0 \in\left[y_{2}\right]$ then
1.1. if $y_{2}^{-}=0$ or $y_{2}^{+}=0$, then $\left.x_{2 \text { min }} \notin\right] x_{2}[$,
1.2. if $y_{2}^{-}=0\left(\right.$ or $\left.y_{2}^{+}=0\right)$ and $\left.y_{2 \text { min }} \in\right] y_{2}[$, then
$\min f=f\left(p_{2 \text { min }}\right)$,

## 1.3. if $0 \in] y_{2}[$, then

$$
\text { 1.3.1. if } \left.y_{1}^{-} \geq 0\left(\text { or } y_{1}^{+} \leq 0\right) \text { and } y_{2 \min } \in\right] y_{2}[\text {, then }
$$

$\min f=f\left(p_{2 \text { min }}\right)$,
1.3.2. if $0 \in] y_{1}\left[\right.$ and both $\left.y_{2 \min 1}, y_{2 \min 2} \in\right] y_{2}[$, then
$\min f=f\left(p_{2 \text { min }}\right)$,
1.3.3. if $0 \in] y_{1}\left[\right.$ and one of $y_{2 \min 1}, y_{2 \min 2}$ lies in $\left[y_{2}\right]$, then $\min f=\min \left(f\left(p_{2 \min }\right), f\left(p_{3 \text { min }}\right)\right)$. 2. If $0 \in\left[x_{2}\right]$ then
2.1. if $x_{2}^{-}=0$ or $x_{2}^{+}=0$, then $\left.y_{2 \min } \notin\right] y_{2}[$,
2.2. if $x_{2}^{-}=0\left(\right.$ or $\left.x_{2}^{+}=0\right)$ and $\left.x_{2 \min } \in\right] x_{2}\left[\right.$, then $\min f=f\left(p_{1 \text { min }}\right)$,
2.3. if $0 \in] x_{2}[$, then
2.3.1. if $x_{1}^{-} \geq 0\left(\right.$ or $\left.x_{1}^{+} \leq 0\right)$ and $\left.x_{2 \min } \in\right] x_{2}[$, then
$\min f=f\left(p_{1 \text { min }}\right)$,

$$
\text { 2.3.2. if } 0 \in] x_{1}\left[\text { and both } x_{2 \min 1}, x_{2 \min 2} \in\right] x_{2}[\text {, then }
$$

$\min f=f\left(p_{1 \text { min }}\right)$,
2.3.3. if $0 \in] x_{1}\left[\right.$ and one of $x_{2 \min 1}, x_{2 \min 2}$ lies in $\left[x_{2}\right]$, then $\min f=\min \left(f\left(p_{1 \text { min }}\right), f\left(p_{3 \text { min }}\right)\right)$.
3. if $0 \notin\left[y_{2}\right]$ and $0 \notin\left[x_{2}\right]$, then
3.1. if $\left.x_{2 \text { min }} \in\right] x_{2}\left[\right.$, then $\left.y_{2 \text { min }} \notin\right] y_{2}\left[\right.$ and $\min f=f\left(p_{1 \text { min }}\right)$,
3.2. if $\left.y_{2 \text { min }} \in\right] y_{2}\left[\right.$, then $\left.x_{2 \text { min }} \notin\right] x_{2}\left[\right.$ and $\min f=f\left(p_{2 \text { min }}\right)$.
4. If $\left.x_{2 \text { min }} \notin\right] x_{2}$ and $\left.y_{2 \text { min }} \notin\right] y_{2}\left[\right.$, then $\min f=f\left(p_{3 \text { min }}\right)$.

Proof. Parts 1.1, 1.3.1 and 3.1 will be proven; the other parts follow by similar arguments.
1.1. Suppose that $y_{2}^{-}=0$ or $y_{2}^{+}=0$. Then $x_{2 \text { min }}=0$, which is impossible.
1.3.1. Let $0 \in] y_{2}\left[\right.$ and $\left.y_{2 \min } \in\right] y_{2}\left[\right.$. Then one must have $x_{2}^{-}>0$ or $x_{2}^{+}<0$. It is sufficient to show that $f\left(p_{2 \text { min }}\right)<f(p), p \in\left\{p_{1 \text { min }}, p_{3 \text { min }}\right\}$.

If $x_{2}^{-}>0$ then, using Eq (3.7), one obtains,

$$
y_{2 \min }=\frac{-x_{1}^{-} x_{2}^{-}-x_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+y_{1}^{2}}}{y_{1}} .
$$

Suppose that $y_{2 \text { min }}<0$, then it must be true that $y_{1}>0$, because $-x_{1}^{-} x_{2}^{-}-x_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+y_{1}^{2}}<0$. This means that $y_{1}=y_{1}^{+}$, and hence $p_{2 \text { min }}=\left(x_{1}^{-}, x_{2}^{-}, y_{1}^{+}, y_{2 \text { min }}\right)$. Using Eq (3.6) with $y_{2}=y_{2}^{-}$, it can be obtained that,

$$
x_{2 \min }=\frac{-y_{1}^{+} y_{2}^{-}+y_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{+}\right)^{2}}}{x_{1}^{-}}
$$

Since $-y_{1}^{+} y_{2}^{-}+y_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{+}\right)^{2}}<0$, then $x_{2 \text { min }} \notin\left[x_{2}\right]$ if $x_{1}^{-}>0$. Suppose that $x_{1}^{-}<0$ and $\left.x_{2 \min } \in\right] x_{2}[$, i.e., $p_{1 \text { min }}=\left(x_{1}^{-}, x_{2 \text { min }}, y_{1}^{+}, y_{2}^{-}\right)$. Plugging $p_{1 \text { min }}$ and $p_{2 \text { min }}$ into the function $f$ one gets

$$
\begin{aligned}
& f\left(p_{1 \min }\right)=\frac{y_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{+}\right)^{2}}}{x_{2 \min }^{2}+\left(y_{2}^{-}\right)^{2}} \\
& f\left(p_{2 \min }\right)=\frac{-x_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{+}\right)^{2}}}{\left(x_{2}^{-}\right)^{2}+y_{2 \min }^{2}}
\end{aligned}
$$

$\left(x_{2}^{-}\right)^{2}<x_{2 \text { min }}^{2}$ and $y_{2 \text { min }}^{2}<\left(y_{2}^{-}\right)^{2}$ implies that $f\left(p_{2 \text { min }}\right)<f\left(p_{1 \text { min }}\right)$. Now suppose that $y_{2 \text { min }}>0$, i.e., $y_{1}=y_{1}^{-}<0$. Then Eq (3.5) with $y_{2}=y_{2}^{+}$, gives

$$
x_{2 \min }=\frac{-y_{1}^{-} y_{2}^{+}-y_{2}^{+} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}^{-}}
$$

Since $-y_{1}^{-}-\sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}<0$, then $\frac{-y_{1}^{-} y_{2}^{-}-y_{2}^{+} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}^{-}}<0$ if $x_{1}^{-}>0$, which means that $\left.x_{2 \text { min }} \notin\right] x_{2}[$. Suppose that $x_{1}^{-}<0$ and $\left.x_{2 \text { min }} \in\right] x_{2}[$. Then,

$$
\begin{aligned}
& f\left(p_{1 \min }\right)=f\left(x_{1}^{-}, x_{2 \min }, y_{1}^{-}, y_{2}^{+}\right)=\frac{-y_{2}^{+} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{2 \min }^{2}+\left(y_{2}^{+}\right)^{2}} \\
& f\left(p_{2 \min }\right)=f\left(x_{1}^{-}, x_{2}^{-}, y_{1}^{-}, y_{2 \min }\right)=\frac{-x_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{\left(x_{2}^{-}\right)^{2}+y_{2 \min }^{2}}
\end{aligned}
$$

On the other hand, $f\left(p_{2 \text { min }}\right)<f\left(p_{1 \text { min }}\right)$ because $\left(x_{2}^{-}\right)^{2}<x_{2 \text { min }}^{2}$ and $y_{2 \text { min }}^{2}<\left(y_{2}^{+}\right)^{2}$. If $x_{2}^{+}<0$ then, using Eq (3.8), one obtains,

$$
y_{2 \text { min }}=\frac{-x_{1}^{+} x_{2}^{+}+x_{2}^{+} \sqrt{\left(x_{1}^{+}\right)^{2}+y_{1}^{2}}}{y_{1}} .
$$

The same computations give that $f\left(p_{2 \text { min }}\right)<f\left(p_{1 \text { min }}\right)$.
Using the fact, $f\left(p_{2 \text { min }}\right) \leq f\left(p_{2}\right)$ for all $p_{2} \in P_{2}$, it can be observed that $f\left(p_{2 \text { min }}\right) \leq f(p)$ for any $p \in B$. 3.1. Let $0 \notin\left[y_{2}\right], 0 \notin\left[x_{2}\right]$ and $\left.x_{2 \text { min }} \in\right] x_{2}[$. This yields four cases: $\left\{x_{2}^{-}>0, y_{2}^{-}>0\right\},\left\{x_{2}^{-}>0, y_{2}^{+}<0\right\},\left\{x_{2}^{+}<0, y_{2}^{-}>0\right\},\left\{x_{2}^{+}<0, y_{2}^{+}<0\right\}$.
The case $\left\{x_{2}^{-}>0, y_{2}^{-}>0\right\}$ will be proven here, the proof for other cases can be given analogously.
Suppose that $x_{2}^{-}>0$ and $y_{2}^{-}>0$. Then, from Eq (3.5), it is known that,

$$
x_{2 \min }=\frac{-y_{1}^{-} y_{2}^{-}-y_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}^{-}}
$$

Since $-y_{1}^{-} y_{2}^{-}-y_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}<0$ and $\left.x_{2 \min } \in\right] x_{2}\left[\right.$, it can be concluded that $x_{1}^{-}<0$. From Eq (3.7), one has,

$$
y_{2 \min }=\frac{-x_{1}^{-} x_{2}^{-}-x_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(x_{1}^{-}\right)^{2}}}{y_{1}^{-}} .
$$

Since $-x_{1}^{-} x_{2}^{-}-x_{2}^{-} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(x_{1}^{-}\right)^{2}}<0$, then if $y_{1}^{-}>0$ implies that $y_{2 \text { min }}<0$, which means that $y_{2 \text { min }} \notin$ $\left[y_{2}\right]$. If $y_{1}^{-}<0$, then

$$
0<\frac{-y_{1}^{-}-\sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}^{-}}<1
$$

From this it follows that,

$$
x_{2 \min }=y_{2}^{-}\left(\frac{-y_{1}^{-}-\sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}^{-}}\right)<y_{2}^{-},
$$

which means that $x_{2}^{-}<y_{2}^{-}$. Moreover, if $y_{1}^{-}<0$,

$$
0<\frac{-x_{1}^{-}-\sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{y_{1}^{-}}<1 .
$$

That is，

$$
y_{2 \min }=x_{2}^{-}\left(\frac{-x_{1}^{-}-\sqrt{\left(x_{1}^{-}\right)^{2}+\left(x_{1}^{-}\right)^{2}}}{y_{1}^{-}}\right)<x_{2}^{-}<y_{2}^{-} .
$$

This proves $y_{2 \text { min }} \notin\left[y_{2}\right]$ ．
The claim that $f\left(p_{1 \text { min }}\right)<f\left(p_{3 \text { min }}\right)$ is obvious．
The computation of $\max f, \min g$ and max $g$ can be analyzed in a similar fashion．The proofs are omitted in order to keep the paper readable．All results are presented in the following algorithms．

| ```Algorithm 1. min f if 0}0\in[\mp@subsup{x}{2}{} if 0\in] 利[ and 0\in] 秝[ if }\mp@subsup{x}{2\mathrm{ min 1 }}{}\in]\mp@subsup{x}{2}{}[\mathrm{ and }\mp@subsup{x}{2\mathrm{ min 2 }}{}\in]\mp@subsup{x}{2}{} min}f=f(\mp@subsup{p}{1 min}{m} else if }\mp@subsup{x}{2\mathrm{ min }1}{}\in]\mp@subsup{x}{2}{}[\mathrm{ or }\mp@subsup{x}{2\mathrm{ min 2 }}{}\in]\mp@subsup{x}{2}{} min}f=\operatorname{min}(f(\mp@subsup{p}{1\mathrm{ min }}{}),f(\mp@subsup{p}{3\mathrm{ min }}{}) else min}f=f(\mp@subsup{p}{3\mathrm{ min }}{} else if }\mp@subsup{x}{2\mathrm{ min }}{}\in]\mp@subsup{x}{2}{} min}f=f(\mp@subsup{p}{1 min}{m} else min}f=f(\mp@subsup{p}{3\mathrm{ min }}{} else if 0}\in[\mp@subsup{y}{2}{} if 0\in] [ [ [ and 0\in] y [ if }\mp@subsup{y}{2\mathrm{ min 1 }}{}\in]\mp@subsup{y}{2}{}[\mathrm{ and }\mp@subsup{y}{2min2}{2}\in]\mp@subsup{y}{2}{} min}f=f(\mp@subsup{p}{2}{2min} else if }\mp@subsup{y}{2\mathrm{ min }1}{}\in]\mp@subsup{y}{2}{}[\mathrm{ or }\mp@subsup{y}{2min2}{2}\in]\mp@subsup{y}{2}{} min}f=\operatorname{min}(f(\mp@subsup{p}{2}{}\operatorname{min}),f(\mp@subsup{p}{3\mathrm{ min }}{}) else min}f=f(\mp@subsup{p}{3\mathrm{ min }}{} else if }\mp@subsup{y}{2\mathrm{ min }}{}\in]\mp@subsup{y}{2}{} min}f=f(\mp@subsup{p}{2}{}\operatorname{min} else min}f=f(\mp@subsup{p}{3\mathrm{ min }}{} else if }\mp@subsup{x}{2\mathrm{ min }}{}\in]\mp@subsup{x}{2}{} min}f=f(\mp@subsup{p}{1 min }{m else if }\mp@subsup{y}{2\mathrm{ min }}{}\in]\mp@subsup{y}{2}{[ min}f=f(\mp@subsup{p}{2\mathrm{ min }}{} else min}f=f(\mp@subsup{p}{3\mathrm{ min }}{})``` | ```Algorithm 2. max f if 0\in[x2] if 0\in] \mp@subsup{x}{1}{}[\mathrm{ and 0 }0]\mp@subsup{x}{2}{}[ if }\mp@subsup{x}{2\mathrm{ max 1 }}{}\in]\mp@subsup{x}{2}{}[\mathrm{ and }\mp@subsup{x}{2 max 2 }{*}]\mp@subsup{x}{2}{} max f=f( (p1 max ) else if }\mp@subsup{x}{2\mathrm{ max 1 }}{}\in]\mp@subsup{x}{2}{}[\mathrm{ or }\mp@subsup{x}{2\mathrm{ max 2 }}{}\in]\mp@subsup{x}{2}{} max}f=\operatorname{max}(f(\mp@subsup{p}{1 max}{*}),f(\mp@subsup{p}{3\mathrm{ max }}{}) else max f=f(p3max ) else if }\mp@subsup{x}{2\mathrm{ max }}{}\in]\mp@subsup{x}{2}{[ max }f=f(\mp@subsup{p}{1 max}{m} else max f=f(p3\operatorname{max}} else if 0\in[y2] if 0\in] [ [ [ and 0\in] y [ if }\mp@subsup{y}{2\mathrm{ max 1 }}{}\in]\mp@subsup{y}{2}{}[\mathrm{ and }\mp@subsup{y}{2\mathrm{ max 2 }}{}\in]\mp@subsup{y}{2}{} max f=f( (p2max ) else if }\mp@subsup{y}{2\operatorname{max 1}}{}\in]\mp@subsup{y}{2}{}[\mathrm{ or }\mp@subsup{y}{2\mathrm{ max 2 }}{}\in]\mp@subsup{y}{2}{} max}f=\operatorname{max}(f(\mp@subsup{p}{2\mathrm{ max }}{}),f(\mp@subsup{p}{3\mathrm{ max }}{}) else max }f=f(\mp@subsup{p}{3\mathrm{ max }}{} else if }\mp@subsup{y}{2\mathrm{ max }}{}\in]\mp@subsup{y}{2}{} max f=f(p2max} else max f=f(p3max ) else if }\mp@subsup{x}{2\mathrm{ max }}{}\in]\mp@subsup{x}{2}{[ max}f=f(\mp@subsup{p}{1 max}{m} else if }\mp@subsup{y}{2\mathrm{ max }}{}\in]\mp@subsup{y}{2}{} max f=f(p (pmax ) else max f}=f(\mp@subsup{p}{3\mathrm{ max }}{})``` |
| :---: | :---: |
| ```Algorithm 3. \(\min g\) if \(0 \in\left[x_{2}\right]\) if \(0 \in] y_{1}[\) and \(0 \in] x_{2}[\) if \(\left.x_{2 \text { min } 1} \in\right] x_{2}\left[\right.\) and \(\left.x_{2 \text { min } 2} \in\right] x_{2}[\) \(\min g=g\left(p_{1 \text { min }}\right)\) else if \(\left.x_{2 \min 1} \in\right] x_{2}\left[\right.\) or \(\left.x_{2 \min 2} \in\right] x_{2}[\) \(\min g=\min \left(g\left(p_{1 \text { min }}\right), g\left(p_{3 \text { min }}\right)\right)\) else \(\min g=g\left(p_{3 \text { min }}\right)\) else if \(\left.x_{2 \text { min }} \in\right] x_{2}\) [ \(\min g=g\left(p_{1 \text { min }}\right)\) else \(\min g=g\left(p_{3 \text { min }}\right)\) else if \(0 \in\left[y_{2}\right]\) if \(0 \in] x_{1}[\) and \(0 \in] y_{2}[\) if \(\left.y_{2 \min 1} \in\right] y_{2}\left[\right.\) and \(\left.y_{2 \min 2} \in\right] y_{2}[\) \(\min g=g\left(p_{2 \text { min }}\right)\) else if \(\left.y_{2 \min 1} \in\right] y_{2}\left[\right.\) or \(\left.y_{2 \min 2} \in\right] y_{2}[\) \(\min g=\min \left(g\left(p_{2 \text { min }}\right), g\left(p_{3 \text { min }}\right)\right)\) else \(\min g=g\left(p_{3 \text { min }}\right)\) else if \(\left.y_{2 \text { min }} \in\right] y_{2}[\) \(\min g=g\left(p_{2} \min \right)\) else \(\min g=g\left(p_{3 \text { min }}\right)\) else if \(\left.x_{2 \text { min }} \in\right] x_{2}\) [ \(\min g=g\left(p_{1 \text { min }}\right)\) else if \(\left.y_{2 \text { min }} \in\right] y_{2}[\) \(\min g=g\left(p_{2 \text { min }}\right)\) else \(\min g=g\left(p_{3 \text { min }}\right)\).``` | ```Algorithm 4. max g if 0\in[x2] if 0\in] y [ and 0\in] x [ if }\mp@subsup{x}{2\mathrm{ max }1}{\in]}\mp@subsup{x}{2}{}[\mathrm{ and }\mp@subsup{x}{2\mathrm{ max 2 }}{2}\in]\mp@subsup{x}{2}{} max g=g(\mp@subsup{p}{1 max}{*}) else if }\mp@subsup{x}{2\mathrm{ max 1 }}{}\in]\mp@subsup{x}{2}{}[\mathrm{ or }\mp@subsup{x}{2\mathrm{ max }2}{}\in]\mp@subsup{x}{2}{} max g}=\operatorname{max}(g(\mp@subsup{p}{1 max}{*}),g(\mp@subsup{p}{3\mathrm{ max }}{}) else max g=g(p3max} else if }\mp@subsup{x}{2\mathrm{ max }}{}\in]\mp@subsup{x}{2}{} max g=g(plmax ) else max g=g(p ( max ) else if 0\in[y2] if 0 \in] 和[ and 0 \in] y  if }\mp@subsup{y}{2\mathrm{ max }1}{}\in]\mp@subsup{y}{2}{}[\mathrm{ and }\mp@subsup{y}{2\mathrm{ max }2}{}\in]\mp@subsup{y}{2}{} max g=g(p ( max ) else if }\mp@subsup{y}{2\mathrm{ max }1}{}\in]\mp@subsup{y}{2}{}[\mathrm{ or }\mp@subsup{y}{2\mathrm{ max }2}{}\in]\mp@subsup{y}{2}{} max g}=\operatorname{max}(g(\mp@subsup{p}{2\mathrm{ max }}{}),g(\mp@subsup{p}{3\mathrm{ max }}{}) else max g=g(p3max ) else if }\mp@subsup{y}{2 max }{\mathrm{ ml y }}\mp@subsup{y}{2}{[ max g=g(p2max} else max g=g(p3max ) else if }\mp@subsup{x}{2\mathrm{ max }}{}\in]\mp@subsup{x}{2}{} max g=g(p}\mp@subsup{p}{1\mathrm{ max }}{} else if }\mp@subsup{y}{2\mathrm{ max }}{}\in]\mp@subsup{y}{2}{[ max g=g(p ( max ) else max g}=g(\mp@subsup{p}{3\mathrm{ max }}{})``` |

## 4. Numerical results

In this section, two numerical examples are provided to show the efficiency and robustness of the proposed algorithm. A comparison of the proposed algorithm with the existing algorithms in [7] and [8] is included.

In the examples, we will adopt two basic cases:
All optimum points are of type $P_{1}$ or $P_{2}$ (Example 1).
All optimum points are of type $P_{3}$ (Example 2).
All other cases fall between these two cases. Therefore, these two examples represent the worst and best case scenarios in the sense of computation time. They demonstrate the fact that the proposed procedure requires up to 256 times less computation time and never more than the method in [8].

Example 1. Consider the two intervals,

$$
\begin{aligned}
& Z_{1}=\left[x_{1}^{-}, x_{1}^{+}\right]+i\left[y_{1}^{-}, y_{1}^{+}\right]=[-3,4]+i[1,2], \\
& Z_{2}=\left[x_{2}^{-}, x_{2}^{+}\right]+i\left[y_{2}^{-}, y_{2}^{+}\right]=[-4,3]+i[-3,-1] .
\end{aligned}
$$

Computing min $f$

$$
\begin{aligned}
x_{2 \min 1} & =\frac{-y_{1}^{+} y_{2}^{+}+y_{2}^{+} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{+}\right)^{2}}}{x_{1}^{-}}=\frac{2-\sqrt{(-3)^{2}+2^{2}}}{(-3)} \\
& =0.535183758487996 \in] x_{2}[ \\
f\left(x_{1}^{-}, x_{2 \min 1}, y_{1}^{+}, y_{2}^{+}\right) & =-2.802775637731994 \\
x_{2 \min 2} & =\frac{-y_{1}^{+} y_{2}^{+}+y_{2}^{+} \sqrt{\left(x_{1}^{+}\right)^{2}+\left(y_{1}^{+}\right)^{2}}}{x_{1}^{+}}=\frac{2-\sqrt{4^{2}+2^{2}}}{4} \\
& =-0.618033988749895 \in] x_{2}[ \\
f\left(x_{1}^{+}, x_{2 \min 2}, y_{1}^{+}, y_{2}^{+}\right) & =-3.236067977499790=\min f .
\end{aligned}
$$

Computing max $f$

$$
\begin{aligned}
x_{2 \max 1} & =\frac{-y_{1}^{-} y_{2}^{+}-y_{2}^{+} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}^{-}}=\frac{1+\sqrt{(-3)^{2}+1^{2}}}{-3} \\
& =-1.387425886722793 \in] x_{2}[ \\
f\left(x_{1}^{-}, x_{2 \max 1}, y_{1}^{-}, y_{2}^{+}\right) & =1.081138830084190 \\
x_{2 \max 2} & =\frac{-y_{1}^{-} y_{2}^{+}-y_{2}^{+} \sqrt{\left(x_{1}^{+}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{x_{1}^{+}}=\frac{1+\sqrt{4^{2}+1^{2}}}{4} \\
& =1.280776406404415 \in] x_{2}[ \\
f\left(x_{1}^{+}, x_{2 \max 2}, y_{1}^{-}, y_{2}^{+}\right) & =1.561552812808830=\max f .
\end{aligned}
$$

Computing ming

$$
\begin{aligned}
x_{2 \min 1} & =\frac{x_{1}^{-} y_{2}^{+}+y_{2}^{+} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{y_{1}^{-}}=\frac{3-\sqrt{(-3)^{2}+1^{2}}}{1} \\
& =-0.162277660168380 \in] x_{2}[ \\
g\left(x_{1}^{-}, x_{2 \min 1}, y_{1}^{-}, y_{2}^{+}\right) & =-3.081138830084190 \\
x_{2 \min 2} & =\frac{x_{1}^{-} y_{2}^{+}+y_{2}^{+} \sqrt{\left(x_{1}^{-}\right)^{2}+\left(y_{1}^{+}\right)^{2}}}{y_{1}^{+}}=\frac{3-\sqrt{(-3)^{2}+2^{2}}}{2} \\
& =-0.302775637731995 \in] x_{2}[ \\
g\left(x_{1}^{-}, x_{2 \min 2}, y_{1}^{+}, y_{2}^{+}\right) & =-3.302775637731994=\min g .
\end{aligned}
$$

## Computing max $g$

$$
\begin{aligned}
x_{2 \max 1} & =\frac{x_{1}^{+} y_{2}^{+}-y_{2}^{+} \sqrt{\left(x_{1}^{+}\right)^{2}+\left(y_{1}^{-}\right)^{2}}}{y_{1}^{-}}=\frac{-4+\sqrt{4^{2}+1^{2}}}{1} \\
& =0.123105625617661 \in] x_{2}[ \\
g\left(x_{1}^{+}, x_{2 \max 1}, y_{1}^{+}, y_{2}^{+}\right) & =4.061552812808831 \\
x_{2 \max 2} & =\frac{x_{1}^{+} y_{2}^{+}-y_{2}^{+} \sqrt{\left(x_{1}^{+}\right)^{2}+\left(y_{1}^{+}\right)^{2}}}{y_{1}^{+}}=\frac{-4+\sqrt{4^{2}+2^{2}}}{2} \\
& =0.236067977499790 \in] x_{2}[ \\
g\left(x_{1}^{+}, x_{2 \max 2}, y_{1}^{+}, y_{2}^{+}\right) & =4.236067977499790=\max g .
\end{aligned}
$$

Thus, the optimal rectangle is

$$
\begin{aligned}
\square\left\{Z_{1} \oslash Z_{2}\right\}= & {[-3.23606797749979,1.56155281280883] } \\
& +i[-3.302775637731994,4.23606797749979] \text { (See Figure 4). }
\end{aligned}
$$

If one uses the existing algorithm in [8], a very large numbers of candidates need to be evaluated in order to get the above results. Particularly for the given example it requires computation of 32 stationary points and evaluation of up to 32 function values of $f$ in order to find $\max f$ or $\min f$, and the same amount of computation is required for finding $\max g$ or $\min g$. That sums up to total of about 240 times more computations compared to the proposed method. On the other hand Figure 4 also shows a comparison between the method introduced in this work and the method in [7]. It can be clearly seen that the computation done by using the procedure in [7] yields to a non-optimal solution.


Figure 4. Example 1.

Example 2. Consider the two intervals,

```
\(Z_{1}=\left[x_{1}^{-}, x_{1}^{+}\right]+i\left[y_{1}^{-}, y_{1}^{+}\right]=[1,2]+i[-2,2]\),
\(Z_{2}=\left[x_{2}^{-}, x_{2}^{+}\right]+i\left[y_{2}^{-}, y_{2}^{+}\right]=[2,3]+i[1,2]\).
Using our algorithms, we get,
\(\min f=\min \left(f\left(t_{1}\right), f\left(t_{2}\right), f\left(t_{3}\right), f\left(t_{4}\right)\right)\)
\(=\min (0,0.10,-0.25,-0.0769)=-0.25\)
where,
\(t_{1}=\left(x_{1}^{-}, x_{2}^{-}, y_{1}^{-}, y_{2}^{-}\right), t_{2}=\left(x_{1}^{-}, x_{2}^{+}, y_{1}^{-}, y_{2}^{-}\right), t_{3}=\left(x_{1}^{-}, x_{2}^{-}, y_{1}^{-}, y_{2}^{+}\right), t_{4}=\left(x_{1}^{-}, x_{2}^{+}, y_{1}^{-}, y_{2}^{+}\right)\)
```

$\max f=\max \left(f\left(r_{1}\right), f\left(r_{2}\right), f\left(r_{3}\right), f\left(r_{4}\right)\right)$
$=\max (1.2,0.8,1,0.7692)=1.2$
where,
$r_{1}=\left(x_{1}^{+}, x_{2}^{-}, y_{1}^{+}, y_{2}^{-}\right), r_{2}=\left(x_{1}^{+}, x_{2}^{+}, y_{1}^{+}, y_{2}^{-}\right), r_{3}=\left(x_{1}^{+}, x_{2}^{-}, y_{1}^{+}, y_{2}^{+}\right), r_{4}=\left(x_{1}^{+}, x_{2}^{+}, y_{1}^{+}, y_{2}^{+}\right)$
$\min g=\min \left(g\left(u_{1}\right), g\left(u_{2}\right), g\left(u_{3}\right), g\left(u_{4}\right)\right)$
$=\min (-1.2,-0.8,-1,-0.7692)=-1.2$
where,
$u_{1}=\left(x_{1}^{+}, x_{2}^{-}, y_{1}^{-}, y_{2}^{-}\right), u_{2}=\left(x_{1}^{+}, x_{2}^{+}, y_{1}^{-}, y_{2}^{-}\right), u_{3}=\left(x_{1}^{+}, x_{2}^{-}, y_{1}^{-}, y_{2}^{+}\right), u_{4}=\left(x_{1}^{+}, x_{2}^{+}, y_{1}^{-}, y_{2}^{+}\right)$
$\max g=\max \left(g\left(v_{1}\right), g\left(v_{2}\right), g\left(v_{3}\right), g\left(v_{4}\right)\right)$
$=\max (0.6,0.5,0.25,0.3077)=0.6$
where,
$v_{1}=\left(x_{1}^{-}, x_{2}^{-}, y_{1}^{+}, y_{2}^{-}\right), v_{2}=\left(x_{1}^{-}, x_{2}^{+}, y_{1}^{+}, y_{2}^{-}\right), v_{3}=\left(x_{1}^{-}, x_{2}^{-}, y_{1}^{+}, y_{2}^{+}\right), v_{4}=\left(x_{1}^{-}, x_{2}^{+}, y_{1}^{+}, y_{2}^{+}\right)$
In this case our algorithm and the existing one in [8] reach the optimal results in the same amount of computation.


Figure 5. Example 2.

## 5. Conclusions

In this paper, complex interval arithmetic using the rectangular form is re-visited. Addition, subtraction and multiplication operations can be performed by using well-known interval arithmetic rules flawlessly. But for the division operation, the optimal rectangular hull problem has not a trivial solution. A fast and accurate way for calculating the optimal rectangular hull is derived and expressed as a simple algorithm. The efficiency of the algorithm is observed by comparison with its ancestors. It is worth noting that although the method is introduced in the complex plane, it can be applied to all two dimensional interval arithmetic problems.

There are many modern applications of interval analysis such as [14-17] in which the method introduced in this paper can be used whenever division is involved. Another possible field of application of the method in this paper is the uncertainty inverse problem where the uncertain parameters are modeled with intervals [18-22]. For instance the method can be successfully implemented in the approach of [18] whenever there are two uncertain parameters in the model.

For future work, the arithmetic where the complex intervals are taken as sectors is a topic that could be improved. Another interesting research area for future work is the multidimensional parallelepiped model for structural uncertainty analysis [23,24]. In [23] the authors unify dependent and independent uncertain variables and as a result the domain forms a multidimensional parallelepiped. In the 2-D case the model results in a parallelogram. The investigation of the arithmetic of parallelogram-valued (or even higher dimension parallelepipeds-valued) quantities appears to be a challenging problem. The solution might possibly be a modification of the method introduced in this paper.

## Conflict of interest

All authors declare no conflicts of interest in this paper

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