Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 6261–6276.
DOI: 10.3934/math.2020403
Received: 27 May 2020
Accepted: 29 July 2020
Published: 06 August 2020

Research article

On the variable exponential fractional Sobolev space $W^{s(\cdot),p(\cdot)}$

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Abstract: In this paper a new kind of variable exponential fractional Sobolev spaces is introduced. For this kind of spaces, some basic properties, such as separability, reflexivity, strict convexity and denseness, are established. At last as an application the existence of solutions for so called $s(\cdot)$-$p(\cdot)$-Laplacian equations is discussed.

Keywords: variable exponent; fractional; Sobolev space
Mathematics Subject Classification: 46B20, 46E35, 46B50

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$. For any $s \in (0, 1)$ and $p \in [1, \infty)$, we can define the fractional Sobolev space $W^{s,p}(\Omega)$. The study on fractional Sobolev space is a classical topic in functional analysis and harmonic analysis and the theory of fractional Sobolev space has been widely applied in different fields, such as optimization, phase transition, anomalous diffusion, material science, non-uniform elliptic problems, gradient potential theory etc (see [10]). In 1991, Kováčik and Rákosník studied variable exponential spaces $L^{p(x)}$ and $W^{1,p(x)}$ (see [17]). Since then, some scholars have successively studied the theories and applications of these spaces (see [7, 9, 11–14, 20, 21] and their related references). With the vigorous development of variable index space theory, in recent years, some scholars have focused their research on the variable exponent fractional Sobolev spaces. In 2017, Kaufmann, Rossi and Vidal extended the constant exponent p of fractional Sobolev space to variable exponent $p(x, y)$ type (not only that, but also another variable exponent $q(x)$), and studied the compact embedding of these spaces and obtained the existence and uniqueness of solutions for non-local problems of $p(x)$-Laplacian equations by means of these spaces (see [16]).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, and $q : \bar{\Omega} \longrightarrow (1, \infty)$ and $p : \bar{\Omega} \times \bar{\Omega} \longrightarrow (1, \infty)$ be two continuous functions bounded away from 1 and ∞. Assume that p is symmetric, i.e. $p(x, y) = p(y, x)$. For $s \in (0, 1)$, the variable exponent Sobolev fractional space is defined as follows in [16]:

$$W^{s,q(x),p(x,y)}(\Omega) := \left\{ u \in L^{q(x)}(\Omega) : \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}} dxdy < \infty \text{ for some } \lambda > 0 \right\}. \tag{1.1}$$

On the spaces defined in this way, scholars set up certain conditions to study trace theorem (see [8]), indefinite weights of $p(x,y)$-Laplace equations (see [19]), non-local eigenvalues with variable exponential growth conditions (see [2]), separability, reflexivity, density and a class of nonlocal fractional problems (see [6]), a priori bounds and multiplicity of solutions for nonlinear elliptic problems involving the fractional $p(\cdot)$-Laplacian (see [15]), extension domains (see [4]), strong comparison principle for the fractional $p(x)$-Laplace operators, sub-super-solution method for the nonlocal equations involving the fractional $p(x)$-Laplacian (see [5]) and so on.

Due to the need of research, some scholars limited the exponent of the variable exponent fractional Sobolev spaces above to $L^{\bar{p}(x)}(\Omega)$, i.e. replace $q(x)$ in the definition with $\bar{p}(x)$, where $\bar{p}(x) = p(x,x)$, see for example [2].

In [3], the author adjusted the definition of (1.1) to give another form of variable exponent fractional Sobolev space:

$$X := \left\{ u \in L^{\bar{p}(x)}(\Omega) : \iint_{Q} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}} dxdy < \infty \text{ for some } \lambda > 0 \right\}, \tag{1.2}$$

where the integral is extended from $\Omega \times \Omega$ to $Q = \mathbb{R}^{2n} \setminus (C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^n \setminus \Omega$. The authors used the closed subspace $X_0 := \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$ to study the $p(x)$-Kirchhoff Dirichlet boundary problem

$$\begin{cases} M(\iint_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}} dxdy)(-\triangle_{p(x)})^s u(x) = \lambda |u(x)|^{r(x)-2} u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.3}$$

where $(-\triangle_{p(x)})^s u(x) = \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}} dy$ for all $x \in \mathbb{R}^n$. Some basic properties, such as reflexivity, completeness, separability, uniform convexity, were also obtained.

In [23], the authors considered such variable order fractional Sobolev space $H_0^{s(\cdot,\cdot)}(\Omega)$, where for any function $u \in L^2(\Omega)$ satisfying $u = 0$ in $\mathbb{R}^n \setminus \Omega$,

$$[u]_{s(\cdot,\cdot)} = \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s(x,y)}} dxdy \right)^{\frac{1}{2}} < \infty.$$

They studied an elliptic equation involving variable exponent driven by the fractional Laplace operator with variable order derivative.

In this paper we want to define a new kind of variable exponent fractional Sobolev spaces similar to the ones in [3] and [23], but with the variable order $s(x)$ and the variable exponent $\frac{p(x)+p(y)}{2}$. Some basic properties of this kind of spaces are discussed and an application on so called $s(\cdot)$-$p(\cdot)$-Laplacian equations is given.

2. Concepts and basic properties of $W^{s(\cdot),p(\cdot)}(\Omega)$

Throughout this paper, without specification, we will generally assume that Ω is a Lebesgue measurable subset of $\mathbb{R}^n$ with positive measure. The set of all Lebesgue measurable functions on Ω is represented by $\mathcal{L}(\Omega)$. We begin with some basic notations and concepts.

Definition 2.1. *Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p(\cdot) : \Omega \to [1, \infty]$ and $\mathcal{S}(\Omega)$ be the set of all Lebesgue measurable functions $s(\cdot) : \Omega \to (0, 1)$.*

Given $p(\cdot) \in \mathcal{P}(\Omega)$, $s(\cdot) \in \mathcal{S}(\Omega)$, denote

$$p^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x), \quad p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x),$$
$$s^+ = \operatorname*{ess\,sup}_{x\in\Omega} s(x), \quad s^- = \operatorname*{ess\,inf}_{x\in\Omega} s(x).$$

For convenience, we set

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

Definition 2.2. *For $p(\cdot) \in \mathcal{P}(\Omega)$ and $u \in \mathcal{L}(\Omega)$, define the modular associated with $p(\cdot)$ by*

$$\rho_{p(\cdot),\Omega}(u) = \int_{\Omega\backslash\Omega_\infty} |u(x)|^{p(x)} dx + \|u\|_{L^\infty(\Omega_\infty)}.$$

Definition 2.3. *For $p(\cdot) \in \mathcal{P}(\Omega)$, $s(\cdot) \in \mathcal{S}(\Omega)$ and $u \in \mathcal{L}(\Omega)$, define φ by*

$$\varphi_{s(\cdot),p(\cdot),\Omega}(u) = \iint_{(\Omega\backslash\Omega_\infty)\times(\Omega\backslash\Omega_\infty)} \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}}}{|x - y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy + \|u\|_{L^\infty(\Omega_\infty)}.$$

It is easy to verify that $\varphi_{s(\cdot),p(\cdot),\Omega}$ is a pseudonorm on $\mathcal{L}(\Omega)$ (see [18]), i.e.
1. $\varphi_{s(\cdot),p(\cdot),\Omega}(0) = 0$,
2. $\varphi_{s(\cdot),p(\cdot),\Omega}(-u) = \varphi_{s(\cdot),p(\cdot),\Omega}(u)$,
3. $\varphi_{s(\cdot),p(\cdot),\Omega}(\alpha u + \beta v) \le \alpha\varphi_{s(\cdot),p(\cdot),\Omega}(u) + \beta\varphi_{s(\cdot),p(\cdot),\Omega}(v)$ for $\alpha, \beta \ge 0, \alpha + \beta = 1$.

Definition 2.4. *For $p(\cdot) \in \mathcal{P}(\Omega)$, the variable exponential Lebesgue space is defined as:*

$$L^{p(\cdot)}(\Omega) := \left\{u \in \mathcal{L}(\Omega) : \exists\lambda > 0, s.t.\ \rho_{p(\cdot),\Omega}(\frac{u}{\lambda}) < \infty\right\},$$

which is a Banach space with the following norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot),\Omega}(\frac{u}{\lambda}) < 1\right\}.$$

See [7, 9, 13, 14, 17].

Definition 2.5. *For $p(\cdot) \in \mathcal{P}(\Omega)$, $s(\cdot) \in \mathcal{S}(\Omega)$, the variable exponential fractional Sobolev space is defined as*

$$W^{s(\cdot),p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : \exists\lambda > 0, s.t.\ \varphi_{s(\cdot),p(\cdot),\Omega}(\frac{u}{\lambda}) < \infty\right\}$$

with the seminorm

$$[u]_{W^{s(\cdot),p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \varphi_{s(\cdot),p(\cdot),\Omega}(\frac{u}{\lambda}) < 1\right\}$$

and the norm

$$\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + [u]_{W^{s(\cdot),p(\cdot)}(\Omega)}.$$

It is easy to verify that under this norm $W^{s(\cdot),p(\cdot)}(\Omega)$ is a Banach space.

Take any Cauchy sequence $\{u_n\} \subset W^{s(\cdot),p(\cdot)}(\Omega)$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that whenever $n, m > N$, we have $\|u_n - u_m\|_{W^{s(\cdot),p(\cdot)}(\Omega)} < \varepsilon$. So $\|u_n - u_m\|_{L^{p(\cdot)}(\Omega)} < \varepsilon$. By the completeness of $L^{p(\cdot)}(\Omega)$, there exists $u_0 \in L^{p(\cdot)}(\Omega)$ such that $\|u_n - u_0\|_{L^{p(\cdot)}(\Omega)} \to 0$ as $n \to \infty$, from which we obtain $\{u_n\}$ converges to u in measure. Since $[u_n - u_m]_{W^{s(\cdot),p(\cdot)}(\Omega)} < \varepsilon$, by Propositions 2.4 below and Fatou Lemma, we have $[u_n - u_0]_{W^{s(\cdot),p(\cdot)}(\Omega)} \to 0$ as $n \to \infty$. That is $\|u_n - u_0\|_{W^{s(\cdot),p(\cdot)}(\Omega)} \to 0$ as $n \to \infty$. Next we prove $u_0 \in W^{s(\cdot),p(\cdot)}(\Omega)$. By Propositions 2.4 below once more, we get n_0 such that

$$\varphi_{s(\cdot),p(\cdot),\Omega}(\frac{u_{n_0}}{\lambda} - \frac{u_0}{\lambda}) < 1, \text{ for all } \lambda > 0.$$

Take $\lambda_0 > 0$ such that $\varphi_{s(\cdot),p(\cdot),\Omega}(\frac{u_{n_0}}{\lambda_0}) < \infty$, then

$$\iint\limits_{(\Omega\backslash\Omega_\infty)\times(\Omega\backslash\Omega_\infty)} \frac{(|u_0(x) - u_0(y)| - |u_{n_0}(x) - u_{n_0}(y)|)^{\frac{p(x)+p(y)}{2}}}{\lambda_0^{\frac{p(x)+p(y)}{2}} |x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy + \|\frac{u_{n_0} - u_0}{\lambda_0}\|_{L^\infty(\Omega_\infty)} < 1. \tag{2.1}$$

In view of the inequalities: for $1 \le p < \infty$ and $a \ge 0, b \ge 0$,

$$(a+b)^p \le 2^{p-1}(a^p + b^p),$$

add

$$\iint\limits_{(\Omega\backslash\Omega_\infty)\times(\Omega\backslash\Omega_\infty)} \frac{|u_{n_0}(x) - u_{n_0}(y)|^{\frac{p(x)+p(y)}{2}}}{\lambda_0^{\frac{p(x)+p(y)}{2}} |x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy$$

to both sides of inequality (2.1), then we get $\varphi_{s(\cdot),p(\cdot),\Omega}(\frac{u_0}{\lambda_0}) < \infty$.

Example. Let Ω be bounded and closed, $s(\cdot) \in \mathcal{S}(\Omega)$ with $0 < s^- \le s^+ < 1$ and $p(\cdot) \in \mathcal{P}(\Omega)$ with $p^+ < \infty$. f is Lipschitz continuous on Ω and $|f| \le 1$, then $f \in W^{s(\cdot),p(\cdot)}(\Omega)$.

Indeed, first we have

$$\rho_{p(\cdot),\Omega}(f) = \int_\Omega |f(x)|^{p(x)} dx \le |\Omega|.$$

Second we have

$$\begin{aligned}
\varphi_{s(\cdot),p(\cdot),\Omega}(f) &= \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy \\
&= \int_\Omega \int_{\Omega\cap\{y\in\mathbb{R}^n : |x-y|\ge 1\}} \frac{|f(x) - f(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy \\
&\quad + \int_\Omega \int_{\Omega\cap\{y\in\mathbb{R}^n : |x-y|<1\}} \frac{|f(x) - f(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy \\
&= I_1 + I_2
\end{aligned}$$

We estimate I_1 and I_2 respectively. As

$$\int_\Omega \int_{\Omega\cap\{y\in\mathbb{R}^n:|x-y|\geq 1\}} \frac{|f(x)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy \leq \int_\Omega \Big(\int_{\{z\in\mathbb{R}^n:|z|\geq 1\}} \frac{1}{|z|^{n+s^-}} dz \Big) dx$$

since $n+s^- > n$, the integral $\int_{\{z\in\mathbb{R}^n:|z|\geq 1\}} \frac{1}{|z|^{n+s^-}} dz$ is convergent. Then we get

$$I_1 \leq 2^{p^+-1} \int_\Omega \int_{\Omega\cap\{y\in\mathbb{R}^n:|x-y|\geq 1\}} \frac{|f(x)|^{\frac{p(x)+p(y)}{2}} + |f(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy < \infty.$$

On the other hand,

$$\begin{aligned} I_2 &\leq \int_\Omega \int_{\Omega\cap\{y\in\mathbb{R}^n:|x-y|<1\}} \frac{M|x-y|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}} dxdy \\ &\leq M \int_\Omega \Big(\int_{\{z\in\mathbb{R}^n:|z|<1\}} \frac{1}{|z|^{n+p^+(s^+-1)}} dz \Big) dx \\ &< \infty, \end{aligned}$$

where M is the Lipschitz constant. Since $n + p^+(s^+ - 1) < n$, the integral $\int_{\{z\in\mathbb{R}^n:|z|<1\}} \frac{1}{|z|^{n+p^+(s^+-1)}} dz$ is convergent. We conclude that $\varphi_{s(\cdot),p(\cdot),\Omega}(f)$ is finite.

In this article, without confusion, $[u]_{W^{s(\cdot),p(\cdot)}(\Omega)}$, $\rho_{p(\cdot),\Omega}(u)$ and $\varphi_{s(\cdot),p(\cdot),\Omega}(u)$ can be abbreviated as $[u]$, $\rho(u)$ and $\varphi(u)$ respectively.

Definition 2.6. *Let $u_k, u \in W^{s(\cdot),p(\cdot)}(\Omega)$. We say that u_k is φ-convergent to u if there exists $\lambda > 0$ such that $\varphi(\lambda(u_k - u)) \to 0$ as $k \to \infty$ and we denote this convergence by $u_k \xrightarrow{\varphi} u$. We say that u_k is $[\cdot]$-convergent to u if $[u_k - u] \to 0$ as $k \to \infty$ and we denote this convergence by $u_k \xrightarrow{[\cdot]} u$.*

Definition 2.7. *Let X be a normed linear space. If every chord of the unit sphere of X has its midpoint below the surface of the unit sphere, then X is called strictly convex.*

For the variable exponent $p(\cdot,\cdot) : \Omega \to [1,\infty]$ which is symmetry, i.e. $p(x,y) = p(y,x)$ on $\Omega\times\Omega$, denote

$$\bar{p}^+ = \operatorname*{ess\,sup}_{(x,y)\in\Omega\times\Omega} p(x,y), \quad \bar{p}^- = \operatorname*{ess\,inf}_{(x,y)\in\Omega\times\Omega} p(x,y),$$

$$\tilde{\Omega}_\infty = \{(x,y) \in \Omega\times\Omega : p(x,y) = \infty\}.$$

In view of Definitions 2.2 and Definitions 2.4, we can define modular $\bar{\rho}_{p(\cdot,\cdot)}$ and variable exponent Lebesgue spaces $L^{p(\cdot,\cdot)}$ on $\Omega\times\Omega$. The conclusions on $L^{p(\cdot)}(\Omega)$ can be moved to $L^{p(\cdot,\cdot)}(\Omega\times\Omega)$.

Proposition 2.1. *If $u \in W^{s(\cdot),p(\cdot)}(\Omega)$ and $[u] > 0$, then $\varphi(\frac{u}{[u]}) \leq 1$. Further, $\varphi(\frac{u}{[u]}) = 1$ for all non-trivial $u \in W^{s(\cdot),p(\cdot)}(\Omega)$ if $p^+_{\Omega\setminus\Omega_\infty} < \infty$.*

Proof. Fix a decreasing sequence $\{\lambda_k\}$ such that $\lambda_k \to [u]$. Then by Fatou Lemma and the definition of $[\cdot]$,

$$\varphi(\frac{u}{[u]}) \leq \liminf_{k\to\infty} \varphi(\frac{u}{\lambda_k}) \leq 1.$$

Now suppose that $p^+_{\Omega\backslash\Omega_\infty} < \infty$, but assume to the contrary that $\varphi(\frac{u}{[u]}) < 1$, then for all λ, $0 < \lambda < [u]$, have

$$\varphi(\frac{u}{\lambda}) = \varphi\Big(\frac{[u]}{\lambda}\frac{u}{[u]}\Big) \le \Big(\frac{[u]}{\lambda}\Big)^{p^+(\Omega\backslash\Omega_\infty)}\varphi(\frac{u}{[u]}).$$

Therefore, we can find λ sufficiently close to $[u]$ such that $\varphi(\frac{u}{\lambda}) < 1$. But by the definition of $[\,\cdot\,]$, we must have $\varphi(\frac{u}{\lambda}) \ge 1$. From this contradiction we see that equality holds. □

Corollary 2.1. *Assume that* $u \in W^{s(\cdot),p(\cdot)}(\Omega)$*.*

1. If $[u] \le 1$*,then* $\varphi(u) \le [u]$*;*

2. If $[u] > 1$*,then* $\varphi(u) \ge [u]$*;*

3. $[u] \le \varphi(u) + 1$*.*

Proof. 1. If $u = 0$, it is immediate that the conclusion holds. Now suppose that $0 < [u] \le 1$, by Proposition 2.1, $\varphi(\frac{u}{[u]}) \le 1$, and so

$$\varphi(u) = \varphi([u]\frac{u}{[u]}) \le [u]\varphi(\frac{u}{[u]}) \le [u].$$

2. If $[u] > 1$, then for all λ, $[u] > \lambda > 1$, by the definition of $[\,\cdot\,]$, we have $\varphi(\frac{u}{\lambda}) \ge 1$ and further $\frac{1}{\lambda}\varphi(u) \ge 1$. Let $\lambda \to [u]$, we come to the conclusion.

3. By 1. and 2., it is immediate. □

Proposition 2.2. $[u] \le 1$ *and* $\varphi(u) \le 1$ *are equivalent in* $W^{s(\cdot),p(\cdot)}(\Omega)$*.*

Proof. If $\varphi(u) \le 1$, by definition of $[\,\cdot\,]$, we obtain directly $[u] \le 1$. On the other hand, if $[u] \le 1$, for any $\lambda > 1$, we have $\varphi(\frac{u}{\lambda}) \le 1$, furthermore by Fatou Lemma $\varphi(u) \le \liminf\limits_{\lambda\to 1^+}\varphi(\frac{u}{\lambda}) \le 1$. □

Proposition 2.3. *Suppose* $|\Omega_\infty| = 0$*, then for any* $u \in W^{s(\cdot),p(\cdot)}(\Omega)$*,*

$$\min\{\varphi(u)^{\frac{1}{p^-}}, \varphi(u)^{\frac{1}{p^+}}\} \le [u] \le \max\{\varphi(u)^{\frac{1}{p^-}}, \varphi(u)^{\frac{1}{p^+}}\}.$$

Proof. If $u = 0$, it is immediate. Consider the case $u \ne 0$. If $p^+ < \infty$, $0 < [u] \le 1$, then we need only to prove that

$$\varphi(u)^{\frac{1}{p^-}} \le [u] \le \varphi(u)^{\frac{1}{p^+}}.$$

If $[u] > 1$, it is similar to get $\varphi(u)^{\frac{1}{p^+}} \le [u] \le \varphi(u)^{\frac{1}{p^-}}$.

By the definition of φ,

$$\frac{\varphi(u)}{[u]^{p^-}} \le \varphi(\frac{u}{[u]}) \le \frac{\varphi(u)}{[u]^{p^+}}.$$

By Proposition 2.1 $\varphi(\frac{u}{[u]}) = 1$, so the desired result is true.

If $p^+ = \infty$, then $\varphi(u)^{\frac{1}{p^+}} = 1$, so the right hand inequality holds and we need only to prove the left hand inequality. By Corollary 2.1, $\varphi(u) \le 1$. Since $|\Omega_\infty| = 0$,

$$\begin{aligned}\varphi\Big(\frac{u}{\varphi(u)^{\frac{1}{p^-}}}\Big) &= \iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{\varphi(u)^{\frac{p(x)+p(y)}{2p^-}}|x-y|^{n+\frac{p(x)s(x)+p(y)s(y)}{2}}}dxdy\\ &\ge \varphi(u)\frac{1}{\varphi(u)} = 1,\end{aligned}$$

from which it follows $\varphi(u)^{\frac{1}{p^-}} \le [u]$. □

Proposition 2.4. *For* $\{u_k\} \subset W^{s(\cdot),p(\cdot)}(\Omega)$*,* $[u_k] \to 0$ *as* $k \to \infty$ *if and only if* $\varphi(\lambda u_k) \to 0$ *for all* $\lambda > 0$*. In particular,* $[\cdot]$*-convergent implies* φ*-convergent.*

Proof. Necessity. For any $0 < \varepsilon < 1, \lambda > 0, h > 1$, there exists $K_0 > 0$ such that whenever $k \geq K_0$, there holds $[\lambda h u_k] < \varepsilon < 1$. By 1. of Corollary 2.1, we have $\varphi(\lambda h u_k) \leq 1$, so

$$\varphi(\lambda u_k) \leq \frac{1}{h}\varphi(\lambda h u_k) \leq \frac{1}{h}.$$

There exists $H > 0$ such that $\frac{1}{h} < \varepsilon$ whenever $h \geq H$. Take $K = \max\{K_0, H\}$, then when $k \geq K$, $\varphi(\lambda u_k) \leq \varepsilon$.

Sufficiency. Assume that $\varphi(\lambda u_k) \to 0$ as $k \to \infty$, there exists $K_0 > 0$ such that $\varphi(\lambda u_k) < 1$ whenever $k \geq K_0$. Further we have $[\lambda u_k] \leq 1$, so

$$[u_k] \leq \frac{1}{\lambda}.$$

For any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that $\frac{1}{\lambda} < \varepsilon$ whenever $\lambda \geq \lambda_0$. Choose $K = \max\{K_0, \lambda_0\}$. When $k \geq K$, there holds $[u_k] \leq \varepsilon$. □

Proposition 2.5. *If* $p^+_{\Omega\backslash\Omega_\infty} < \infty$*, then* $[\cdot]$*-convergent and* φ*-convergent are equivalent in* $W^{s(\cdot),p(\cdot)}(\Omega)$*.*

Proof. Necessity. By Proposition 2.4, it is immediate.

Sufficiency. Let $u_k \xrightarrow{\varphi} 0$ as $k \to \infty$, then there exists $\lambda_0 > 0$ such that $\varphi(\lambda_0 u_k) \to 0$. By Proposition 2.4, we need only to prove that for any $\lambda > 0$, $\lambda u_k \xrightarrow{\varphi} 0$ as $k \to \infty$. Noticing that $\varphi(\lambda u_k) \leq ((\frac{\lambda}{\lambda_0})^{p^+(\Omega\backslash\Omega_\infty)} + \frac{\lambda}{\lambda_0})\varphi(\lambda_0 u_k)$, we come to the conclusion. □

Proposition 2.6. *If* $|\Omega| < +\infty$ *and* $p^+ < \infty$*, then for* $u \in W^{s(\cdot),p(\cdot)}(\Omega)$ *and* $\{u_k\} \subset W^{s(\cdot),p(\cdot)}(\Omega)$*, the following statements are equivalent:*

1. $u_k \xrightarrow{\|\cdot\|} u$.
2. $u_k \xrightarrow{\rho} u$ *and* $u_k \xrightarrow{\varphi} u$.
3. $u_k \to u$ *in measure and for some* $\gamma > 0$ *and* $\delta > 0$, $\rho(\gamma u_k) \to \rho(\gamma u)$, $\varphi(\delta u_k) \to \varphi(\delta u)$.

Proof. The equivalence between statements 1 and 2 can be obtained from Theorem 2.69 in [7] and Proposition 2.5. Now we prove the equivalence between statements 2 and 3.

If statement 2 holds, by Theorem 2.69 in [7], we just have to prove that for some $\delta > 0$, $\varphi(\delta u_k) \to \varphi(\delta u)$. Since $u_k \to u$ in measure, thus

$$|u_k(x) - u_k(y)|^{\frac{p(x)+p(y)}{2}} \to |u(x) - u(y)|^{\frac{p(x)+p(y)}{2}}$$

on $\Omega \times \Omega$ in measure.

Moreover, if $|\Omega| < +\infty$, then

$$\frac{|u_k(x) - u_k(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} \to \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}$$

on $\Omega \times \Omega$ in measure.

By the inequality

$$\begin{aligned}&\frac{|u_k(x)-u_k(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}\\ \leq &2^{p^+-1}\Big(\frac{|(u_k(x)-u(x))-(u_k(y)-u(y))|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}+\frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}\Big)\end{aligned}$$

and Vitali Convergence Theorem we deduce that $\varphi(u_k)\to\varphi(u)$, so statement 3 holds.

On the other hand, assume that statement 3 holds. Now suppose that $u_k\to u$ in measure and for some $\delta>0$, $\varphi(\delta u_k)\to\varphi(\delta u)$. We may assume without loss of generality that $\delta=1$. Then we have

$$|(u_k(x)-u(x))-(u_k(y)-u(y))|^{\frac{p(x)+p(y)}{2}}\to 0$$

on $\Omega\times\Omega$ in measure and

$$\frac{|(u_k(x)-u(x))-(u_k(y)-u(y))|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}\to 0$$

on $\Omega\times\Omega$ in measure. Combining the inequalities above, we get

$$\begin{aligned}&\frac{|(u_k(x)-u(x))-(u_k(y)-u(y))|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}\\ \leq &2^{p^+-1}\Big(\frac{|u_k(x)-u_k(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}+\frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}\Big)\end{aligned}$$

and $\varphi(u_k)\to\varphi(u)$, i.e. $u_k\xrightarrow{\varphi}u$. □

Corollary 2.2. *If $p^+_{\Omega\setminus\Omega_\infty}<\infty$, then for any $\varepsilon>0$, there exists $\delta>0$ such that for any $u\in W^{s(\cdot),p(\cdot)}(\Omega)$ with $\varphi(u)\leq\delta$, we have $[u]<\varepsilon$.*

It is immediate by Proposition 2.5.

Proposition 2.7. *φ is lower semicontinuous, i.e. if $u_k\xrightarrow{[\cdot]}u$ as $k\to\infty$, then $\varphi(u)\leq\liminf\limits_{k\to\infty}\varphi(u_k)$.*

Proof. We prove in following two cases.

1. $\varphi(u)<\infty$.

Let $u_k,u\in W^{s(\cdot),p(\cdot)}(\Omega)$, $u_k\xrightarrow{[\cdot]}u$ as $k\to\infty$. By Propositio 2.4, $\lim\limits_{k\to\infty}\varphi(\lambda(u_k-u))=0$ for any $\lambda>0$. Let $\varepsilon\in(0,\frac{1}{2})$. By the convexity of φ,

$$\begin{aligned}\varphi((1-\varepsilon)u)&=\varphi\Big(\frac{1}{2}u+\frac{1-2\varepsilon}{2}(u-u_k)+\frac{1-2\varepsilon}{2}u_k\Big)\\&\leq\frac{1}{2}\varphi(u)+\frac{1}{2}\varphi\Big((1-2\varepsilon)(u-u_k)+(1-2\varepsilon)u_k\Big)\\&\leq\frac{1}{2}\varphi(u)+\frac{2\varepsilon}{2}\varphi\Big(\frac{1-2\varepsilon}{2\varepsilon}(u-u_k)\Big)+\frac{1-2\varepsilon}{2}\varphi(u_k).\end{aligned}$$

Letting $k \to \infty$, by Fatou Lemma, we have

$$\varphi((1-\varepsilon)u) \leq \frac{1}{2}\varphi(u) + \frac{1-2\varepsilon}{2} \liminf_{k\to\infty} \varphi(u_k).$$

Next letting $\varepsilon \to 0^+$, by Fatou Lemma,

$$\varphi(u) \leq \frac{1}{2}\varphi(u) + \frac{1}{2} \liminf_{k\to\infty} \varphi(u_k),$$

from which we come to the conclusion.

2. $\varphi(u) = \infty$.

It is immediate that the conclusion holds if $\liminf_{k\to\infty} \varphi(u_k) = \infty$. Now suppose $\liminf_{k\to\infty} \varphi(u_k) < \infty$. Denote $\lambda_0 = \sup\{\lambda > 0, \varphi(\lambda u) < \infty\}$. Because $u \in W^{s(\cdot),p(\cdot)}(\Omega)$, $\lambda_0 > 0$. By $\varphi(u) = \infty$, we know $\lambda_0 \leq 1$.

Next we prove that $\lambda_0 \notin (0,1)$. Assume that $\lambda_0 \in (0,1)$, then choose $\lambda_1 \in (\lambda_0, 1)$ and $\alpha \in (0,1)$ such that

$$\frac{\lambda_1 - \lambda_0}{\lambda_0} + \alpha + \lambda_0 = 1.$$

We have

$$\begin{aligned}\varphi(\lambda_1 u) &= \varphi\Big((\lambda_1 - \lambda_0)u + \lambda_0(u - u_k) + \lambda_0 u_k\Big)\\ &\leq \frac{\lambda_1 - \lambda_0}{\lambda_0}\varphi(\lambda_0 u) + \alpha\varphi\Big(\frac{\lambda_0}{\alpha}(u - u_k)\Big) + \lambda_0\varphi(u_k).\end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned}\varphi(\lambda_1 u) &\leq \frac{\lambda_1 - \lambda_0}{\lambda_0}\varphi(\lambda_0 u) + \lambda_0 \liminf_{k\to\infty} \varphi(u_k)\\ &\leq (1-\alpha) \liminf_{k\to\infty} \varphi(u_k) < \infty,\end{aligned}$$

which contradicts the definition of λ_0. Therefore $\lambda_0 \notin (0,1)$, i.e. $\lambda_0 \leq 1$. As $\lambda_0 = 1$, for any $\lambda \in (0,1)$, $\varphi(\lambda u) < \infty$. According to the conclusion of first case

$$\varphi(\lambda u) \leq \liminf_{k\to\infty} \varphi(\lambda u_k) \leq \liminf_{k\to\infty} \varphi(u_k)$$

by Fatou Lemma. Further we get

$$\varphi(u) \leq \liminf_{\lambda\to 1^-} \varphi(\lambda u) \leq \liminf_{k\to\infty} \varphi(u_k),$$

then we complete the proof. □

Theorem 2.1. *1. If $p^+ < \infty$, then $W^{s(\cdot),p(\cdot)}(\Omega)$ is separable. 2. If $1 < p^- \leq p^+ < \infty$, then $W^{s(\cdot),p(\cdot)}(\Omega)$ is reflexive.*

Proof. We only prove the first conclusion, the second conclusion is similar. By $p^+ < \infty$, we know that $L^{p(\cdot)}(\Omega)$ and $L^{\frac{p(\cdot)+p(\cdot)}{2}}(\Omega \times \Omega)$ are separable. By Theorem 1.22 in [1], we have $L^{p(\cdot)}(\Omega) \times L^{\frac{p(\cdot)+p(\cdot)}{2}}(\Omega \times \Omega)$ is also separable. Define the mapping

$$
\begin{aligned}
T:\ W^{s(\cdot),p(\cdot)}(\Omega) &\longrightarrow L^{p(\cdot)}(\Omega)\times L^{\frac{p(\cdot)+p(\cdot)}{2}}(\Omega\times\Omega)\\
u &\longmapsto \left(u(x), \frac{u(x)-u(y)}{|x-y|^{\frac{2n}{p(x)+p(y)}+\frac{p(x)s(x)+p(y)s(y)}{p(x)+p(y)}}}\right).
\end{aligned}
$$

It is easy to show that $T:\ W^{s(\cdot),p(\cdot)}(\Omega)\to T(W^{s(\cdot),p(\cdot)}(\Omega))$ is an isometric mapping. As $W^{s(\cdot),p(\cdot)}(\Omega)$ is a Banach space, $T(W^{s(\cdot),p(\cdot)}(\Omega))$ is a closed subspace in $L^{p(\cdot)}(\Omega)\times L^{\frac{p(\cdot)+p(\cdot)}{2}}(\Omega\times\Omega)$ and by Theorem 1.21 in [1], $T(W^{s(\cdot),p(\cdot)}(\Omega))$ is separable, i.e. $W^{s(\cdot),p(\cdot)}(\Omega)$ is separable. □

Theorem 2.2. *If* $1<p(x)\le p^+<\infty$, *then* $W^{s(\cdot),p(\cdot)}(\Omega)$ *is strictly convex.*

Proof. Here we use the following equivalent definition of strictly convex spaces. Let $(X,\|\cdot\|)$ be normed linear space, $(X,\|\cdot\|)$ is called strictly convex if for every $u\in X, v\in X, u\neq 0, v\neq 0$, the equality $\|u+v\|=\|u\|+\|v\|$ implies $u=\lambda v$, where λ is positive.

For every $u\in W^{s(\cdot),p(\cdot)}(\Omega), v\in W^{s(\cdot),p(\cdot)}(\Omega), u\neq 0, v\neq 0, \|u+v\|_{W^{s(\cdot),p(\cdot)}(\Omega)}=\|u\|_{W^{s(\cdot),p(\cdot)}(\Omega)}+\|v\|_{W^{s(\cdot),p(\cdot)}(\Omega)}$, we assert that

$$\|u+v\|_{L^{p(\cdot)}(\Omega)}=\|u\|_{L^{p(\cdot)}(\Omega)}+\|v\|_{L^{p(\cdot)}(\Omega)}.$$

Indeed by the definition of $\|\cdot\|_{W^{s(\cdot),p(\cdot)}(\Omega)}$, we have

$$\|u+v\|_{L^{p(\cdot)}(\Omega)}+[v+v]_{W^{s(\cdot),p(\cdot)}(\Omega)}=\|u\|_{L^{p(\cdot)}(\Omega)}+[u]_{W^{s(\cdot),p(\cdot)}(\Omega)}+\|v\|_{L^{p(\cdot)}(\Omega)}+[v]_{W^{s(\cdot),p(\cdot)}(\Omega)}.$$

If $\|u+v\|_{L^{p(\cdot)}(\Omega)}<\|u\|_{L^{p(\cdot)}(\Omega)}+\|v\|_{L^{p(\cdot)}(\Omega)}$, we obtain

$$[u+v]_{W^{s(\cdot),p(\cdot)}(\Omega)}>[u]_{W^{s(\cdot),p(\cdot)}(\Omega)}+[v]_{W^{s(\cdot),p(\cdot)}(\Omega)}$$

which is contradict to the fact $[u+v]_{W^{s(\cdot),p(\cdot)}(\Omega)}\le[u]_{W^{s(\cdot),p(\cdot)}(\Omega)}+[v]_{W^{s(\cdot),p(\cdot)}(\Omega)}$.

By Theorem 1 in [21], $(L^{p(\cdot)}(\Omega),\|\cdot\|_{L^{p(\cdot)}(\Omega)})$ is strictly convex, then there exists $\lambda>0$ such that $u=\lambda v$. □

Remark. Singer introduced k-strict convexity in [22]. It is defined as follows: the normed space $(X,\|\cdot\|)$ is k-strictly convex, if for any $x_0,x_1,\cdots,x_k\in X$, $\|x_0+x_1+\cdots+x_k\|=\|x_0\|+\|x_1\|+\cdots+\|x_k\|$ implies $x_0,x_1,\cdots,x_k$ is linearly dependent. It is easy to see that strictly convex is 1-strictly convex. If the normed space $(X,\|\cdot\|)$ is k-strictly convex, then for any $m\ge k$, the normed space $(X,\|\cdot\|)$ must be m-strictly convex. So if $1<p(x)\le p^+<\infty$, $W^{s(\cdot),p(\cdot)}(\Omega)$ is k-strictly convex.

Theorem 2.3. *If* $|\Omega|<+\infty$ *and* $p^+<\infty$, *then the set of all bounded measurable functions is dense in* $W^{s(\cdot),p(\cdot)}(\Omega)$.

Proof. For any $u\in W^{s(\cdot),p(\cdot)}(\Omega)$, define a sequence of functions

$$u_k(x)=\begin{cases}u(x), & \text{if } |u(x)|\le k,\\ k, & \text{if } u(x)>k,\\ -k, & \text{if } u(x)<-k,\end{cases}\qquad k=1,2,\cdots.$$

Then we have that
1. $u_k(x) \to u(x)$, a.e. in Ω;
2. $|u_k(x)| \le |u(x)|$;
3. $|u_k(x) - u_k(y)| \le |u(x) - u(y)|$, for any $x, y \in \Omega$.

By Lebesgue Dominated Convergence Theorem, we have that $\rho(u_k) \to \rho(u)$, $\varphi(u_k) \to \varphi(u)$. By Proposition 2.6, we have $u_k \xrightarrow{\|\cdot\|} u$. □

3. An application

In this section, we discuss the Dirichlet boundary value problems of $s(x)$-$p(x)$-Laplacian equations. First let $W_0^{s(\cdot),p(\cdot)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{s(\cdot),p(\cdot)}(\Omega)$, i.e.

$$W_0^{s(\cdot),p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{s(\cdot),p(\cdot)}(\Omega)}}.$$

Define the $s(x)$-$p(x)$-Laplacian operator $\mathcal{F}$ as

$$\mathcal{F}u(x) := \int_\Omega \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}-2}(u(x) - u(y))}{|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dy.$$

Let Ω be a bounded domain in $\mathbb{R}^n$, $p(\cdot)$ and $s(\cdot)$ be continuous, $1 < p^- \le p^+ < \infty$. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies the following conditions:

(f1) There exist $q(x)$ with $1 < q(x) \le q^+ < p^-$ and constant $C > 0$ such that $|f(x,t)| \le C(1 + |t|^{q(x)-1})$ for a.e. $x \in \Omega$ and each $t \in \mathbb{R}$.

(f2) There exists $\mu > 1$ such that $\mu F(x,t) \le f(x,t)t$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, where

$$F(x,t) = \int_0^t f(x,\tau)d\tau.$$

(f3) For a.e. $x \in \Omega$, $f(x,t)$ is monotonically decreasing with respect to t.

Consider

$$\begin{cases} \mathcal{F}u(x) = f(x,u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Definition 3.1. *We say that $u \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ is a weak solution of problem* (3.1) *if for all $v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ we have*

$$\iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy = \int_\Omega f(x,u(x))v(x)dx.$$

Theorem 3.1. *Problem* (3.1) *has a unique weak solution in $W_0^{s(\cdot),p(\cdot)}(\Omega)$* .

Corresponding to the problem (3.1), consider the energy functional $I : W_0^{s(\cdot),p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy - \int_\Omega F(x,u(x))dx.$$

We know that the critical point of I is the weak solution of the problem (3.1), so we only examine the critical point of I. Before proving Theorem 3.1, we give two theorems to be used. Theorem 3.2 can be inferred from Theorems 1.3 and 1.7 in [24], and Theorem 3.3 is derived from [7].

Theorem 3.2. *Let X be a real reflexive Banach space. If the real functional $I : X \to \mathbb{R}$ is coercive, strictly convex, and has bounded Gâteaux differential in X, then I has a unique minimum point, which is of course also a critical point.*

Theorem 3.3. *Let $p(\cdot)$, $q(\cdot) \in \mathcal{P}(\Omega)$ and suppose $|\Omega \setminus \Omega_\infty^{q(\cdot)}| < \infty$. Then $L^{p(\cdot)} \subset L^{q(\cdot)}$ if and only if $q(x) \le p(x)$ a.e.. Furthermore*

$$\|f\|_{q(\cdot)} \le (1 + |\Omega \setminus \Omega_\infty^{q(\cdot)}|)\|f\|_{p(\cdot)}.$$

Our task is to verify that I is coercive, strictly convex, and has bounded Gâteaux differential in $W_0^{s(\cdot),p(\cdot)}(\Omega)$, so that the only minimum point of I is the critical point, which is the weak solution of the problem (3.1).

Proof of Theorem 3.1.

1. Existence of bounded Gâteaux differential.

Let $I(u) = \psi(u) + \phi(u)$, with

$$\psi(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy,$$

$$\phi(u) = \int_\Omega F(x, u(x))dx.$$

We consider the Gâteaux derivative of ψ and ϕ respectively.

First let $u, h \in W_0^{s(\cdot),p(\cdot)}(\Omega)$. Given $x, y \in \Omega$ and $0 < |t| < 1$, by Mean Value Theorem, there exists $\theta \in (0, 1)$ such that

$$\begin{aligned}
&\frac{|(u(x) - u(y)) + t(h(x) - h(y))|^{\frac{p(x)+p(y)}{2}} - |u(x) - u(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|t||x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} \\
&= \frac{|(u(x) - u(y)) + \theta t(h(x) - h(y))|^{\frac{p(x)+p(y)}{2}-2}(u(x) - u(y))(h(x) - h(y))}{|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} \\
&\le \frac{||u(x) - u(y)| + |h(x) - h(y)||^{\frac{p(x)+p(y)}{2}-2}(u(x) - u(y))(h(x) - h(y))}{|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}.
\end{aligned}$$

The Hölder inequality implies that

$$\int_\Omega \int_\Omega \frac{||u(x) - u(y)| + |h(x) - h(y)||^{\frac{p(x)+p(y)}{2}-2}(u(x) - u(y))(h(x) - h(y))}{|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy < \infty.$$

Hence by Lebesgue Dominated Convergence Theorem, we obtain

$$< \psi'(u), h >= \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{\frac{p(x)+p(y)}{2}-2}(u(x) - u(y))(h(x) - h(y))}{|x - y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy.$$

On the other hand, for any $h \in W_0^{s(\cdot),p(\cdot)}(\Omega)$ consider

$$\begin{aligned}< \phi'(u), h > &= \lim_{t\to 0}\frac{\phi(u+th)-\phi(u)}{t}\\ &= \lim_{t\to 0}\int_\Omega f(x, u+\theta th)hdx, \qquad\qquad 0\le\theta\le 1.\end{aligned}$$

Since (f1) implies

$$\begin{aligned}|f(x,u+\theta th)h| &\le C|1+|u+\theta th|^{q(x)-1}||h|\\ &\le C\Big[\frac{q(x)-1}{q(x)}|1+|u+\theta th|^{q(x)-1}|^{\frac{q(x)}{q(x)-1}}+\frac{1}{q(x)}|h|^{q(x)}\Big]\\ &\le C\Big[\frac{q^+}{q^-}2^{\frac{1}{q^--1}}(1+|u|^{q(x)}+|h|^{q(x)})+\frac{1}{q^-}|h|^{q(x)}\Big]\\ &\le C\Big[1+|u|^{q(x)}+|h|^{q(x)}\Big],\end{aligned}$$

by (f1) once more and Theorem 3.3, we have

$$W_0^{s(\cdot),p(\cdot)}(\Omega)\subset L^{p(\cdot)}(\Omega)\subset L^{q(\cdot)}(\Omega).$$

According to Lebesgue Dominated Convergence Theorem, we know that

$$< \phi'(u), h >= \int_\Omega f(x,u(x))h(x)dx.$$

It is immediate that I' is linear, so now we verify that I' is a bounded functional of $h\in W_0^{s(\cdot),p(\cdot)}(\Omega)$. By Hölder inequality,

$$\begin{aligned}| < \psi'(u), h > | &\le \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}-1}|h(x)-h(y)|}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}}dxdy\\ &\le C\left\|\frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}-1}}{|x-y|^{(n+\frac{s(x)p(x)+s(y)p(y)}{2})\frac{p(x)+p(y)-2}{p(x)+p(y)}}}\right\|_{L^{\frac{p(x)+p(y)}{p(x)+p(y)-2}}(\Omega\times\Omega)}\\ &\quad\times\left\|\frac{|h(x)-h(y)|}{|x-y|^{(n+\frac{s(x)p(x)+s(y)p(y)}{2})\frac{2}{p(x)+p(y)}}}\right\|_{L^{\frac{p(x)+p(y)}{2}}(\Omega\times\Omega)}\\ &= C[u][h]\\ &\le C\|u\|\|h\|.\end{aligned}$$

By Hölder inequality, (f1) and Theorem 3.3,

$$\begin{aligned}| < \phi'(u), h > | &\le \int_\Omega |f(x,u(x))||h(x)|dx\\ &\le C(\int_\Omega |h(x)|dx+\int_\Omega |u|^{q(x)-1}|h(x)|dx)\\ &\le C\|h\|_{L^1(\Omega)}+C\|u\|_{L^{q(\cdot)}(\Omega)}\|h\|_{L^{q(\cdot)}(\Omega)}\\ &\le C(1+\|u\|)\|h\|.\end{aligned}$$

2. I is coercive.

By Young inequality,

$$
\begin{aligned}
I(u) &= \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy - \int_\Omega F(x,u(x))dx \\
&\geq \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy - \int_\Omega |f(x,u(x))\frac{u(x)}{\mu}|dx \\
&\geq \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{\frac{p(x)+p(y)}{2}|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy - \int_\Omega \frac{C}{\mu}(|u(x)| + |u(x)|^{q(x)})dx \\
&\geq \frac{1}{p^+} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{\frac{p(x)+p(y)}{2}}}{|x-y|^{n+\frac{s(x)p(x)+s(y)p(y)}{2}}} dxdy - \frac{C|\Omega|}{\mu} - \frac{C}{\mu}\int_\Omega \frac{|u(x)|^{q(x)}}{q(x)} - \int_\Omega |u(x)|^{q(x)}dx.
\end{aligned}
$$

If we assume $\|u\| > 1$, then we have

$$
I(u) \geq \frac{1}{p^+}\|u\|^{p^-} - (\frac{C}{\mu} + 1)\|u\|^{q^+} - \frac{C|\Omega|}{\mu}.
$$

So as $\|u\| \to +\infty$, $I(u) \to +\infty$.

3. I is strictly convex.

By $p^- \geq 1$, we know that any $u, v \in W_0^{s(\cdot),p(\cdot)}(\Omega)$, $\psi(\frac{u+v}{2}) < \frac{1}{2}\psi(u) + \frac{1}{2}\psi(v)$.

By (f3), we have $\phi(\frac{u+v}{2}) \geq \frac{1}{2}\phi(u) + \frac{1}{2}\phi(v)$, so we assert that I is strictly convex.

Therefore, according to Theorem 3.2, we get the unique minimum value point of I, which is the weak solution of the problem (3.1). □

4. Conclusions

We define a class of variable exponent fractional Sobolev spaces $W^{s(\cdot),p(\cdot)}(\Omega)$, which is a subspace of $L^{p(\cdot)}(\Omega)$, and has variable order $s(x)$ and variable exponent $\frac{p(x)+p(y)}{2}$. $W^{s(\cdot),p(\cdot)}(\Omega)$ is a Banach space under the given norm. We give some basic properties, such as the closed unit ball is equivalent in the sense of $[\cdot]$ and φ, and that the $[\cdot]$-convergent and φ-convergent are equivalent, norm convergent is equivalent to the ρ-convergent and the φ-convergent. If the exponent $p(x)$ satisfies certain conditions, we obtain that $W^{s(\cdot),p(\cdot)}(\Omega)$ is reflexive, separable, strictly convex and the set of all bounded measurable functions is dense in $W^{s(\cdot),p(\cdot)}(\Omega)$. As an application, we obtain the existence and uniqueness of weak solutions in $W_0^{s(\cdot),p(\cdot)}(\Omega)$ for Dirichlet boundary value problems of $s(x) - p(x)$-Laplacian equations.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant No. 11771107).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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