Mathematics

## Research article

# On stability and instability of standing waves for the inhomogeneous fractional Schrödinger equation 

Jiayin Liu*

School of Mathematics and Information Science, North Minzu University, Yinchuan, 750021, China

* Correspondence: Email: xecd@163.com; Tel: +8618895079120; Fax: +8609512067916.


#### Abstract

In this paper, we consider the stability and instability of standing waves for the


 inhomogeneous fractional Schrödinger equation$$
i \partial_{t} \psi=(-\Delta)^{s} \psi-|x|^{-b}|\psi|^{2 p} \psi
$$

By applying the profile decomposition of bounded sequences in $H^{s}$ and variational methods, in the $L^{2}$-subcritical case, i.e., $0<p<\frac{4 s-2 b}{N}$, we prove that the standing waves are orbitally stable. In the $L^{2}$-critical case, i.e., $p=\frac{4 s-2 b}{N}$, we show that the standing waves are strongly unstable by blow-up.

Keywords: inhomogeneous fractional Schrödinger equation; stability; instability Mathematics Subject Classification: 35Q55, 35B44

## 1. Introduction

In recent years, there has been a great deal of interest in using fractional Laplacians to model the physical phenomena. By extending the Feynman path integral from the Brownian-like to the Lévylike quantum mechanical paths, Laskin in $[1,2]$ deduce the following fractional nonlinear Schrödinger equation (NLS)

$$
\begin{equation*}
i \partial_{t} \psi=(-\Delta)^{s} \psi+f(\psi), \tag{1.1}
\end{equation*}
$$

where $0<s<1$ and $f(\psi)$ is the nonlinearity. The fractional differential operator $(-\Delta)^{s}$ is defined by $(-\Delta)^{s} \psi=\mathcal{F}^{-1}\left[|\xi|^{2 s} \mathcal{F}(\psi)\right]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and inverse Fourier transform, respectively.

Recently, equations of this type received much attention, see [3-18] for power-type nonlinearity $|\psi|^{p} \psi$, [19-24] for the Hartree-type nonlinearity $\left(|x|^{-\gamma} *|\psi|^{2}\right) \psi$, and [19, 20, 25-29] for the Choquard -type nonlinearity $\left(I_{\alpha} *|\psi|^{p}\right)|\psi|^{p-2} \psi$, where the symbol $*$ denotes the convolution.

In this paper, we consider the stability and instability of standing waves for the following inhomogeneous fractional Schrödinger equation

$$
\left\{\begin{array}{l}
i \partial_{t} \psi=(-\Delta)^{s} \psi-|x|^{-b}|\psi|^{p} \psi,(t, x) \in[0, T) \times \mathbb{R}^{N},  \tag{1.2}\\
\psi(0, x)=\psi_{0}(x),
\end{array}\right.
$$

where $\psi(t, x):[0, T) \times \mathbb{R}^{N} \rightarrow \mathbb{C}$ is the complex valued function, $x=\left(x_{1}, x_{2}, \cdots, x_{n}\right), \psi_{0} \in H^{s}$, $0<s<1,0<T \leq \infty, 0<b<\min \{2 s, N\}$.

Equation (1.2) arises various physical contexts in the description of nonlinear waves such as propagation of a laser beam and plasma waves. When $b=0$, it appears in nonlinear optics, plasma physics, and fluid mechanics, see [30]. When $b<0$, it can be considered as modeling inhomogeneities in the medium. The inhomogeneous nonlinearity arises due to the effect of changes in the field intensity on the wave propagation characteristics of the medium and the nonlinear weight can be looked as the proportional to the electron density, see $[31,32]$.

Note that if $\psi(t, x)$ is a solution of (1.2), then

$$
\psi_{\lambda}(t, x)=\lambda^{\frac{2 s-b}{p}} \psi\left(\lambda^{2 s} t, \lambda x\right) \text { for all } \lambda>0
$$

is also a solution of (1.2). Computing the homogeneous Sobolev norm, we have

$$
\left\|\psi_{\lambda}(t)\right\|_{\dot{H}^{s}}=\lambda^{s+\frac{2 s-b}{p}-\frac{N}{2}}\|\psi(t)\|_{\dot{H}^{s}} .
$$

Thus, the critical Sobolev index is given by $s_{c}=\frac{N}{2}-\frac{2 s-b}{p}$. When $s_{c}<0$, equation (1.2) is $L^{2}-$ subcritical. The smallest power for which blow-up may occur is $p=\frac{4 s-2 b}{N}$, which is referred to $L^{2}$-critical case corresponding to $s_{c}=0$. When $0<s_{c}<s$, (1.2) is $L^{2}$-supercritical and $H^{s}$-subcritical. When $s_{c}=s$, (1.2) is $H^{s}$-critical. In this paper, we are interested in the $L^{2}$-subcritical and critical cases. Therefore, we restrict our attention to the case $s_{c} \leq 0$. Rewriting this last condition in terms of $p$, we obtain

$$
p \in\left(0, \frac{4 s-2 b}{N}\right] .
$$

To avoid $p$ to be negative, we assume the technical restriction $0<b<\min \{2 s, N\}$.
When $s=1$, the well-posedness for (1.2) was first studied by Genoud-Stuart in [ [33], Appendix] by using the argument of Cazenave [34], see also [35]. Farah in [36] established a Gagliardo-Nirenberg type estimate and use it to obtain sufficient conditions for global existence and blow-up in $H^{1}$. Afterwards, Farah and Guzman in [37,38] proved that the above global solution is scattering under the radial condition of the initial data. Dinh in [39] established several blow-up criteria for (1.2) with $s=1$.

When $s<1$ and $b<0$, Saanouni in [27] studied the well-posedness issues and stability of standing waves for (1.2). But to the best of our knowledge, there are no any results about the stability and instability of standing waves for (1.2) with $b>0$. The main purpose of this paper is to study the stability and instability of standing waves for (1.2).

The standing waves of (1.2) are solutions of the form $e^{i \omega t} u$, where $\omega \in \mathbb{R}$ is a frequency and $u \in H^{s}$ is a nontrivial solution to the elliptic equation

$$
\begin{equation*}
(-\Delta)^{s} u+\omega u-|x|^{-b}|u|^{p} u=0 . \tag{1.3}
\end{equation*}
$$

Note also that (1.3) can be written as $S_{\omega}^{\prime}(u)=0$, where $S_{\omega}(u)$ is the action functional defined by

$$
\begin{equation*}
S_{\omega}(u):=E(u)+\frac{\omega}{2}\|u\|_{L^{2}}^{2}, \tag{1.4}
\end{equation*}
$$

where the energy functional $E(u)$ is defined by

$$
\begin{equation*}
E(u)=\frac{1}{2}\|u\|_{\dot{H}^{s}}^{2}-\frac{1}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}|u(x)|^{p+2} d x . \tag{1.5}
\end{equation*}
$$

We denote the set of non-trivial solutions of (1.3) by

$$
\mathcal{A}_{\omega}:=\left\{u \in H^{s} \backslash\{0\}: \quad S_{\omega}^{\prime}(u)=0\right\} .
$$

Definition 1.1. (Ground states). A function $u \in \mathcal{A}_{\omega}$ is called a ground state for (1.3) if it is a minimizer of $S_{\omega}$ over the set $\mathcal{A}_{\omega}$. The set of ground states is denoted by $\mathcal{G}_{\omega}$. In particular,

$$
\mathcal{G}_{\omega}=\left\{u \in \mathcal{A}_{\omega}, \quad S_{\omega}(u) \leq S_{\omega}(v), \forall v \in \mathcal{A}_{\omega}\right\} .
$$

The first part of this paper concerns the stability of standing waves in the $L^{2}$-subcritical case $0<$ $p<\frac{4 s-2 b}{N}$. To this end, applying the ideas by Cazenave and Lions in [40], for $c>0$, we consider the following minimizing problem

$$
\begin{equation*}
m(c)=\inf \left\{E(v): v \in H^{s},\|v\|_{L^{2}}^{2}=c\right\} . \tag{1.6}
\end{equation*}
$$

We will see later (Lemma 3.2) that the above minimizing problem is well-defined. Moreover, all minimizing sequences of (1.6) are pre-compact, and then the above infimum is attained. Let us denote

$$
\mathcal{M}_{c}:=\left\{u: E(u)=m(c),\|u\|_{L^{2}}^{2}=c\right\} .
$$

By the Euler-Lagrange Theorem, we see that if $u \in \mathcal{M}_{c}$, then there exists $\omega>0$ such that $u$ is a solution of (1.3). Note also that if $u$ is a solution of (1.3), then $\psi(t, x)=e^{i \omega t} u(x)$ is a solution to (1.2). Moreover, if $u \in \mathcal{M}_{c}$, i.e., $u$ is a minimizer of (1.6), then $\left\|e^{i \omega t} u\right\|_{L^{2}}^{2}=\|u\|_{L^{2}}^{2}=c$ and $E\left(e^{i \omega t} u\right)=E(u)=$ $m(c)$ for all $t \geq 0$. Thus, $e^{i \omega t} u$ is also a minimizer of (1.6), i.e., $e^{i \omega t} u \in \mathcal{M}_{c}$. One usually calls $e^{i \omega t} u$ the orbit of $u$.

Our first result is the following orbital stability of standing waves for (1.2) in the $L^{2}$-subcritical case.
Theorem 1.2. Let $N \geq 2, \frac{N}{2 N-1} \leq s<1,0<p<\frac{4 s-2 b}{N}$, and $0<b<\min \{2 s, N\}$. Then, the set $\mathcal{M}_{c}$ is not empty, and it is orbitally stable in the following sense: for any $\varepsilon>0$, there exists $\delta>0$ such that for any initial data $\psi_{0}$ satisfying

$$
\inf _{u \in \mathcal{M}_{c}}\left\|\psi_{0}-u\right\|_{H^{s}}<\delta
$$

the corresponding solution $\psi(t)$ to (1.2) with initial data $\psi_{0}$ satisfies

$$
\inf _{u \in \mathcal{M}_{c}}\|\psi(t)-u\|_{H^{s}}<\varepsilon
$$

for all $t>0$.

Compared with the results in $[19,20,34]$, the main difficulty of the proof is the lack of space translation invariance due to the inhomogeneous nonlinearity $|x|^{-b}|u|^{2 p} u$. The usual method is to compare with the limit equation. However, due to $|x|^{-b} \rightarrow 0$ as $|x| \rightarrow \infty$, we cannot adopt the usual method to study this problem. In order to overcome this difficulty, we apply the profile decomposition in $H^{s}$ to study the compactness of all minimizing sequences. In particular, in order to exclude the vanishing, we must prove the boundedness of the translation sequence. Moreover, one can apply the same argument to prove the stability of standing waves for (1.2) with $s=1$.

Finally, motivated by the ideas in [23, 29, 34, 41-43], we study the strong instability result in the $L^{2}$-critical case $p=\frac{4 s-2 b}{N}$.

Theorem 1.3. Let $N \geq 2, \frac{N}{2 N-1} \leq s<1, p=\frac{4 s-2 b}{N}$, and $0<b<\min \{2 s, N\}$. Then the standing wave $\psi(t, x)=e^{i \omega t} u_{\omega}(x)$ is unstable in the following sense: there exists $\left\{\psi_{0, n}\right\} \subset H^{s}$ such that $\psi_{0, n} \rightarrow u_{\omega}$ in $H^{s}$ as $n \rightarrow \infty$ and the corresponding solution $\psi_{n}$ of (1.2) with initial data $\psi_{0, n}$ blows up in finite or infinite time for any $n \geq 1$.

This paper is organized as follows: in Section 2, we firstly collect some lemmas such as the local well-posedness of (1.2), the profile decomposition of bounded sequences in $H^{s}$. In Section 3, we give the proof of the stability result stated in Theorem (1.2). Finally, we study the strong instability of standing waves in Section 4.

## 2. Preliminaries

In this section, we recall some preliminary results that will be used later. Firstly, let us recall the local theory for the Cauchy problem (1.2). The local well-posedness for (1.2) with $b=0$ in the energy space $H^{s}$ was first studied by Hong and Sire in [22]. The proof is based on Strichartz estimates and the contraction mapping argument. Note that for non-radial data, Strichartz estimates have a loss of derivatives. Fortunately, this loss of derivatives can be compensated by using Sobolev embedding. However, it leads to a weak local well-posedness in the energy space compared to the classical nonlinear Schrödinger equation. We refer the reader to [13, 22] for more details. One can remove the loss of derivatives in Strichartz estimates by considering radially symmetric data. However, it needs a restriction on the validity of $s$, namely $\frac{N}{2 N-1} \leq s<1$. In this paper, we can obtain the following local well-posedness for (1.2) with radial $H^{s}$ initial data. The proof is standard, see [13, 22]. So we omit it.

Theorem 2.1. Let $N \geq 2, \frac{N}{2 N-1} \leq s<1,0<p<\frac{4 s-2 b}{N-2 s}$ and $0<b<\min \{2 s, N\}$. Then for any $\psi_{0} \in H^{s}$ radial, there exists $T=T\left(\left\|\psi_{0}\right\|_{H^{s}}\right)$ such that (1.2) admits a unique solution $\psi \in C\left([0, T], H^{s}\right)$. Let $\left[0, T^{*}\right)$ be the maximal time interval on which the solution $\psi$ is well-defined, if $T^{*}<\infty$, then $\|\psi(t)\|_{\dot{H}^{s}} \rightarrow \infty$ as $t \uparrow T^{*}$. Moreover, for all $0 \leq t<T^{*}$, the solution $\psi(t)$ satisfies the following conservations of mass and energy

$$
\begin{equation*}
M(\psi(t)):=\|\psi(t)\|_{L^{2}}=\left\|\psi_{0}\right\|_{L^{2}}, \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
E(\psi(t)):=\frac{1}{2} \int_{\mathbb{R}^{N}}\left|(-\Delta)^{s / 2} \psi(t, x)\right|^{2} d x-\frac{1}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}|\psi(t, x)|^{p+2} d x=E\left(\psi_{0}\right) . \tag{2.2}
\end{equation*}
$$

Next, we recall the profile decomposition of bounded sequences in $H^{s}$, which has been established in [24].

Lemma 2.2. Let $N \geq 2,0<s<1$. If $\left\{u_{n}\right\}_{n=1}^{\infty}$ is a bounded sequence in $H^{s}$, then there exist a subsequence of $\left\{u_{n}\right\}_{n=1}^{\infty}\left(\right.$ still denoted by $\left.\left\{u_{n}\right\}_{n=1}^{\infty}\right)$, a family $\left\{x_{n}^{j}\right\}_{j=1}^{\infty}$ of sequences in $\mathbb{R}^{N}$ and a sequence $\left\{U^{j}\right\}_{j=1}^{\infty}$ in $H^{s}$ such that
(i) for every $k \neq j,\left|x_{n}^{k}-x_{n}^{j}\right| \rightarrow+\infty$, as $n \rightarrow \infty$;
(ii) for every $l \geq 1$ and every $x \in \mathbb{R}^{N}$, we have

$$
\begin{equation*}
u_{n}(x)=\sum_{j=1}^{l} U^{j}\left(x-x_{n}^{j}\right)+r_{n}^{l}, \tag{2.3}
\end{equation*}
$$

with $\lim \sup _{n \rightarrow \infty}\left\|r_{n}^{l}\right\|_{L^{q}} \rightarrow 0$ as $l \rightarrow \infty$ for every $q \in\left(2, \frac{2 N}{N-2 s}\right)$. Moreover,

$$
\begin{gather*}
\left\|u_{n}\right\|_{L^{2}}^{2}=\sum_{j=1}^{l}\left\|U^{j}\right\|_{L^{2}}^{2}+\left\|r_{n}^{l}\right\|_{L^{2}}^{2}+o(1)  \tag{2.4}\\
\left\|(-\Delta)^{s / 2} u_{n}\right\|_{L^{2}}^{2}=\sum_{j=1}^{l}\left\|(-\Delta)^{s / 2} U^{j}\right\|_{L^{2}}^{2}+\left\|(-\Delta)^{s / 2} r_{n}^{l}\right\|_{L^{2}}^{2}+o(1)  \tag{2.5}\\
\int_{\mathbb{R}^{N}}|x|^{-b}\left|\sum_{j=1}^{l} U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} d x=\sum_{j=1}^{l} \int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} d x+o(1) \tag{2.6}
\end{gather*}
$$

where $\circ(1)=\circ_{n}(1) \rightarrow 0$ as $n \rightarrow \infty$.
Proof. For the proof of (2.3)-(2.5), see [24]. We need only prove (2.6). Using the elementary inequality

$$
\|\left.\sum_{j=1}^{l} a_{j}\right|^{p}-\left.\sum_{j=1}^{l}\left|a_{j}\right|^{p}\left|\leq C \sum_{j \neq k}^{l}\right| a_{j}\right|^{p-1}\left|a_{k}\right|,
$$

and the pair orthogonality of sequence $\left\{x_{n}^{j}\right\}_{j=1}^{\infty}$, we can estimate as follows:

$$
\begin{aligned}
& \left.\left|\int_{\mathbb{R}^{N}}\right| x\right|^{-b}\left|\sum_{j=1}^{l} U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} d x-\sum_{j=1}^{l} \int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} d x \mid \\
= & \left.\int_{\mathbb{R}^{N}}|x|^{-b}| | \sum_{j=1}^{l} U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2}-\sum_{j=1}^{l}\left|U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} \mid d x \\
\leq & C \int_{\mathbb{R}^{N}}|x|^{-b} \sum_{j \neq k}^{l}\left|U^{j}\left(x-x_{n}^{j}\right)\right|^{p+1}\left|U^{k}\left(x-x_{n}^{k}\right)\right| d x \\
= & \left.C \int_{\mathbb{R}^{N}}\left|x+x_{n}^{j}\right|\right|^{-b} \sum_{j \neq k}^{l}\left|U^{j}(x)\right|^{p+1}\left|U^{k}\left(x+x_{n}^{j}-x_{n}^{k}\right)\right| d x \rightarrow 0,
\end{aligned}
$$

as $n \rightarrow \infty$. This completes the proof.

Finally, we recall the following sharp Gagliardo-Nirenberg inequality, which has been established in [44].

Lemma 2.3. [44] Let $0<s<1,0<p<\frac{4 s-2 b}{N-2 s}$ and $0<b<\min \{2 s, N\}$. Then, for all $u \in H^{s}$,

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}|x|^{-b}|u(x)|^{p+2} d x \leq C_{o p t}\|u\|_{H^{s}}^{\frac{N_{p}+2 b}{2 s b}}\|u\|_{L^{2}}^{p+2-\frac{N_{p+2}+2 b}{2 s}}, \tag{2.7}
\end{equation*}
$$

where the best constant $C_{\text {opt }}$ is given by

$$
C_{o p t}=\left(\frac{N p+2 b}{2 s(p+2)-(N p+2 b)}\right)^{\frac{4 s-\left(N_{p}+2 b\right)}{4 s}} \frac{2 s(p+2)}{(N p+2 b)\|Q\|_{L^{2}}^{p}},
$$

where $Q$ is the ground state of (1.3) with $\omega=1$. Moreover, the following Pohozaev's identities hold true:

$$
\begin{equation*}
\|Q\|_{\dot{H}^{s}}^{2}=\frac{N p+2 b}{2 s(p+2)} \int_{\mathbb{R}^{N}}|x|^{-b}|Q|^{p+2} d x=\frac{N p+2 b}{2 s(p+2)-(N p+2 b)}\|Q\|_{L^{2}}^{2} . \tag{2.8}
\end{equation*}
$$

## 3. Orbital stability of standing waves

In order to study the orbital stability of standing waves of (1.2), we firstly establish the following global existence of (1.2) by using (2.1), (2.2), (2.7). The proof is standard, see e.g., [34,45]. So we omit it.

Lemma 3.1. Let $N \geq 2, \frac{N}{2 N-1} \leq s<1,0<p<\frac{4 s-2 b}{N}$, and $0<b<\min \{2 s, N\}$. Then, the solution $\psi$ of (1.2) exists globally.

Next, by applying the profile decomposition of bounded sequences in $H^{s}$, we can solve the variational problem (1.6) and obtain the following result.

Lemma 3.2. Let $N \geq 2,0<s<1,0<p<\frac{2 s-b}{N}$, and $0<b<\min \{2 s, N\}$. Then, there exists $u_{0} \in H^{s}$ such that $m(c)=E\left(u_{0}\right)$.

Proof. Firstly, we prove that the minimizing problem (1.6) is well-defined and there exists $C_{0}>0$ such that

$$
\begin{equation*}
m(c) \leq-C_{0}<0 . \tag{3.1}
\end{equation*}
$$

We deduce from the inequality (2.7) that there exists a constant $C\left(\|Q\|_{L^{2}}\right)$ such that

$$
\begin{aligned}
E(u) & =\frac{1}{2} \int_{\mathbb{R}^{N}}\left|(-\Delta)^{s / 2} u\right|^{2} d x-\frac{1}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}|u(x)|^{p+2} d x \\
& \geq \frac{1}{2}\left\|(-\Delta)^{s / 2} u\right\|_{L^{2}}^{2}-C\left(\|Q\|_{L^{2}}\right)\left\|(-\Delta)^{s / 2} u\right\|_{L^{2}}^{\frac{N_{p}+2 b}{2 s}}\|u\|_{L^{2}}^{p+2-\frac{N_{p+2 b}}{2 s}}
\end{aligned}
$$

Since $p<\frac{4 s-2 b}{N}$, it follows that $\frac{N p+2 b}{2 s}<2$. Thus, we deduce from the Young inequality that for all $0<\varepsilon<\frac{1}{2}$, there exists a constant $C\left(\varepsilon,\|Q\|_{L^{2}}, c\right)$ such that

$$
C\left(\|Q\|_{L^{2}}\right)\left\|(-\Delta)^{s / 2} u\right\|_{L^{2}}^{\frac{N_{p+2 b}}{2 s}}\|u\|_{L^{2}}^{p+2-\frac{N_{p+2 b}}{2 s}} \leq \varepsilon\left\|(-\Delta)^{s / 2} u\right\|_{L^{2}}^{2}+C\left(\varepsilon,\|Q\|_{L^{2}}, c\right) .
$$

This implies that

$$
\begin{equation*}
E(u) \geq\left(\frac{1}{2}-\varepsilon\right)\left\|(-\Delta)^{s / 2} u\right\|_{L^{2}}^{2}-C\left(\varepsilon,\|Q\|_{L^{2}}, c\right) . \tag{3.2}
\end{equation*}
$$

Therefore, $E(u)$ has a lower bound and the variational problem (1.6) is well-defined.
Now, let $u \in H^{s}$ be a fixed function and $\|u\|_{L^{2}}^{2}=c$. Set $u_{\lambda}=\lambda^{\frac{N}{2}} u(\lambda x)$. It follows easily that

$$
\left\|u_{\lambda}\right\|_{L^{2}}^{2}=\|u\|_{L^{2}}^{2}=c,
$$

and

$$
E\left(u_{\lambda}\right)=\frac{\lambda^{2 s}}{2} \int_{\mathbb{R}^{N}}\left|(-\Delta)^{s / 2} u\right|^{2} d x-\frac{\lambda^{\frac{N_{p+b}}{2}}}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}|u(x)|^{p+2} d x .
$$

Since $N p+b<N p+2 b<4 s$, we can choose $\lambda>0$ sufficiently small such that there exists $C_{0}>0$ such that $E\left(u_{\lambda}\right) \leq-C_{0}<0$. Hence, (3.1) is true.

Secondly, let $\left\{u_{n}\right\}_{n=1}^{\infty}$ be the minimizing sequence of the variational problem (1.6) such that

$$
\begin{equation*}
E\left(u_{n}\right) \rightarrow m(c) \text { and }\left\|u_{n}\right\|_{L^{2}}^{2}=c . \tag{3.3}
\end{equation*}
$$

This implies that for $n$ large enough, $E\left(u_{n}\right)<m(c)+1$. Thus, it follows from (3.2) that for all $0<\varepsilon<\frac{1}{2}$

$$
\left(\frac{1}{2}-\varepsilon\right)\left\|(-\Delta)^{s / 2} u_{n}\right\|_{L^{2}}^{2} \leq m(c)+1+C\left(\varepsilon,\|Q\|_{L^{2}}, c\right) .
$$

This yield that $\left\{u_{n}\right\}_{n=1}^{\infty}$ is bounded in $H^{s}$.
Next, we claim that there exists $C_{1}>0$ such that

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \int_{\mathbb{R}^{N}}|x|^{-b}\left|u_{n}(x)\right|^{p+2} d x \geq C_{1} \tag{3.4}
\end{equation*}
$$

Assume by contradiction that (3.4) is false, we get

$$
0>m(c)=\lim _{n \rightarrow \infty} E\left(u_{n}\right)=\frac{1}{2} \int_{\mathbb{R}^{N}}\left|(-\Delta)^{s / 2} u_{n}\right|^{2} d x \geq 0 .
$$

Thus, we have (3.4).
Thirdly, applying the profile decomposition of bounded sequences in $H^{s}$, we prove that the infimum of the variational problem (1.6) can be attained. Apply Lemma 2.2 to the minimizing sequence $\left\{u_{n}\right\}_{n=1}^{\infty}$. Up to a subsequence, $u_{n}$ can be decomposed as

$$
\begin{equation*}
u_{n}(x)=\sum_{j=1}^{l} U^{j}\left(x-x_{n}^{j}\right)+r_{n}^{l}, \tag{3.5}
\end{equation*}
$$

with $\lim \sup _{n \rightarrow \infty}\left\|r_{n}^{l}\right\|_{L^{q}} \rightarrow 0$ as $l \rightarrow \infty$ for every $q \in\left(2, \frac{2 N}{N-2 s}\right)$.
Now, injecting (3.5) into the energy functional $E\left(u_{n}\right)$, it follows from (2.4)-(2.6) that

$$
\begin{equation*}
E\left(u_{n}\right)=\sum_{j=1}^{l} E\left(U^{j}\left(\cdot-x_{n}^{j}\right)\right)+E\left(r_{n}^{l}\right)+o_{n}(1), \tag{3.6}
\end{equation*}
$$

where $\circ_{n}(1) \rightarrow 0$ as $n \rightarrow \infty$. For every $U^{j}\left(\cdot-x_{n}^{j}\right)(1 \leq j \leq l)$, take the scaling transform $U_{\lambda_{j}}^{j}(x)=$ $\lambda_{j} U^{j}\left(x-x_{n}^{j}\right)$ with $\lambda_{j}=\frac{\sqrt{c}}{\left\|U^{\prime}\right\|_{L^{2}}}$. It follows easily that

$$
\begin{equation*}
\left\|U_{\lambda_{j}}^{j}\right\|_{L^{2}}^{2}=c, \tag{3.7}
\end{equation*}
$$

and

$$
\begin{aligned}
E\left(U_{\lambda_{j}}^{j}\right) & =\frac{\lambda_{j}^{2}}{2} \int_{\mathbb{R}^{N}}\left|(-\Delta)^{s / 2} U^{j}\right|^{2} d x-\frac{\lambda_{j}^{p+2}}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} d x \\
& =\lambda_{j}^{2} E\left(U^{j}\left(\cdot-x_{n}^{j}\right)\right)-\frac{\lambda_{j}^{2}\left(\lambda_{j}^{p}-1\right)}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} d x .
\end{aligned}
$$

This implies that

$$
\begin{equation*}
E\left(U^{j}\left(\cdot-x_{n}^{j}\right)\right)=\frac{E\left(U_{\lambda_{j}}^{j}\right)}{\lambda_{j}^{2}}+\frac{\lambda_{j}^{p}-1}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} d x . \tag{3.8}
\end{equation*}
$$

Similarly, for the term $E\left(r_{n}^{l}\right)$, we can obtain

$$
\begin{align*}
E\left(r_{n}^{l}\right) & =\frac{\left\|r_{n}^{l}\right\|_{L^{2}}^{2}}{c} E\left(\frac{\sqrt{c}}{\left\|r_{n}^{l}\right\|_{L^{2}}} r_{n}^{l}\right)+\frac{\left(\frac{\sqrt{c}}{\|r\|_{n}^{\prime}}\right)^{p}-1}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}\left|r_{n}^{l}(x)\right|^{p+2} d x+o(1) \\
& \geq \frac{\left\|r_{n}^{l}\right\|_{L^{2}}^{2}}{c} E\left(\frac{\sqrt{c}}{\left\|r_{n}^{l}\right\|_{L^{2}}} r_{n}^{l}\right)+o_{n}(1) . \tag{3.9}
\end{align*}
$$

Since $\left\|U_{\lambda_{j}}^{j}\right\|_{L^{2}}^{2}=\left\|\frac{\sqrt{c}}{\left\|r_{n}\right\|_{L^{2}}} r_{n}^{l}\right\|_{L^{2}}^{2}=c$, we deduce from the definition of $m(c)$ that

$$
\begin{equation*}
E\left(U_{\lambda_{j}}^{j}\right) \geq m(c), \text { and } E\left(\frac{\sqrt{c}}{\left\|r_{n}^{l}\right\|_{L^{2}}} r_{n}^{l}\right) \geq m(c) \tag{3.10}
\end{equation*}
$$

Thus, we infer from (3.6), (3.8) and (3.9) that

$$
\begin{align*}
E\left(u_{n}\right) & \geq \sum_{j=1}^{l}\left(\frac{E\left(U_{\lambda_{j}}^{j}\right)}{\lambda_{j}^{2}}+\frac{\lambda_{j}^{p}-1}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} d x\right)+\frac{\left\|r_{n}^{l}\right\|_{L^{2}}}{c} E\left(\frac{\sqrt{c}}{\left\|r_{n}^{l}\right\|_{L^{2}}} r_{n}^{l}\right)+o_{n}(1) \\
& \geq \sum_{j=1}^{l} \frac{\left\|U^{j}\right\|_{L^{2}}^{2}}{c} m(c)+\inf _{j \geq 1} \frac{\lambda_{j}^{p}-1}{p}\left(\sum_{j=1}^{l} \int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j}\left(x-x_{n}^{j}\right)\right|^{p+2} d x\right)+\frac{\left\|r_{n}^{l}\right\|_{L^{2}}}{c} m(c)+o_{n}(1) . \tag{3.11}
\end{align*}
$$

Note that the series $\sum_{j=1}^{\infty}\left\|U^{j}\right\|_{L^{2}}^{2}$ is convergent, there exists $j_{0} \geq 1$ such that

$$
\begin{equation*}
\left\|U^{j_{0}}\right\|_{L^{2}}^{2}=\sup \left\{\left\|U^{j}\right\|_{L^{2}}^{2}, j \geq 1\right\} \text { and } \inf _{j \geq 1} \lambda_{j}=\lambda_{j_{0}}=\frac{\sqrt{c}}{\left\|U^{j_{0}}\right\|_{L^{2}}} . \tag{3.12}
\end{equation*}
$$

Let $n \rightarrow \infty$ and $l \rightarrow \infty$ in (3.11), it follows from (3.4) that

$$
m(c) \geq m(c)+C_{1}\left(\left(\frac{\sqrt{c}}{\left\|U^{j_{0}}\right\|_{L^{2}}}\right)^{p}-1\right)
$$

which implies

$$
\left\|U^{j_{0}}\right\|_{L^{2}}^{2} \geq c
$$

Hence, $\left\|U^{j_{0}}\right\|_{L^{2}}^{2}=c$ and there exists only one term $U^{j_{0}} \neq 0$ in the decomposition (3.5). We consequently rewrite (3.5) as

$$
u_{n}(x)=U^{j_{0}}\left(x-x_{n}^{j_{0}}\right)+r_{n}(x) .
$$

Note that

$$
\left\|u_{n}\right\|_{L^{2}}^{2}=\left\|U^{j_{0}}\right\|_{L^{2}}^{2}+\left\|r_{n}\right\|_{L^{2}}^{2}+o_{n}(1)
$$

and $\left\|u_{n}\right\|_{L^{2}}^{2}=\left\|U^{j_{0}}\right\|_{L^{2}}^{2}=c$, we get

$$
\lim _{n \rightarrow \infty}\left\|r_{n}\right\|_{L^{2}}=0
$$

This shows that $r_{n} \rightarrow 0$ in $L^{2}$. This, together with the Gagliardo-Nirenberg inequality, implies that $\lim _{n \rightarrow \infty}\left\|r_{n}\right\|_{L^{q+2}}^{q+2}=0$, for all $q \in\left(0, \frac{4}{N-2}\right)$. We consequently obtain

$$
\int_{\mathbb{R}^{N}}|x|^{-b}\left|r_{n}(x)\right|^{p+2} d x \rightarrow 0
$$

Applying the lower semi-continuity of norm, it follows that $\lim _{\inf }^{n \rightarrow \infty}$ $E\left(r_{n}\right) \geq 0$, and thus

$$
\begin{aligned}
\liminf _{n \rightarrow \infty} E\left(U^{j_{0}}\left(\cdot-x_{n}^{j_{0}}\right)\right) & \leq \liminf _{n \rightarrow \infty} E\left(U^{j_{0}}\left(\cdot-x_{n}^{j_{0}}\right)\right)+\liminf _{n \rightarrow \infty} E\left(r_{n}\right) \\
& \leq \liminf _{n \rightarrow \infty}\left(E\left(U^{j_{0}}\left(\cdot-x_{n}^{j_{0}}\right)\right)+E\left(r_{n}\right)\right) \\
& =\liminf _{n \rightarrow \infty} E\left(u_{n}\right)=m(c) .
\end{aligned}
$$

On the other hand, since $\left\|U^{j_{0}}\left(\cdot-x_{n}^{j_{0}}\right)\right\|_{L^{2}}^{2}=\left\|U^{j_{0}}\right\|_{L^{2}}^{2}=c$ for all $n \geq 1$, we have $E\left(U^{j_{0}}\left(\cdot-x_{n}^{j_{0}}\right)\right) \geq m(c)$ for all $n \geq 1$. Therefore,

$$
\liminf _{n \rightarrow \infty} E\left(U^{j_{0}}\left(\cdot-x_{n}^{j_{0}}\right)\right)=m(c) .
$$

We next prove that the sequence $\left(x_{n}^{j_{0}}\right)_{n \geq 1}$ is bounded. Indeed, if it is not true, then up to a subsequence, we assume that $\left|x_{n}^{j_{0}}\right| \rightarrow \infty$ as $n \rightarrow \infty$. Without loss of generality, we assume that $U^{j_{0}}$ is continuous and compactly supported. We have

$$
\int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j_{0}}\left(x-x_{n}^{j_{0}}\right)\right|^{p+2} d x=\int_{\text {supp }\left(U^{j_{0}}\right)}\left|x+x_{n}^{j_{0}}\right|^{-b}\left|U^{j_{0}}(x)\right|^{p+2} d x
$$

Since $\left|x_{n}^{j_{0}}\right| \rightarrow \infty$ as $n \rightarrow \infty$, we see that $\left|x+x_{n}^{j_{0}}\right| \geq\left|x_{n}^{j_{0}}\right|-|x| \rightarrow \infty$ as $n \rightarrow \infty$ for all $x \in \operatorname{supp}\left(U^{j_{0}}\right)$. This shows that

$$
\int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j_{0}}\left(x-x_{n}^{j_{0}}\right)\right|^{p+2} d x \rightarrow 0
$$

as $n \rightarrow \infty$. This yields

$$
\liminf _{n \rightarrow \infty} E\left(U^{j_{0}}\left(\cdot-x_{n}^{j_{0}}\right)\right)=\frac{1}{2}\left\|U^{j_{0}}\right\|_{\dot{H}^{s}}=m(c) .
$$

By the definition of $E\left(U^{j 0}\right)$, we obtain

$$
E\left(U^{j_{0}}\right)+\frac{1}{p+2} \int_{\mathbb{R}^{N}}|x|^{-b}\left|U^{j_{0}}\right|^{p+2} d x=m(c),
$$

which implies that $E\left(U^{j_{0}}\right)<m(c)$, which is an contradiction with $E\left(U^{j_{0}}\right) \geq m(c)$ due to $\left\|U^{j_{0}}\right\|_{L^{2}}^{2}=c$. Therefore, the sequence $\left(x_{n}^{j_{0}}\right)_{n \geq 1}$ is bounded, and up to a subsequence, we assume that $x_{n}^{j_{0}} \rightarrow x_{0}$ as $n \rightarrow \infty$.

We now write

$$
u_{n}(x)=\tilde{U}^{j_{0}}(x)+\tilde{r}_{n}(x),
$$

where $\tilde{U}^{j_{0}}(x)=U^{j_{0}}\left(x-x^{j_{0}}\right)$ and $\tilde{r}_{n}(x):=U^{j_{0}}\left(x-x_{n}^{j_{0}}\right)-U^{j_{0}}\left(x-x^{j_{0}}\right)+r_{n}(x)$. Using the fact $\left\|u_{n}\right\|_{L^{2}}^{2}=$ $\left\|U^{j_{0}}\right\|_{L^{2}}^{2}=c$, it is easy to see that

$$
\tilde{r}_{n} \rightharpoonup 0 \text { in } H^{s} \text { and } \tilde{r}_{n} \rightarrow 0 \text { in } L^{2} .
$$

The first observation on $\tilde{r}_{n}$ allows us to write

$$
E\left(u_{n}\right)=E\left(\tilde{U}^{j_{0}}\right)+E\left(\tilde{r}_{n}\right)+o_{n}(1) .
$$

Again, by using the lower semi-continuity of norm and the fact $\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{N}}|x|^{-b}\left|\tilde{r}_{n}\right|^{p+2} d x=0$, we get that $\lim \inf _{n \rightarrow \infty} E\left(\tilde{r}_{n}\right) \geq 0$. Hence, using the fact that $\left\|\tilde{U}^{j_{0}}\right\|_{L^{2}}^{2}=c$, we infer that

$$
\begin{aligned}
m(c)=\liminf _{n \rightarrow \infty} E\left(u_{n}\right) & \geq \liminf _{n \rightarrow \infty}\left(E\left(\tilde{U}^{j_{0}}\right)+E\left(\tilde{r}_{n}\right)\right) \\
& \geq E\left(\tilde{U}^{j_{0}}\right)+\liminf _{n \rightarrow \infty} E\left(\tilde{r}_{n}\right) \\
& \geq E\left(\tilde{U}^{j_{0}}\right) \geq m(c) .
\end{aligned}
$$

Therefore, $E\left(\tilde{U}^{j_{0}}\right)=m(c)$ which completes the proof.

Proof of Theorem 1.2. Firstly, by Lemma 3.1, we see that the solution $\psi$ of (1.2) exists globally. Assume by contradiction that there exist $\varepsilon_{0}$ and a sequence $\left\{\psi_{0, n}\right\}_{n=1}^{\infty}$ such that

$$
\begin{equation*}
\inf _{u \in \mathcal{M}_{c}}\left\|\psi_{0, n}-u\right\|_{H^{s}}<\frac{1}{n} \tag{3.13}
\end{equation*}
$$

and there exists $\left\{t_{n}\right\}_{n=1}^{\infty}$ such that the corresponding solution sequence $\left\{\psi_{n}\left(t_{n}\right)\right\}_{n=1}^{\infty}$ of (1.2) satisfies

$$
\begin{equation*}
\inf _{u \in \mathcal{M}_{c}}\left\|\psi_{n}\left(t_{n}\right)-u\right\|_{H^{s}} \geq \varepsilon_{0} . \tag{3.14}
\end{equation*}
$$

Firstly, we claim that there exists $v \in \mathcal{M}_{c}$ such that

$$
\lim _{n \rightarrow \infty}\left\|\psi_{0, n}-v\right\|_{H^{s}}=0
$$

Indeed, by (3.13), there exists $\left\{v_{n}\right\}_{n=1}^{\infty} \subset \mathcal{M}_{c}$ such that

$$
\begin{equation*}
\left\|\psi_{0, n}-v_{n}\right\|_{H^{s}}<\frac{2}{n} \tag{3.15}
\end{equation*}
$$

Due to $\left\{v_{n}\right\}_{n=1}^{\infty} \subset \mathcal{M}_{c},\left\{v_{n}\right\}_{n=1}^{\infty}$ is a minimizing sequence of (1.6). By the argument of Lemma 3.2, there exists $v \in \mathcal{M}_{c}$ such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|v_{n}-v\right\|_{H^{s}}=0 \tag{3.16}
\end{equation*}
$$

Then the claim follows from (3.15) and (3.16) immediately. Hence,

$$
\lim _{n \rightarrow \infty}\left\|\psi_{0, n}\right\|_{L^{2}}^{2}=\|v\|_{L^{2}}^{2}=c, \quad \lim _{n \rightarrow \infty} E\left(\psi_{0, n}\right)=E(v)=m(c) .
$$

By the conservation of mass and energy, we have

$$
\lim _{n \rightarrow \infty}\left\|\psi_{n}\left(t_{n}\right)\right\|_{2}^{2}=c, \quad \lim _{n \rightarrow \infty} E\left(\psi_{n}\left(t_{n}\right)\right)=E(v)=m(c) .
$$

By the argument of Lemma 3.2, $\left\{\psi_{n}\left(t_{n}\right)\right\}_{n=1}^{\infty}$ is bounded in $H^{s}$. Set

$$
\tilde{\psi}_{n}=\frac{\sqrt{c} \psi_{n}\left(t_{n}\right)}{\left\|\psi_{n}\left(t_{n}\right)\right\|_{L^{2}}} .
$$

Then $\left\|\tilde{\psi}_{n}\right\|_{2}^{2}=c$ and

$$
\begin{aligned}
E\left(\tilde{\psi}_{n}\right) & =\frac{1}{2} \frac{c}{\left\|\psi_{n}\left(t_{n}\right)\right\|_{L^{2}}}\left\|\psi_{n}\left(t_{n}\right)\right\|_{H^{s}}^{2}-\frac{1}{p+2} \frac{c^{\frac{p+2}{2}}}{\left\|\psi_{n}\left(t_{n}\right)\right\|_{L^{2}}^{p+2}} \int_{\mathbb{R}^{N}}|x|^{-b}\left|\psi_{n}\left(t_{n}\right)(x)\right|^{p+2} d x \\
& =\frac{c}{\left\|\psi_{n}\left(t_{n}\right)\right\|_{L^{2}}^{2}} E\left(\psi_{n}\left(t_{n}\right)\right)+\frac{1}{p+2}\left(\frac{c}{\left\|\psi_{n}\left(t_{n}\right)\right\|_{L^{2}}^{2}}-\frac{c^{\frac{p+2}{2}}}{\left\|\psi_{n}\left(t_{n}\right)\right\|_{L^{2}}^{p+2}}\right) \int_{\mathbb{R}^{N}}|x|^{-b}\left|\psi_{n}\left(t_{n}\right)\right|^{p+2} d x .
\end{aligned}
$$

which implies that

$$
\lim _{n \rightarrow \infty} E\left(\tilde{\psi}_{n}\right)=E\left(\psi_{n}\left(t_{n}\right)\right)=m(c) .
$$

Hence, $\tilde{\psi}_{n}$ is a minimizing sequence of (1.6), and By the argument of Lemma 3.2, there exists $\tilde{v} \in \mathcal{M}_{c}$ such that

$$
\tilde{\psi}_{n} \rightarrow \tilde{v} \text { in } H^{s} .
$$

By the definition of $\tilde{\psi}_{n}$, it follows that

$$
\tilde{\psi}_{n}-\psi_{n}\left(t_{n}\right) \rightarrow 0 \text { in } H^{s} .
$$

We consequently obtain that

$$
\psi_{n}\left(t_{n}\right) \rightarrow \tilde{v} \text { in } H^{s} .
$$

which contradicts (3.14). This completes the proof.

## 4. Strong instability of standing waves

In this section, we will prove the instability of standing waves for (1.2). We firstly recall the following blow-up criterion for (1.2) established in [44].

Lemma 4.1. Let $N \geq 2, s \in\left(\frac{1}{2}, 1\right), p=\frac{4 s-2 b}{N}, 0<b<\min \{2 s, N\}$ and $p<2 s$. Suppose that $\psi(t) \in C\left([0, T) ; H^{2 s}\right)$ is a radial solution of (1.2). Furthermore, we suppose that

$$
E\left[\psi_{0}\right]<0,
$$

then $\psi(t)$ blows up in finite time in the sense that $T<\infty$ must hold, or $\psi(t)$ blows up infinite time such that

$$
\left\|(-\Delta)^{\frac{s}{2}} \psi(t)\right\|_{L^{2}}>C t \text { for all } t \geq t_{*}
$$

with some constants $C>0$ and $t_{*}>0$ that depend only on $\psi_{0}, s, N$.

Applying this blow-up criterion, we can prove the the instability of standing waves for (1.2).
Proof of Theorem 1.3. Firstly, we deduce from Pohozaev's identities (2.8) that $E(u)=0$, where $u$ is the ground state solution of (1.3). Thus, if we can construct initial data $\psi_{0, n}$ such that $E\left(\psi_{0, n}\right)<0$ and $\psi_{0, n} \rightarrow u$ in $H^{s}$, as $n \rightarrow \infty$, then the corresponding solution $\psi_{n}$ blows up in finite or infinite time by applying Lemma 4.1.

Let $\left\{c_{n}\right\} \subseteq \mathbb{C}$ be such that $\left|c_{n}\right|>1$ and $\lim _{n \rightarrow \infty}\left|c_{n}\right|=1$, and $\left\{\lambda_{n}\right\} \subseteq \mathbb{R}^{+}$be such that $\lim _{n \rightarrow \infty}\left|\lambda_{n}\right|=1$. We take the initial data

$$
\psi_{0, n}(x)=c_{n} u_{\omega}^{\lambda_{n}}(x)=c_{n} \lambda_{n}^{\frac{N}{2}} u\left(\lambda_{n} x\right) .
$$

Then, we have that for all $n \geq 1$

$$
\begin{gathered}
\lim _{n \rightarrow \infty}\left\|\psi_{0, n}\right\|_{L^{2}}=\lim _{n \rightarrow \infty} \mid c_{n}\|u\|_{L^{2}}=\|u\|_{L^{2}}, \\
\lim _{n \rightarrow \infty}\left\|\psi_{0, n}\right\|_{\dot{H}^{s}}=\lim _{n \rightarrow \infty}\left|c_{n}\right| \lambda_{n}^{s}\|u\|_{\dot{H}^{s}}=\|u\|_{\dot{H}^{s}} .
\end{gathered}
$$

Thus, we deduce from Brezis-Lieb's lemma that $\psi_{0, n} \rightarrow u_{\omega}$ in $H^{s}$ as $n \rightarrow \infty$.
On the other hand, we deduce from Pohozaev's identities (2.8) that

$$
\begin{aligned}
E\left(\psi_{0, n}\right) & \left.\left.=\frac{1}{2} \int_{\mathbb{R}^{N}} \right\rvert\,-\Delta\right)\left.^{s / 2} \psi_{0, n}(x)\right|^{2} d x-\frac{N}{2 N+4 s-2 b} \int_{\mathbb{R}^{N}}|x|^{-b}\left|\psi_{0, n}(x)\right|^{\frac{2 N+4 s-2 b}{N}} d x \\
& =\frac{1}{2}\left|c_{n}\right|^{2} \lambda_{n}^{2 s}\|u\|_{\dot{H}^{s}}^{2}-\frac{N}{2 N+4 s-2 b}\left|c_{n}\right|^{\frac{2 N+4 s-2 b}{N}} \lambda_{n}^{2 s} \int_{\mathbb{R}^{N}}|x|^{-b}|u(x)|^{\frac{2 N+s-2 b}{N}} d x \\
& =\frac{1}{2}\left|c_{n}\right|^{2} \lambda_{n}^{2 s}\|u\|_{\dot{H}^{s}}^{2}-\frac{1}{2}\left|c_{n}\right|^{\frac{2 N+4 s-2 b}{N}} \lambda_{n}^{2 s}\|u\|_{\dot{H}^{s}}^{2} \\
& =\frac{1}{2}\left(\left|c_{n}\right|^{2}-\left|c_{n}\right|^{\frac{2 N+4 s-2 b}{N}}\right) \lambda_{n}^{2 s}\|u\|_{\dot{H}^{s}}^{2}<0,
\end{aligned}
$$

for all $n \geq 1$. Therefore, applying Lemma 4.1, we deduce that the solution $\psi_{n}(t)$ of (1.2) with initial data $\psi_{0, n}$ blows up in finite or infinite time. This completes the proof.

## 5. Conclusions

In this paper, we investigate the stability and instability of standing waves for the inhomogeneous fractional Schrödinger equation. In the $L^{2}$-subcritical case, i.e., $0<p<\frac{4 s-2 b}{N}$, we prove that the standing waves are orbitally stable by using the profile decomposition theory and variational methods. In the $L^{2}$-critical case, i.e., $p=\frac{4 s-2 b}{N}$, we show that the standing waves are strongly unstable by blowup. In particular, by using our methods in this paper, one can easily study the stability and instability of standing waves for the inhomogeneous Schrödinger equation, i.e., $s=1$.

## Acknowledgments

The authors would like to express their sincere thanks to the referees for the valuable comments and suggestions which helped to improve the original paper. This work is supported by the NSF of Ningxia Hui Autonomous Region of China (No. 2018AAC03129), and the General Research Projects of North Minzu University (No.2020XYZSX03), the NSF of China (No. 11701012), and the FirstClass Disciplines Foundation of Ningxia (No. NXYLXK2017B09).

## Conflict of interest

All authors declare no conflicts of interest in this paper.

## References

1. N. Laskin, Fractional quantum mechanics and Lèvy path integrals, Phys. Lett. A, 268 (2000), 298-304.
2. N. Laskin, Fractional Schrödinger equations, Phys. Rev. E, 66 (2002), 056108.
3. T. Boulenger, D. Himmelsbach, E. Lenzmann, Blowup for fractional NLS, J. Funct. Anal., 271 (2016), 2569-2603.
4. B. Wang, Y. Wang, Fractional white noise functional soliton solutions of a wick-type stochastic fractional NLSE, Appl. Math. Lett., 110 (2020), 106583.
5. G. Wu, C. Dai, Nonautonomous soliton solutions of variable-coefficient fractional nonlinear Schrödinger equation, Appl. Math. Lett., 106 (2020), 106365.
6. C. Dai, J. Zhang, Controlling effect of vector and scalar crossed double-Ma breathers in a partially nonlocal nonlinear medium with a linear potential, Nonlinear Dyn., 100 (2020), 1621-1628.
7. L. Yu, G. Wu, Y. Wang, et al. Traveling wave solutions constructed by Mittag-Leffler function of a $(2+1)$-dimensional space-time fractional NLS equation, Results Phys., 17 (2020), 103156.
8. Y. Cho, H. Hajaiej, G. Hwang, et al. On the Cauchy problem of fractional Schrödinger equations with Hartree type nonlimearity, Funkcial. Ekvac., 56 (2013), 193-224.
9. Y. Cho, H. Hajaiej, G. Hwang, et al. On the orbital stability of fractional Schrödinger equations, Commun. Pure Appl. Anal., 13 (2013), 1267-1282.
10. Y. Cho, G. Hwang, S. Kwon, et al. Profile decompositions and Blow-up phenomena of mass critical fractional Schrödinger equations, Nonlinear Anal., 86 (2013), 12-29.
11. Y. Cho, G. Hwang, S. Kwon, et al. On finite time blow-up for the mass-critical Hartree equations, Proc. Roy. Soc. Edinburgh Sect. A, 145 (2015), 467-479.
12. Y. Cho, G. Hwang, S. Kwon, et al. Well-posedness and ill-posedness for the cubic fractional Schrödinger equations, Discrete Contin. Dyn. Syst., 35 (2015), 2863-2880.
13. V. D. Dinh, Well-posedness of nonlinear fractional Schrödinger and wave equations in Sobolev spaces, Int. J. Appl. Math., 31 (2018), 483-525.
14. V. D. Dinh, A study on blowup solutions to the focusing $L^{2}$-supercritical nonlinear fractional Schrödinger equation, J. Math. Phys., 59 (2018), 071506.
15. V. D. Dinh, B. Feng, On fractional nonlinear Schrödinger equation with combined power-type nonlinearities, Discrete Contin. Dyn. Syst., 39 (2019), 4565-4612.
16. B. Feng, On the blow-up solutions for the fractional nonlinear Schrödinger equation with combined power-type nonlinearities, Comm. Pure Appl. Anal., 17 (2018), 1785-1804.
17. B. Feng, J. Ren, Q. Wang, Existence and instability of normalized standing waves for the fractional Schrödinger equations in the $L^{2}$-supercritical case, J. Math. Phys., 61 (2020), 071511.
18. S. Zhu, Existence of stable standing waves for the fractional Schrödinger equations with combined nonlinearities, J. Evol. Equations, 17 (2017), 1003-1021.
19. B. Feng, H. Zhang, Stability of standing waves for the fractional Schrödinger-Choquard equation, Comput. Math. Appl., 75 (2018), 2499-2507.
20. B. Feng, H. Zhang, Stability of standing waves for the fractional Schrödinger-Hartree equation, J. Math. Anal. Appl., 460 (2018), 352-364.
21. Q. Guo, S. Zhu, Sharp threshold of blow-up and scattering for the fractional Hartree equation, J. Differ. Equations, 264 (2018), 2802-2832.
22. Y. Hong, Y. Sire, On fractional Schrödinger equations in Sobolev spaces, Commun. Pure Appl. Anal., 14 (2015), 2265-2282.
23. J. Zhang, S. Zhu, Stability of standing waves for the nonlinear fractional Schrödinger equation, J. Dyn. Differ. Equations, 29 (2017), 1017-1030.
24. S. Zhu, On the blow-up solutions for the nonlinear fractional Schrödinger equation, J. Differ. Equations, 261 (2016), 1506-1531.
25. B. Feng, R. Chen, J. Ren, Existence of stable standing waves for the fractional Schrödinger equations with combined power-type and Choquard-type nonlinearities, J. Math. Phys., 60 (2019), 051512.
26. T. Saanouni, Remarks on damped fractional Schrödinger equation with pure power nonlinearity, J. Math. Phys., 56 (2015), 061502.
27. T. Saanouni, Remarks on the inhomogeneous fractional nonlinear Schrödinger equation, J. Math. Phys., 57 (2016), 081503.
28. T. Saanouni, Strong instability of standing waves for the fractional Choquard equation, J. Math. Phys., 59 (2018), 081509.
29. B. Feng, R. Chen, J. Liu, Blow-up criteria and instability of normalized standing waves for the fractional Schrödinger-Choquard equation, Adv. Nonlinear Anal., 10 (2021), 311-330.
30. I. V. Barashenkov, N. V. Alexeeva, E. V. Zemlianaya, Two and three dimensional oscillons in nonlinear Faradey resonance, Phys. Rev. Lett., 89 (2002), 104101.
31. T. S. Gill, Optical guiding of laser beam in nonuniform plasma, Pramana, 55 (2000), 835-842.
32. C. S. Liu, V. K. Tripathi, Laser guiding in an axially nonuniform plasma channel, Phys. Plasmas, 1 (1994), 3100-3103.
33. F. Genoud, C.A. Stuart, Schrödinger equations with a spatially decaying nonlinearity: Existence and stability of standing waves, Discrete Contin. Dyn. Syst., 21 (2008), 137-286.
34. T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 10 (2003).
35. F. Genoud, Bifurcation and stability of travelling waves in self-focusing planar waveguides, Adv. Nonlinear Stud., 10 (2010), 357-400.
36. L. G. Farah, Global well-posedness and blow-up on the energy space for the inhomogeneous nonlinear Schrödinger equation, J. Evol. Equations, 16 (2016), 193-208.
37. L. G. Farah, C. M. Guzman, Scattering for the radial focusing INLS equation in higher dimensions, 2017. preprint arXiv:1703.10 988.
38. L. G. Farah, C. M. Guzman, et al. Scattering for the radial 3D cubic focusing inhomogeneous nonlinear Schrödinger equation, J. Differ. Equations, 262 (2017), 4175-4231.
39. V. D. Dinh, Blowup of $H^{1}$ solutions for a class of the focusing inhomogeneous nonlinear Schrödinger equation, Nonlinear Anal., 174 (2018), 169-188.
40. T. Cazenave, P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Commun. Math. Phys., 85 (1982), 549-561.
41. A. Bensouilah, V. D. Dinh, S. H. Zhu, On stability and instability of standing waves for the nonlinear Schrödinger equation with an inverse-square potential, J. Math. Phys., 59 (2018), 101505.
42. B. Feng, R. Chen, Q. Wang, Instability of standing waves for the nonlinear Schrödinger-Poisson equation in the $L^{2}$-critical case, J. Dyn. Differ. Equations, 32 (2020), 1357-1370.
43. B. Feng, J. Liu, H. Niu, et al. Strong instability of standing waves for a fourth-order nonlinear Schrödinger equation with the mixed dispersions, Nonlinear Anal., 196 (2020), 111791.
44. C. Peng, D. Zhao, Global existence and blowup on the energy space for the inhomogeneous fractional nonlinear Schrödinger equation, Discrete Contin. Dyn. Syst. B, 24 (2019), 3335-3356.
45. B. Feng, On the blow-up solutions for the nonlinear Schrödinger equation with combined powertype nonlinearities, J. Evol. Equations, 18 (2018), 203-220.
© 2020 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)
