AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 6124–6134.
DOI: 10.3934/math.2020393
Received: 06 June 2020
Accepted: 15 July 2020
Published: 28 July 2020

Research article

On entire solutions of certain type of nonlinear differential equations

Fengrong Zhang, Linlin Wu, Jing Yang and Weiran Lü*

Department of Mathematics, China University of Petroleum, Qingdao 266580, P.R. China

* **Correspondence:** Email: luwr@upc.edu.cn.

Abstract: In this paper, we shall extend some results regarding the growth estimate of entire solutions of certain type of linear differential equations to that of nonlinear differential equations. Moreover, our results will include several known results for linear differential equations obtained earlier as special cases.

Keywords: entire solution; differential equation; mathieu equation; hyper-order; nevanlinna theory
Mathematics Subject Classification: 34M10, 30D05, 30D35

1. Introduction and main results

In studying differential equations in the complex plane $\mathbb{C}$, it is always an interesting and quite difficult problem to prove the existence or uniqueness of the entire or meromorphic solution of a given differential equation. Note that for the past five or more decades, Nevanlinna theory of meromorphic functions has been used extensively to tackle problems and derive many interesting results regarding existence and growth of meromorphic solutions of differential equations in complex plane(see, e.g., [13, 16, 20] and [27]). Herein, we assume that the reader is familiar with the fundamental results of Nevanlinna theory and its standard notations such as $m(r, f)$, $N(r, f)$, $T(r, f)$, $S(r, f)$ and etc, see e.g., [11, 24]. However, for the convenience of the reader, we shall repeat some notations needed below.

Given a meromorphic function f, recall that $\alpha \not\equiv 0, \infty$ is a small function with respect to f, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$, possibly outside a set E of r of finite linear measure.

It will be useful to recall the definition of the order $\rho(f)$ and the hyper-order $\rho_2(f)$ of a meromorphic function f. The order is defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

In addition, the hyper-order $\rho_2(f)$ is defined as

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Moreover, let $k \geq 1$ be an integer, we need the following facts, which can be found, e.g., in [11, 24] and references therein.

$$m(r, \frac{f^{(k)}}{f}) = S(r,f) = \begin{cases} O(\log r)\ (r\to\infty), & \text{if } \rho(f) < \infty \\ O(\log(rT(r,f)))\ (r\to\infty, r\notin E), & \text{if } \rho(f) = \infty. \end{cases}$$

A differential polynomial $Q(z,f)$ in f is a finite sum of products of f, derivatives of f, with all the coefficients being small functions of f. Namely

$$Q(z,f) = \sum_{\lambda\in I} a_\lambda f^{\lambda_0}(f')^{\lambda_1}\cdots(f^{(n)})^{\lambda_n}, \tag{1.1}$$

where I is a finite index set. The degree of a single term in $Q(z,f)$ will now be defined as $|\lambda| := \lambda_0 + \lambda_1 + \cdots + \lambda_n$. Of course, the maximal degree of $Q(z,f)$ will then be defined as $|q| := \max_{\lambda\in I}|\lambda|$.

Before stating our main results, we recall some previous results concerning algebraic differential equations. An algebraic differential equation is of the form

$$P(z,w,w',\cdots,w^{(k)}) = 0, \tag{1.2}$$

where P is a polynomial in each of its variables. The equation is of order k, if $w^{(k)}$is the highest derivative appearing in P. An important part of the theory of algebraic differential equations is to investigate the order $\rho(w)$ of solutions w meromorphic in $\mathbb{C}$, preferably in terms of P only. For $k = 1$, Gol'dberg [9] proved that $w(z)$ must be of finite order.

In 1933, Yosida [25] applied the Nevanlinna theory of meromorphic functions to differential equations in the complex plane for the first time and generalized a Malmquists theorem.

The following result was shown by Wittich ([22], pp. 64–65), which can be stated as follows:

Theorem A. Let $P(z,w,w',\cdots,w^{(l)}) = \sum_{n_0\cdots n_l} a_{n_0\cdots n_l}(z)w^{n_0}(w')^{n_1}\cdots(w^{(l)})^{n_l} = 0$ be an algebraic differential equation, where $a_{n_0 n_1\cdots n_l}(z)$ are polynomials. Then the above equation has no transcendental entire solution if only one term appears in the equation with a maximal degree.

Since then many interesting results related to the growth or existence of meromorphic solutions of certain types of generalized differential equations were derived or obtained, see, e.g., [1, 2, 6, 8, 18, 19]. Since 1970's, Nevanlinna's value distribution theory (particularly Clunie type of lemmas relating equations involving differential polynomials) have been used or utilized by some authors (see, e.g., [15, 16]) to tackle the nonlinear differential equations of the form

$$f^n + P_d(z,f) = h,$$

where $P_d(z,f)$ denotes a polynomial in f and its derivatives with a total degree $d \leq n-1$, with small functions of f as the coefficients, and h is a given entire or meromorphic function.

Zhang and Liao [26] obtained the following result.

Theorem B. Let a, p_1, p_2 and λ be nonzero constants. Then the equation

$$f^3(z) + af'(z) = p_1 e^{\lambda z} + p_2 e^{-\lambda z}$$

does not have any transcendental entire solution.

In the paper, we first shall study the relevant problems for the following type of differential equation:

$$Q(z, f) = a_1(z)e^{b_1(z)} + a_2(z)e^{b_2(z)} + p(z), \tag{1.3}$$

where $Q(z, f)$ denotes a differential polynomial in f.

Theorem 1.1. Assume that a differential polynomial $Q(z, f)$ (with no constant term, i.e., $Q(z, 0) \equiv 0$) contains just one term of maximal degree, $p\,(\not\equiv 0)$ is a small function of the exponential function e^z, and a_i, $b_i\,(i = 1, 2)$ are nonzero polynomials with $b_1 \not\equiv b_2$. If f is an admissible entire solution to Eq (1.3), then we have $\rho(f) = \max\{\deg(b_1), \deg(b_2)\}$.

For example, the differential equation

$$256f^4 - 64ff'' = e^{4z} + e^{-4z} - 2$$

has a transcendental entire solution $f(z) = (e^z + e^{-z})/4$ and $\rho(f) = 1$.

It is not difficult to verify that $f(z) = \sin z + 1$ is a solution to the equation

$$4f^3 - 12(f'')^2 + 3f'' - 12f = -\sin(3z) - 8,$$

and $\rho(f) = 1$.

The equation $f^2 + 8f'' - 8f = 16e^{2z} + 4e^{-2z} - 16$ has exactly two entire solutions, namely $f_1(z) = 4e^z - 2e^{-z}$ and $f_2(z) = -4e^z + 2e^{-z}$. Obviously, $\rho(f_1) = \rho(f_2) = 1$. In fact, this equation has no other meromorphic solutions satisfying $N(r, f) = S(r, f)$ (see, e.g., [14]).

Theorem 1.1 fails if $Q(z, f)$ contains more than one term of the maximal degree. Indeed, the differential equation

$$f' - (\cos z)f = -z\cos z + 1$$

is solved by $f(z) = e^{\sin z} + z$, and we would get $\rho(f) = +\infty$.

Next, we will deal with the growth of entire transcendental solutions of nonlinear differential equations. Before stating our next result, we recall some previous results concerning the second order homogeneous linear differential equation of type

$$f'' + P(e^z)f' + Q(e^z)f = 0, \tag{1.4}$$

where $P(e^z)$ and $Q(e^z)$ are polynomials in e^z and they are not both constants. It is well known that every solution f of (1.4) is an entire function.

In fact, this type differential equation was discussed by Mathieu in 1868 in connection with the problem of vibrations of an elliptic membrane in the following manner:

Assume that the membrane, which is in the plane XOY when it is in equilibrium, is vibrating with frequency p. Then, if we write

$$V = u(x, y)\cos(pt + \epsilon),$$

the equation of two dimensional wave motion

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{p^2}{c^2}u = 0. \tag{1.5}$$

By a slight transformation, Eq (1.5) will become a linear differential equation, of the second order, of the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + (a + 16q\cos 2z)u = 0, \tag{1.6}$$

where a, q are constants.

Mathieu Eq (1.6) has some important applications, which can be found, e.g., in [21] and references therein for interested readers.

Thereafter up to the present, in this direction, many good results were obtained, see, e.g., [5, 12]. It is also worth mentioned that when they studied the growth on (1.4), the so called central index method was used. The purpose of the present paper is to extend these works (see, e.g., [3, 4, 7, 10, 17]) concerning the nature of solutions of linear differential equations to nonlinear differentials. Moreover, our results will include several known results for linear differential equations obtained earlier as special cases.

Before proceeding further, we give an idea of the problem we are dealing with the growth of the solution to (1.6) by the following assertion.

According to (1.6) and the fact (see, e.g., [11]):

$$T(r, \mathrm{e}^{P(z)}) \sim T(r, \mathrm{e}^{a_k z^k}) \sim \frac{|a_k|}{\pi} r^k \ (r \to \infty),$$

where $P(z) = a_k z^k + a_{k-1}z^{k-1} + \cdots + a_0$ is a polynomial with $a_k(\neq 0)$, then, any solution $u \not\equiv 0$ of (1.6) satisfies

$$m(r, \frac{u''}{u}) = m(r, a + 16q\cos 2z) = T(r, a + 16q\cos 2z) \sim \frac{4}{\pi} r \ (r \to \infty),$$

which shows that $\rho(u) = \infty$.

Motivated by this, our next aim is to discuss the growth of solutions to some more general forms of Eq (1.6). To bring about our results from the more general hypotheses without complicated calculations will probably be the most interesting feature of this note. For convenience, in what follows, we set

$$M_i(z) = a_{0i}z^{k_i} + a_{1i}z^{k_i-1} + \cdots + a_{ki}, \ k^* := \max\{k_i, \ 1 \le i \le m\},$$

where $a_{0i}, a_{1i}, \cdots, a_{ki}$ are constants with $a_{0i} \neq 0\,(i = 1, 2, \cdots, m)$.

$$N_j(z) = b_{0j}z^{l_j} + b_{1j}z^{l_j-1} + \cdots + b_{lj}, \ l^* := \max\{l_j, 1 \le j \le n\},$$

where $b_{0j}, b_{1j}, \cdots, b_{lj}$ are constants with $b_{0j} \neq 0\,(j = 1, 2, \cdots, n)$.

$$P_j(f) = f^{\lambda_{0j}}(f')^{\lambda_{1j}} \cdots (f^{(q)})^{\lambda_{qj}}, \ d_j = \lambda_{0j} + \lambda_{1j} + \cdots + \lambda_{qj},$$

where $\lambda_{0j}, \lambda_{1j}, \cdots, \lambda_{qj}$ $(j = 1, 2, \cdots, n)$ are nature numbers.

Theorem 1.2 Suppose that $\alpha_i(z)(i = 1, \cdots, m)$, $\beta_j(z)(j = 1, \cdots, n)$ are nonzero polynomials and $k^* > l^*$. If $d_1 = d_2 = \cdots = d_n = d \geq 1$, then every entire solution $f(\not\equiv 0)$ of the equation

$$\sum_{j=1}^{n} \beta_j(z) e^{N_j(z)} P_j(f) + \sum_{i=1}^{m} \alpha_i(z) e^{M_i(z)} f^d = 0 \tag{1.7}$$

satisfies $\rho(f) = \infty$ and $\rho_2(f) = k^*$.

The following examples show that the conclusion of Theorem 1.2 can occur.

Example 1.1 The equation

$$f' - (\cos z) f = 0$$

is satisfied by $f(z) = e^{\sin z}$. It is easy to see that $\rho(f) = \infty$ and $\rho_2(f) = 1$.

Example 1.2 Consider the nonlinear equation

$$f'' f' - (e^{2z} + e^{3z}) f^2 = 0.$$

Then $f(z) = e^{e^z}$ solves the above equation and $\rho(f) = \infty$, $\rho_2(f) = 1$.

Example 1.3 The nonlinear equation

$$f' f + (f')^2 - (2z e^{z^2} + 4z^2 e^{2z^2}) f^2 = 0$$

has an entire solution $f(z) = e^{e^{z^2}}$ with $\rho(f) = \infty$ and $\rho_2(f) = 2$.

Example 1.4 The linear differential equation

$$f''' - f' - e^{3z} f = 0$$

has three entire solutions $f_j(z) = e^{c_j e^z - z}$, in which $c_j^3 = 1$, $j = 1, 2, 3$. Obviously, $\rho(f_j) = \infty$, $\rho_2(f_j) = 1$ $(j = 1, 2, 3)$.

The following examples show that the condition $k^* > l^*$ is necessary.

Example 1.5 The equation

$$(z^2 + z e^z) f'' - 2 e^z f' - 6 f = 0$$

is satisfied by $f(z) = z^3$, and $\rho(f) = 0$.

Example 1.6 It is clear that the differential equation

$$e^z f'' + f' - (1 + e^z) f = 0$$

has a solution $f = e^z$, but $\rho(f) = 1$, and $\rho_2(f) = 0$.

2. Some Lemmas

In order to prove our results, we also need the following results.

The determinant $\omega(f_1, \cdots, f_n)$ is called the Wronskian of $f_1, \cdots, f_n$, and which is given by

$$\omega(f_1, \cdots, f_n) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Lemma 2.1 (see, e.g., Proposition 1.4.2 in [13]) Let $f_1, \cdots, f_n$ be meromorphic functions. Then $\omega(f_1, \cdots, f_n)$ vanishes identically if and only if $f_1, \cdots, f_n$ are linearly dependent.

Lemma 2.2 (see, e.g., Lemma 1.9 in [24]) Suppose that $g_1, g_2, \cdots, g_n$ are entire functions and $a_0, a_1, a_2, \cdots, a_n$ are meromorphic functions such that

$$T(r, a_j) = o(\sum_{k=1}^{n} T(r, e^{g_k}))\ (r \to \infty,\ r \notin E,\ j = 0, 1, 2, \cdots, n),$$

where E is a set whose linear measure is finite. If

$$\sum_{j=1}^{n} a_j e^{g_j} \equiv a_0,$$

then there exist constants $c_1, c_2, \cdots, c_n$, at least one of them is not zero, such that

$$\sum_{j=1}^{n} c_j a_j e^{g_j} \equiv 0.$$

Lemma 2.3 (see, e.g., pp.35, Corollary in [24]) Let f and g be meromorphic functions such that $T(r, f) = O(T(r, g))\,(r \to \infty, r \notin E)$, where E is a set of finite measure. Then $\rho(f) \leq \rho(g)$.

The following Lemma is crucial to the proofs of our main results.

Lemma 2.4 Let $F(z) = \sum_{i=1}^{n} a_i(z)e^{b_i(z)}$, and a_i, $b_i\,(i = 1, 2, \cdots, n)$ be nonzero polynomials with $b_i \not\equiv b_j, i \neq j$. Then

$$\rho(F) = \deg(b_s),$$

in which $\deg b_s := \max\{\deg b_i,\ i = 1, 2, \cdots, n\}$.

Proof: First, by Lemma 2.1 and Lemma 2.2, we assume without loss of generality, that $a_1(z)e^{b_1(z)}$, $a_2(z)e^{b_2(z)}, \cdots, a_n(z)e^{b_n(z)}$ are linear independence. Thus we see that

$$\omega = \begin{vmatrix} a_1e^{b_1} & a_2e^{b_2} & \cdots & a_ne^{b_n} \\ (a_1e^{b_1})' & (a_2e^{b_2})' & \cdots & (a_ne^{b_n})' \\ \vdots & \vdots & \cdots & \vdots \\ (a_1e^{b_1})^{(n-1)} & (a_2e^{b_2})^{(n-1)} & \cdots & (a_ne^{b_n})^{(n-1)} \end{vmatrix} \not\equiv 0,$$

and which is normally written as $\omega = \omega(a_1 e^{b_1}, a_2 e^{b_2}, \cdots, a_n e^{b_n})$.

Again, by repeated differentiation in both sides of $F(z) = \sum_{i=1}^{n} a_i(z) e^{b_i(z)}$, we have

$$\begin{cases} F = a_{1\,1} e^{b_1} + a_{1\,2} e^{b_2} + \cdots + a_{1\,n} e^{b_n} \\ F' = a_{2\,1} e^{b_1} + a_{2\,2} e^{b_2} + \cdots + a_{2\,n} e^{b_n} \\ \cdots \qquad \cdots \qquad \cdots \\ F^{(n-1)} = a_{n\,1} e^{b_1} + a_{n\,2} e^{b_2} + \cdots + a_{n\,n} e^{b_n} \end{cases} \tag{2.1}$$

with $a_{1j} = a_j$, $a_{kj} = a'_{k-1\,j} + a_{k-1\,j} b'_j$, $k = 2, 3, \cdots, n$, $j = 1, 2, \cdots, n$.

On the other hand, one may show that

$$a_{kj} = (a_{k-1\,j} e^{b_j})' e^{-b_j},\ k = 2, 3, \cdots, n,\ \ j = 1, 2, \cdots, n. \tag{2.2}$$

Further, we set $D = \begin{vmatrix} a_{1\,1} & a_{1\,2} & \cdots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \cdots & a_{2\,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n\,1} & a_{n\,2} & \cdots & a_{n\,n} \end{vmatrix}$.

It follows from the properties of determinant and (2.2) that $D \not\equiv 0$. For sake of brevity and simplicity, we, as a rule, now make the following reasonable assumption: $\deg b_1 = \max\{\deg b_i,\ i = 1, 2, \cdots, n\}$. Thus, by Cramer's ruler, (2.1) gives

$$e^{b_1} = \frac{D_1}{D} \tag{2.3}$$

with $D_1 = \begin{vmatrix} F & a_{1\,2} & \cdots & a_{1\,n} \\ F' & a_{2\,2} & \cdots & a_{2\,n} \\ \vdots & \vdots & \cdots & \vdots \\ F^{(n-1)} & a_{n\,2} & \cdots & a_{n\,n} \end{vmatrix}$.

In view of (2.3) and the expression of D_1, we find

$$e^{b_1} = \frac{A_{11}}{D} F + \frac{A_{21}}{D} F' + \cdots + \frac{A_{n1}}{D} F^{(n-1)}, \tag{2.4}$$

where $A_{11}, A_{21}, \cdots, A_{n1}$ are co-factors of the first column of D_1. In view of a_i, b_i $(i = 1, 2, \cdots, n)$ are nonzero polynomials, we deduce that $\frac{A_{11}}{D}, \frac{A_{21}}{D}, \cdots, \frac{A_{n1}}{D}$ are small functions of e^{b_1}. In addition, it is easy to see by (2.4) that they are also small functions of F.

Consequently, it follows from the logarithmic derivative lemma and (2.4) that

$$m(r, \frac{e^{b_1}}{F}) = S(r, F). \tag{2.5}$$

Moreover, note that $e^{b_1} = F\,\frac{e^{b_1}}{F}$ and (2.5), we would get

$$T(r, e^{b_1}) = m(r, e^{b_1}) \le T(r, F) + S(r, F).$$

Therefore, $T(r, e^{b_s}) = m(r, e^{b_s}) \le T(r, F) + S(r, F)$ is satisfies, and then Lemma 2.3 immediately implies $b_s = \rho(e^{b_s}) \le \rho(F)$. It is easy to see that

$$T(r, F) \le nT(r, e^{b_s}) + S(r, e^{b_s}).$$

Consequently, on ground of Lemma 2.3, $\rho(F) \leq \rho(e^{b_s}) = b_s$ is correct.

This completes the proof of Lemma 2.4.

Lemma 2.5 ([13]) Let f be a meromorphic solution of an algebraic equation

$$P(z, f, f', \cdots, f^{(n)}) = 0, \tag{2.6}$$

where P is a polynomial in $f, f', \cdots, f^{(n)}$ with meromorphic coefficients small with respect to f. If a complex constant c does not satisfy Eq (2.6), then

$$m(r, \frac{1}{f-c}) = S(r, f).$$

3. Proofs of Theorems

Proof of Theorem 1.1.

Now, we no loss in generality in supposing b_1 and b_2 are not constants. If (1.3) has an entire solution f, we then from (1.3) and Lemma 2.4 get

$$T(r, Q) \geq \max\{T(r, e^{b_1}),\ T(r, e^{b_2})\} + S(r, Q),$$

this shows that p is a small function of Q, and we see that p is also a small function of f.

Again, by (1.3), we have

$$Q' = (a_1' + a_1 b_1')e^{b_1} + (a_2' + a_2 b_2')e^{b_2} + p' := t_1 e^{b_1} + t_2 e^{b_2} + p',$$

which, and (1.3), gives

$$t_1 Q - a_1 Q' = (a_2 t_1 - a_1 t_2)e^{b_2} + t_1 p - a_1 p' := s_1 e^{b_2} + p_1, \tag{3.1}$$

and therefore

$$t_1' Q + (t_1 - a_1')Q' - a_1 Q'' = (s_1' + s_1 b_2')e^{b_2} + p_1'. \tag{3.2}$$

According to (3.1) and (3.2), we obtain

$$(s_1' + s_1 b_2')(t_1 Q - a_1 Q') - s_1[t_1' Q + (t_1 - a_1')Q' - a_1 Q''] = s_1' p_1 - s_1 p_1' + s_1 p_1 b_2'. \tag{3.3}$$

If $s_1' p_1 - s_1 p_1' + s_1 p_1 b_2' \equiv 0$, then $A p_1 = s_1 e^{b_2}$ for a constant A, it is a contradiction. Now, we may assume that $s_1' p_1 - s_1 p_1' + s_1 p_1 b_2' \not\equiv 0$. Thus applying Lemma 2.5 (where $c = 0$ is used) to (3.3), we have

$$m(r, \frac{1}{f}) = S(r, f). \tag{3.4}$$

Let N denote the maximal degree of $Q(z, f)$. Now we claim that

$$m(r, Q) = N\, m(r, f) + S(r, f). \tag{3.5}$$

To prove (3.5), we rewrite $Q(z, f)$ as

$$Q(z, f) = \sum_{j=1}^{N} P_j f^j, \tag{3.6}$$

where $P_j\,(j = 1, 2, \cdots, N)$ denote a polynomial of $f^{(k)}/f, k \in \mathbb{N}$. Obviously, it follows by (3.4) and (3.6) that

$$m(r, Q) \leq N\,m(r, f) + S(r, f). \tag{3.7}$$

On the other hand, according to (3.6)

$$|Q(z, f)| = |f^N||P_N + P_{N-1}\frac{1}{f} + \cdots + P_1\frac{1}{f^{N-1}}| \tag{3.8}$$

is correct. By the same methods as used in [23], we then by (3.8) have

$$N\,m(r, f) \leq m(r, Q) + S(r, f) \tag{3.9}$$

and thereby (3.5) immediately follows from (3.7) and (3.9).

Therefore, we have $N\,m(r, f) = \max\{T(r, e^{b_1}), T(r, e^{b_2})\} + S(r, f)$, which implies $\rho(f) = \max\{\deg(b_1), \deg(b_2)\}$, and Theorem 1.1 follows.

Proof of Theorem 1.2.

We first of all rewrite (1.7) as

$$\sum_{j=1}^{n} \beta_j(z) e^{N_j(z)} \frac{P_j(f)}{f^d} + \sum_{i=1}^{m} \alpha_i(z) e^{M_i(z)} = 0. \tag{3.10}$$

Suppose that f is of finite order. Since $\alpha_i(z)(i = 1, \ldots, m)$, $\beta_j(z)(j = 1, \ldots, n)$ are nonzero polynomials, it follows by Lemma 2.4 and (3.10) that

$$O(r^{k^*}) \leq O(r^{l^*}) + O(\log r),$$

which is impossible. Thus the order of f must be infinite and then it is easily shown in this case there exists an integer $l\,(1 \leq l \leq n)$ such that

$$m(r, f^{(l)}/f) = O(r^{k^*}),$$

which gives

$$S(r, f) = O(\log(rT(r, f))\,(r \to \infty,\ r \notin E),$$

where E is a set whose linear measure is not greater than 2. Thus, we find

$$m(r, f^{(l)}/f) = O(\log rT(r, f)) = O(r^{k^*})\,(r \to \infty,\ r \notin E),$$

or

$$r^{k^*} = O(\log rT(r, f))\,(r \to \infty,\ r \notin E),$$

which and Lemma 2.3 will lead to the conclusion that the order of $\log T(r, f)$ is k^*, which will lead to the conclusion that hyper-order of f must be k^*. The assertion follows.

Finally, we would like to pose the following conjecture, for further studies.

Conjecture. The asserts of the present paper remain to be valid for meromorphic solutions f satisfying $N(r, f) = S(r, f)$ or $N(r, f)$ is of finite order.

4. Conclusions

Using the theory of meromorphic functions and the Cramer's rule, this paper extends some results on entire solutions of certain type of linear differential equations to that of nonlinear differential equations. Some examples show that the existence of solutions for such equations. Meanwhile, a conjecture is posed for further studies.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which led to considerable improvement of the article.

This work was supported by the National Natural Science Foundation of China (Nos. 11602305, 11601521) and the Fundamental Research Funds for the Central Universities (Nos. 18CX02045A, 17CX02048).

Conflict of interest

The authors declare no conflict of interest.

References

1. S. Bank, *Some results on analytic and meromorphic solutions of algebraic differential equations*, Adv. Math., **15** (1975), 41–62.
2. S. Bank, J. Langley, *Oscillation theory for higher order linear differential equations with entire coeffients*, Complex Var. Elliptic Eq., **16** (1991), 163–175.
3. Z. X. Chen, *The growth of solutions of* $f'' + e^{-z}f' + Q(z)f = 0$ *where the order* $(Q) = 1$, Sci. China Ser. A, **45** (2002), 290–300.
4. Z. X. Chen, K. H. Shon, *The hyper order of solution of second order differential equations and subnomal solutions of periodic equations*, Taiwanese J. Math., **14** (2010), 611–628.
5. Z. X. Chen, C. C. Yang, *Some further results on the zeros and growths of entire solutions of second order linear differential equations*, Kodai Math. J., **22** (1999), 273–285.
6. A. E. Eremenko, *Meromorphic solutions of algebraic differential equations*, Russ. Math. Surv., **37** (1982), 61–95.
7. M. Frei, *Über die subnormalen Löungen der Differentialgleichung* $\omega'' + e^{-z}\omega' + (konst.)\omega = 0$, Comment. Math. Helv. **36** (1962), 1–8.
8. F. Gackstatter, I. Laine, *Zur Theorie der gewAhnlichen Differentialgleichungen im Komplexen*, Ann. Polo. Math., **38** (1980), 259–287.
9. A. A. Gol'dberg, *On the single valued integrals of differential equations of the first order*, Ukrainian Math. J., **8** (1956), 254–261 (Russian).
10. G. Gundersen, *On the question of whether* $f'' + e^{-z}f' + B(z)f = 0$ *can admit a solution* $f \not\equiv 0$ *of finite order*, Proc. Roy. Soc. Edinburgh Sect. A, **102** (1986), 9–17.

11. W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.
12. S. Hellerstein, J. Miles, J. Rossi, *On the growth of solutions of certain linear differential equations*, Ann. Acad. Sci. Fenn., Ser. A 1 Math., **17** (1992), 343–365.
13. I. Laine, *Nevanlinna theory and complex differential equations*, Walter de Gruyter, Berlin/New York, 1993.
14. P. Li, *Entire solutions of certain type of differential equations II*, J. Math. Anal. Appl., **375** (2011), 310–319.
15. P. Li, C. C. Yang, *On the nonexistence of entire solutions of certain type of nonlinear differential equations*, J. Math. Anal. Appl., **320** (2006), 827–835.
16. L. W. Liao, C. C. Yang, J. J. Zhang, *On meromorphic solutions of certain type of non-linear differential equations*, Ann. Acad. Sci. Fenn. Math., **38** (2013), 581–593.
17. M. Ozawa, *On a solution of* $\omega'' + e^{-z}\omega' + (az + b)\omega = 0$, Kodai Math. J., **3** (1980), 295–309.
18. N. Steinmetz, *Meromorphic solutions of second-order algebraic differential equations*, Complex Var. Elliptic Eq., **13** (1989), 75–83.
19. N. Toda, *On algebroid solutions of algebraic differential equations in the complex plane II*, J. Math. Soc. Japan, **45** (1993), 705–717.
20. J. Wang, W. R. Lü, *The fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients*, Acta Math. Appl. Sinica, **27** (2004), 72–80 (Chinese).
21. E. T. Whittaker, G. N. Watson, *A course of modern analysis*, Cambridge University Press, Cambridge, 1927.
22. H. Wittich, *Neuere Untersuchungen über eindeutige analytische Funktionen*, Springer, Berlin Heidelberg, 1968.
23. C. C. Yang, *On deficiencies of differential polynomials*, Math. Z., **116** (1970), 197–204.
24. C. C. Yang, H. X. Yi, *Uniqueness theory of meromorphic functions*, Science Press, Beijing/New York, 2003.
25. K. Yosida, *A generalization of Malmquist's theorem*, Japan. J. Math., **9** (1933), 253–256.
26. J. Zhang, L. W. Liao, *On entire solutions of a certain type of nonlinear differential and difference equations*, Taiwanese J. Math., **15** (2011), 2145–2157.
27. J. J. Zhang, X. Q. Lu, L. W. Liao, *On exact transcendental meromorphic solutions of nonlinear complex differential equations*, Houston J. Math., **45** (2019), 439–453.