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*Research article*

# Some unified bounds for exponentially $tgs$-convex functions governed by conformable fractional operators

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**Abstract:** In the article, we introduce the concept of the exponentially $tgs$-convex function and discover two new conformable fractional integral identities concerning the first-order differentiable convex mappings. By using these identities, we establish several new right-sided Hermite-Hadamard type inequalities for the exponentially $tgs$-convex functions via conformable fractional integrals. Our outcomes for conformable fractional integral operators are also applied to some special means.

**Keywords:** integral inequality; exponentially $tgs$-convex function; conformable fractional integral operator; Hermite-Hadamard inequality
**Mathematics Subject Classification:** 26D15, 26D10, 90C23

## 1. Introduction

On different time ranges, fractional calculus has a great impact due to a diversity of applications that have contributed to several fields of technical sciences and engineering [1–12]. One of the principal options behind the popularity of the area is that fractional-order differentiations and integrations are more beneficial tools in expressing real-world matters than the integer-order ones. Various studies in the literature, on distinct fractional operators such as the classical Riemann-Liouville, Caputo, Katugamploa, Hadamard, and Marchaud versions have shown versatility in modeling and control applications across various disciplines. However, such forms of fractional derivatives may not be able to explain the dynamic performance accurately, hence, many authors are found to be sorting out new fractional differentiations and integrations which have a kernel depending upon a function and this makes the range of definition expanded [13, 14]. Furthermore, models based on these fractional operators provide excellent results to be compared with the integer-order differentiations [15–27].

The derivatives in this calculus seemed complicated and lost some of the basic properties that usual derivatives have such as the product rule and the chain rule. However, the semigroup properties of these operators behave well in some cases. Recently, the authors in [28] defined a new well-behaved simple derivative called "conformable fractional derivative" which depends just on the basic limit definition of the derivative. It will define the derivative of higher-order (i.e., order $\delta > 1$) and also define the integral of order $0 < \delta \leq 1$ only. It will also prove the product rule and the mean value theorem and solve some (conformable) differential equations where the fractional exponential function $e^{\frac{\vartheta^{\delta}}{\delta}}$ plays an important rule. Inequalities and their utilities assume a crucial job in the literature of pure and applied mathematics [29–37]. The assortment of distinct kinds of classical variants and their modifications were built up by using the classical fractional operators.

Convexity and its applications exist in almost every field of mathematics due to impermanence in several areas of science, technology in nonlinear programming and optimization theory. By utilizing the idea of convexity, numerous variants have been derived by researchers, for example, Hardy, Opial, Ostrowski, Jensen and the most distinguished one is the Hermite-Hadamard inequality [38–41].

Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $\mathcal{Q} : \mathcal{I} \to \mathbb{R}$ be a convex function. Then the double inequality

$$(l_2 - l_1)\mathcal{Q}\Big(\frac{l_1 + l_2}{2}\Big) \leq \int_{l_1}^{l_2} \mathcal{Q}(z)dz \leq (l_2 - l_1)\frac{\mathcal{Q}(l_1) + \mathcal{Q}(l_2)}{2}, \tag{1.1}$$

holds for all $l_1, l_2 \in \mathcal{I}$ with $l_1 \neq l_2$. Clearly, if $\mathcal{Q}$ is concave on $\mathcal{I}$, then one has the reverse of inequality (1.1). By taking into account fractional integral operators, several lower and upper bounds for the mean value of a convex function can be obtained by utilizing of inequality (1.1).

Exponentially convex functions have emerged as a significant new class of convex functions, which have potential applications in technology, data science, and statistics. In [42], Bernstein introduced the concept of exponentially convex function in covariance formation, then the idea of an exponentially convex function is extended by inserting the condition of $r$-convexity [43]. Following this tendency, Jakšetić and Pečarić introduced various kinds of exponentially convex functions in [44] and have contemplated the applications in Euler-Radau expansions and Stolarsky means. Our aim is to utilize the exponential convexity property of the functions as well as the absolute values of their derivatives in order to establish estimates for conformable fractional integral introduced by Abdeljawed [45] and Jarad et al. [46].

Following the above propensity, we present a novel technique for establishing new generalizations of Hermite-Hadamard inequalities that correlate with exponentially $tgs$-convex functions and conformable fractional operator techniques in this paper. The main purpose is that our consequences, which are more consistent and efficient, are accelerated via the fractional calculus technique. In addition, our consequences also taking into account the estimates for Hermite-Hadamard inequalities for exponentially $tgs$-convex functions. We also investigate the applications of the two proposed conformable fractional operator to exponentially $tgs$-convex functions and fractional calculus. The proposed numerical experiments show that our results are superior to some related results.

## 2. Preliminaries

Before coming to the main results, we provide some significant definitions, theorems and properties of fractional calculus in order to establish a mathematically sound theory that will serve the purpose of the current article.

Awan et al. [47] proposed a new class of functions called exponentially convex functions.

**Definition 2.1.** *(See [47]) A positive real-valued function $\mathcal{Q} : \mathcal{K} \subset \mathbb{R} \rightarrow (0,\infty)$ is said to be exponentially convex on $\mathcal{K}$ if the inequality*

$$\mathcal{Q}(\vartheta l_1 + (1-\vartheta)l_2) \leq \vartheta \frac{\mathcal{Q}(l_1)}{e^{\alpha l_1}} + (1-\vartheta)\frac{\mathcal{Q}(l_2)}{e^{\alpha l_2}}, \tag{2.1}$$

*holds for all $l_1, l_2 \in \mathbb{R}, \alpha \in \mathbb{R}$ and $\vartheta \in [0,1]$.*

Now, we introduce a novel concept of convex function which is known as the exponentially $tgs$-convex function.

**Definition 2.2.** *A positive real-valued function $\mathcal{Q} : \mathcal{K} \subset \mathbb{R} \rightarrow (0,\infty)$ is said to be exponentially $tgs$-convex on $\mathcal{K}$ if the inequality*

$$\mathcal{Q}(\vartheta l_1 + (1-\vartheta)l_2) \leq \vartheta(1-\vartheta)\Big[\frac{\mathcal{Q}(l_1)}{e^{\alpha l_1}} + \frac{\mathcal{Q}(l_2)}{e^{\alpha l_2}}\Big], \tag{2.2}$$

*holds for all $l_1, l_2 \in \mathbb{R}, \alpha \in \mathbb{R}$ and $\vartheta \in [0,1]$.*

The conformable fractional integral operator was introduced by Abdeljawad [45].

**Definition 2.3.** *(See [45]) Let $\rho \in (n, n+1]$ and $\delta = \rho - n$. Then the left and right-sided conformable fractional integrals of order $\rho > 0$ is defined by*

$$\mathcal{J}^{\rho}_{l_1^+}\mathcal{Q}(z) = \frac{1}{n!}\int_{l_1}^{z} (z-\vartheta)^n(\vartheta - l_1)^{\rho-1}\mathcal{Q}(\vartheta)d\vartheta \tag{2.3}$$

*and*

$$\mathcal{J}^{\rho}_{l_2^-}\mathcal{Q}(z) = \frac{1}{n!}\int_{z}^{l_2} (\vartheta - z)^n(l_2 - \vartheta)^{\rho-1}\mathcal{Q}(\vartheta)d\vartheta. \tag{2.4}$$

Next, we demonstrate the following fractional integral operator introduced by Jarad et al. [46].

**Definition 2.4.** *(See [46]) Let $\delta \in \mathbb{C}$ and $\Re(\delta) > 0$. Then the left and right-sided fractional conformable integral operators of order $\rho > 0$ are stated as:*

$$\mathcal{J}^{\rho,\delta}_{l_1^+}\mathcal{Q}(z) = \frac{1}{\Gamma(\delta)}\int_{l_1}^{z} \Big(\frac{(z-l_1)^{\rho} - (\vartheta - l_1)^{\rho}}{\rho}\Big)^{\delta-1}\frac{\mathcal{Q}(\vartheta)}{(\vartheta - l_1)^{1-\rho}}d\vartheta \tag{2.5}$$

*and*

$$\mathcal{J}^{\rho,\delta}_{l_2^-}\mathcal{Q}(z) = \frac{1}{\Gamma(\delta)}\int_{l_1}^{z} \Big(\frac{(l_2-z)^{\rho} - (l_2 - \vartheta)^{\rho}}{\rho}\Big)^{\delta-1}\frac{\mathcal{Q}(\vartheta)}{(l_2 - \vartheta)^{1-\rho}}d\vartheta. \tag{2.6}$$

Recalling some special functions which are known as beta and incomplete beta function.

$$\mathbb{B}(l_1, l_2) = \int_0^1 \vartheta^{l_1-1}(1-\vartheta)^{l_2-1} d\vartheta,$$

$$\mathbb{B}_v(l_1, l_2) = \int_0^v \vartheta^{l_1-1}(1-\vartheta)^{l_2-1} d\vartheta, \qquad v \in [0,1].$$

Further, the following relationship holds between classical Beta and incomplete Beta functions:

$$\mathbb{B}(l_1, l_2) = \mathbb{B}_v(l_1, l_2) + \mathbb{B}_{1-v}(l_1, l_2),$$

$$\mathbb{B}_v(l_1+1, l_2) = \frac{l_1 \mathbb{B}_v(l_1, l_2) - (\frac{1}{2})^{l_1+l_2}}{l_1 + l_2}$$

and

$$\mathbb{B}_v(l_1, l_2+1) = \frac{l_2 \mathbb{B}_v(l_1, l_2) - (\frac{1}{2})^{l_1+l_2}}{l_1 + l_2}.$$

## 3. Hermite-Hadamard type inequality for exponentially tgs-convex functions via conformable fractional integrals

Throughout the article, let $\mathcal{I} = [l_1, l_2]$ be an interval in real line $\mathbb{R}$. In this section, we shall demonstrate some integral versions of exponentially tgs-convex functions via conformable fractional integrals.

**Theorem 3.1.** *For $\rho \in (n, n+1])$ with $\rho > 0$ and let $Q : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be an exponentially tgs-convex function such that $Q \in L_1([l_1, l_2])$, then the following inequalities hold:*

$$\begin{aligned} &\frac{4\Gamma(\rho-n)}{\Gamma(\rho+1)} Q\left(\frac{l_1+l_2}{2}\right) \\ &\leq \frac{1}{(l_2-l_1)^\rho}\left[\mathcal{J}^\rho_{l_1^+}\frac{Q(l_2)}{e^{\alpha l_2}} + \mathcal{J}^\rho_{l_2^-}\frac{Q(l_1)}{e^{\alpha l_1}}\right] \\ &\leq \frac{2(n+1)\Gamma(\rho-n+1)}{\Gamma(\rho+3)}\left(\frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}}\right). \end{aligned} \tag{3.1}$$

*Proof.* By using exponentially tgs-convexity of $Q$, we have

$$Q\left(\frac{x+y}{2}\right) \leq \frac{1}{4}\left(\frac{Q(x)}{e^{\alpha x}} + \frac{Q(y)}{e^{\alpha y}}\right). \tag{3.2}$$

Let $x = \vartheta l_1 + (1-\vartheta) l_2$ and $y = (1-\vartheta) l_1 + \vartheta l_2$, we get

$$4Q\left(\frac{l_1+l_2}{2}\right) \leq \frac{Q(\vartheta l_1 + (1-\vartheta) l_2)}{e^{\alpha Q(\vartheta l_1 + (1-\vartheta) l_2)}} + \frac{Q(\vartheta l_2 + (1-\vartheta) l_1)}{e^{\alpha[(1-\vartheta) l_1 + \vartheta l_2]}}. \tag{3.3}$$

If we multiply (3.3) by $\frac{1}{n!}\vartheta^n(1-\vartheta)^{\rho-n-1}$ with $\vartheta \in (0,1), \rho > 0$ and then integrating the resulting estimate with respect to $\vartheta$ over $[0,1]$, we find

$$\begin{aligned}
&\frac{4}{n!}Q\Big(\frac{l_1+l_2}{2}\Big)\int_0^1 \vartheta^n(1-\vartheta)^{\rho-n-1}d\vartheta \\
&\leq \frac{1}{n!}\int_0^1 \vartheta^n(1-\vartheta)^{\rho-n-1}\frac{Q(\vartheta l_1+(1-\vartheta)l_2)}{e^{\alpha Q(\vartheta l_1+(1-\vartheta)l_2)}}d\vartheta \\
&+\frac{1}{n!}\int_0^1 \vartheta^n(1-\vartheta)^{\rho-n-1}\frac{Q(\vartheta l_2+(1-\vartheta)l_1)}{e^{\alpha[(1-\vartheta)l_1+\vartheta l_2]}}d\vartheta \\
&= I_1+I_2 \qquad (3.4)
\end{aligned}$$

By setting $u=\vartheta l_1+(1-\vartheta)l_2$, we have

$$\begin{aligned}
I_1 &= \frac{1}{n!}\int_0^1 \vartheta^n(1-\vartheta)^{\rho-n-1}\frac{Q(\vartheta l_1+(1-\vartheta)l_2)}{e^{\alpha Q(\vartheta l_1+(1-\vartheta)l_2)}}d\vartheta \\
&= \frac{1}{n!(l_2-l_1)^\rho}\int_{l_1}^{l_2}(l_2-1)^n(u-l_1)^{\rho-m-1}\frac{Q(u)}{e^{\alpha u}}du \\
&= \frac{1}{(l_2-l_1)^\rho}\mathcal{J}^\rho_{l_1^+}\frac{Q(l_2)}{e^{\alpha l_2}}. \qquad (3.5)
\end{aligned}$$

Analogously, by setting $v=\vartheta l_2+(1-\vartheta)l_1$, we have

$$\begin{aligned}
I_2 &= \frac{1}{n!}\int_0^1 \vartheta^n(1-\vartheta)^{\rho-n-1}Q(\vartheta l_2+(1-\vartheta)l_1)d\vartheta \\
&= \frac{1}{n!(l_2-l_1)^\rho}\int_{l_1}^{l_2}(v-l_1)^n(l_2-v)^{\rho-n-1}\frac{Q(v)}{e^{\alpha v}}dv \\
&= \frac{1}{(l_2-l_1)^\rho}J^\rho_{l_2^-}\frac{Q(l_1)}{e^{\alpha l_1}}. \qquad (3.6)
\end{aligned}$$

Thus by using (3.5) and (3.6) in (3.4), we get the first inequality of (3.1).

Consider

$$Q(\vartheta l_1+(1-\vartheta)l_2) \leq \vartheta(1-\vartheta)\Big(\frac{Q(l_1)}{e^{\alpha l_1}}+\frac{Q(l_2)}{e^{\alpha l_2}}\Big)$$

and

$$Q(\vartheta l_2+(1-\vartheta)l_1) \leq \vartheta(1-\vartheta)\Big(\frac{Q(l_1)}{e^{\alpha l_1}}+\frac{Q(l_2)}{e^{\alpha l_2}}\Big).$$

By adding

$$Q(\vartheta l_1+(1-\vartheta)l_2)+Q(\vartheta l_2+(1-\vartheta)l_1) \leq 2\vartheta(1-\vartheta)\Big(\frac{Q(l_1)}{e^{\alpha l_1}}+\frac{Q(l_2)}{e^{\alpha l_2}}\Big). \qquad (3.7)$$

If we multiply (3.7) by $\frac{1}{n!}\vartheta^n(1-\vartheta)^{\rho-n-1}$ with $\vartheta \in (0,1), \rho > 0$ and then integrating the resulting inequality with respect to $\vartheta$ over $[0,1]$, we get

$$\frac{1}{(l_2-l_1)^\rho}\Big[\mathcal{J}^\rho_{l_1^+}\frac{Q(l_2)}{e^{\alpha l_2}}+\mathcal{J}^\rho_{l_2^-}\frac{Q(l_1)}{e^{\alpha l_1}}\Big]$$

$$\leq \frac{2(n+1)\Gamma(\rho-n+1)}{\Gamma(\rho+3)}\Big(\frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}}\Big), \tag{3.8}$$

which is the required result. □

Some special cases of above theorem are stated as follows:

**Corollary 3.1.** *Choosing $\alpha = 0$, then Theorem 3.1 reduces to a new result*

$$\frac{4\Gamma(\rho-n)}{\Gamma(\rho+1)} Q\Big(\frac{l_1+l_2}{2}\Big)$$

$$\leq \frac{1}{(l_2-l_1)^{\rho}}\Big[\mathcal{J}^{\rho}_{l_1^+}Q(l_2) + \mathcal{J}^{\rho}_{l_2^-}Q(l_1)\Big]$$

$$\leq \frac{2(n+1)\Gamma(\rho-n+1)}{\Gamma(\rho+3)}\Big(Q(l_1) + Q(l_2)\Big).$$

*Remark* 3.1. Choosing $\rho = n + 1$ and $\alpha = 0$, then Theorem 3.1 reduces to Theorem 3.1 in [19].

## 4. Hermite-Hadamard type inequality for differentiable exponentially tgs-convex functions via conformable fractional integrals

Our next result is the following lemma which plays a dominating role in proving our coming results.

**Lemma 4.1.** *For $\rho \in (n, n+1])$ with $\rho > 0$ and let $Q : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on $\mathcal{I}^{\circ}$(interior of $\mathcal{I}$) with $l_1 < l_2$ such that $Q' \in L_1([l_1, l_2])$, then the following inequality holds:*

$$\mathbb{B}(n+1, \rho-n)\Big(\frac{Q(l_1)+Q(l_2)}{2}\Big) - \frac{n!}{2(l_2-l_1)^{\rho}}[\mathcal{J}^{\rho}_{l_1^+}Q(l_2) + \mathcal{J}^{\rho}_{l_2^-}Q(l_1)]$$

$$= \int_0^1 (\mathbb{B}_{1-u}(n+1, \rho-n) - \mathbb{B}_u(n+1, \rho-n))Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta. \tag{4.1}$$

*Proof.* It suffices that

$$\int_0^1 (\mathbb{B}_{1-u}(n+1, \rho-n) - \mathbb{B}_u(n+1, \rho-n))Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta$$

$$= \int_0^1 \mathbb{B}_{1-u}(n+1, \rho-n)Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta$$

$$- \int_0^1 \mathbb{B}_u(n+1, \rho-n)Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta$$

$$= S_1 - S_2 \tag{4.2}$$

Then by integration by parts, we have

$$S_1 = \int_0^1 \mathbb{B}_{1-u}(n+1, \rho-n)Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta$$

$$= \int_0^1 \Big( \int_0^{1-u} v^n (1-v)^{\rho-n-1} dv \Big) Q'(\vartheta l_1 + (1-\vartheta) l_2) d\vartheta$$

$$= \frac{1}{l_2 - l_1} \mathbb{B}(n+1, \rho-n) Q(l_2)$$

$$- \frac{1}{l_2 - l_1} \int_0^1 (1-u)^n u^{\rho-n-1} Q(\vartheta l_1 + (1-\vartheta) l_2) d\vartheta$$

$$= \frac{1}{l_2 - l_1} \mathbb{B}(n+1, \rho-n) Q(l_2)$$

$$- \frac{1}{l_2 - l_1} \int_{l_2}^{l_1} \Big( \frac{l_1 - z}{l_1 - l_2} \Big)^n \Big( \frac{z - l_2}{l_1 - l_2} \Big)^{\rho-n-1} \frac{Q(z)}{l_1 - l_2} dz$$

$$= \frac{1}{l_2 - l_1} \mathbb{B}(n+1, \rho-n) Q(l_2) - \frac{n!}{(l_2 - l_1)^{\rho+1}} \mathcal{J}^{\rho}_{l_2^-} Q(l_1). \tag{4.3}$$

Analogously

$$S_2 = \int_0^1 \mathbb{B}_u(n+1, \rho-n) Q'(\vartheta l_1 + (1-\vartheta) l_2) d\vartheta$$

$$= \int_0^1 \Big( \int_0^{u} v^m (1-v)^{\rho-n-1} dv \Big) Q'(\vartheta l_1 + (1-\vartheta) l_2) d\vartheta$$

$$= -\frac{1}{l_2 - l_1} \mathbb{B}(n+1, \rho-n) Q(l_1)$$

$$+ \frac{1}{l_2 - l_1} \int_0^1 (u)^n (1-u)^{\rho-n-1} Q(\vartheta l_1 + (1-\vartheta) l_2) d\vartheta$$

$$= -\frac{1}{l_2 - l_1} \mathbb{B}(n+1, \rho-n) Q(l_1)$$

$$+ \frac{1}{l_2 - l_1} \int_{l_2}^{l_1} \Big( \frac{z - l_2}{l_1 - l_2} \Big)^n \Big( \frac{l_1 - z}{l_1 - l_2} \Big)^{\rho-n-1} \frac{Q(z)}{l_1 - l_2} dz$$

$$= -\frac{1}{l_2 - l_1} \mathbb{B}(n+1, \rho-n) Q(l_1) - \frac{n!}{(l_2 - l_1)^{\rho+1}} \mathcal{J}^{\rho}_{l_1^+} Q(l_2). \tag{4.4}$$

By substituting values of $S_1$ and $S_2$ in (4.2) and then If we multiply by $\frac{l_2 - l_1}{2}$, we get (4.1). □

For the sake of simplicity, we use the following notation:

$$\Upsilon_Q(\rho; \mathbb{B}; n; l_1, l_2) = \mathbb{B}(n+1, \rho-n) \Big( \frac{Q(l_1) + Q(l_2)}{2} \Big) - \frac{n!}{2(l_2 - l_1)^{\rho}} \Big[ \mathcal{J}^{\rho}_{l_1^+} Q(l_2) + \mathcal{J}^{\rho}_{l_2^-} Q(l_1) \Big].$$

**Theorem 4.2.** *For $\rho \in (n, n+1])$ with $\rho > 0$ and let $Q : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ with $l_1 < l_2$ such that $Q' \in L_1([l_1, l_2])$. If $|Q'|^r$, with $r \geq 1$, is an exponentially tgs-convex function, then the following inequality holds:*

$$|\Upsilon_Q(\rho; \mathbb{B}; n; l_1, l_2)| \leq \frac{l_2 - l_1}{2} \Big( \mathbb{B}(n+1, \rho-n+1) - \mathbb{B}(n+1, \rho-n) + \mathbb{B}(n+2, \rho-n) \Big)^{1-\frac{1}{r}}$$

$$\times \Big( \frac{e^{\alpha r l_2} |Q'(l_1)|^r + e^{\alpha r l_1} |Q'(l_2)|^r}{6 e^{\alpha r l_1} e^{\alpha r l_2}} \Big)^{\frac{1}{r}}. \tag{4.5}$$

*Proof.* Utilizing exponentially tgs-convex function of $|\mathcal{Q}'|^r$, Lemma 4.1 and Hölder's inequality, one obtains

$$\begin{aligned}
&|\Upsilon_{\mathcal{Q}}(\rho;\mathbb{B};n;l_1,l_2)|\\
&=\Big|\frac{l_2-l_1}{2}\int_0^1(\mathbb{B}_{1-u}(n+1,\rho-n)-\mathbb{B}_u(n+1,\rho-n))\mathcal{Q}'(\vartheta l_1+(1-\vartheta)l_2)d\vartheta\Big|\\
&\le\frac{l_2-l_1}{2}\Big(\int_0^1(\mathbb{B}_{1-u}(n+1,\rho-n)-\mathbb{B}_u(n+1,\rho-n))d\vartheta\Big)^{1-\frac{1}{r}}\\
&\quad\times\Big(\int_0^1|\mathcal{Q}'(\vartheta l_1+(1-\vartheta)l_2)|^r d\vartheta\Big)^{\frac{1}{r}}\\
&\le\frac{l_2-l_1}{2}\Big(\mathbb{B}(n+1,\rho-n+1)-\mathbb{B}(n+1,\rho-n)+\mathbb{B}(n+2,\rho-n)\Big)^{1-\frac{1}{r}}\\
&\quad\times\Big(\int_0^1\vartheta(1-\vartheta)\Big(\Big|\frac{\mathcal{Q}'(l_1)}{e^{\alpha l_1}}\Big|^r+\Big|\frac{\mathcal{Q}'(l_2)}{e^{\alpha l_2}}\Big|^r\Big)d\vartheta\Big)^{\frac{1}{r}}\\
&\le\frac{l_2-l_1}{2}\Big(\mathbb{B}(n+1,\rho-n+1)-\mathbb{B}(n+1,\rho-n)+\mathbb{B}(n+2,\rho-n)\Big)^{1-\frac{1}{r}}\\
&\quad\times\Big(\frac{e^{\alpha r l_2}|\mathcal{Q}'(l_1)|^r+e^{\alpha r l_1}|\mathcal{Q}'(l_2)|^r}{6e^{\alpha r l_1}e^{\alpha r l_2}}\Big)^{\frac{1}{r}},
\end{aligned}\tag{4.6}$$

which is the required result. □

**Theorem 4.3.** *For $\rho\in(n,n+1]$ with $\rho>0$ and let $\mathcal{Q}:\mathcal{I}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ with $l_1<l_2$ such that $\mathcal{Q}'\in L_1([l_1,l_2])$. If $|\mathcal{Q}'|^r$, with $r,s>1$ such that $\frac{1}{s}+\frac{1}{r}=1$, is exponentially tgs-convex function, then the following inequality holds:*

$$\begin{aligned}
|\Upsilon_{\mathcal{Q}}(\rho;\mathbb{B};n;l_1,l_2)|&\le\frac{l_2-l_1}{2}\Big(2\int_0^{\frac{1}{2}}\Big(\int_u^{1-u}v^n(1-v)^{\rho-n-1}dv\Big)^s du\Big)^{\frac{1}{s}}\\
&\quad\times\Big(\frac{e^{\alpha r l_2}|\mathcal{Q}'(l_1)|^r+e^{\alpha r l_1}|\mathcal{Q}'(l_2)|^r}{6e^{\alpha r l_1}e^{\alpha r l_2}}\Big)^{\frac{1}{r}}.
\end{aligned}\tag{4.7}$$

*Proof.* Utilizing exponentially tgs-convex function of $|\mathcal{Q}'|^r$ and well-known Hölder inequality, one obtains

$$\begin{aligned}
&|\Upsilon_{\mathcal{Q}}(\rho;\mathbb{B};n;l_1,l_2)|\\
&=\Big|\frac{l_2-l_1}{2}\int_0^1(\mathbb{B}_{1-u}(n+1,\rho-n)-\mathbb{B}_u(n+1,\rho-n))\mathcal{Q}'(\vartheta l_1+(1-\vartheta)l_2)d\vartheta\Big|\\
&\le\frac{l_2-l_1}{2}\Big(\int_0^1|\mathbb{B}_{1-u}(n+1,\rho-n)-\mathbb{B}_n(n+1,\rho-n)|^s d\vartheta\Big)^{\frac{1}{s}}\\
&\quad\times\Big(\int_0^1|\mathcal{Q}'(\vartheta l_1+(1-\vartheta)l_2)|^r d\vartheta\Big)^{\frac{1}{r}}\\
&\le\frac{l_2-l_1}{2}\Big(\int_0^{\frac{1}{2}}(\mathbb{B}_{1-u}(n+1,\rho-n)-\mathbb{B}_u(n+1,\rho-n))^s du
\end{aligned}$$

$$+\int_{\frac{1}{2}}^{1}(\mathbb{B}_u(n+1,\rho-n)-\mathbb{B}_{1-u}(n+1,\rho-n))^s du\Big)^{\frac{1}{s}}\Big(\int_0^1 \vartheta(1-\vartheta)\Big(\frac{|\mathcal{Q}'(l_1)|^r}{e^{\alpha r l_1}}+\frac{|\mathcal{Q}'(l_2)|^q}{e^{\alpha r l_2}}\Big)d\vartheta\Big)^{\frac{1}{r}}$$

$$=\frac{l_2-l_1}{2}\Big(\int_0^{\frac{1}{2}}\Big(\int_u^{1-u} v^n(1-v)^{\rho-n-1}dv\Big)^s dv+\int_{\frac{1}{2}}^{1}\Big(\int_{1-u}^{u} v^n(1-v)^{\rho-n-1}dv\Big)^s dv\Big)^{\frac{1}{s}}$$

$$\times\Big(\frac{e^{\alpha r l_2}|\mathcal{Q}'(l_1)|^r+e^{\alpha r l_1}|\mathcal{Q}'(l_2)|^r}{6e^{\alpha r l_1}e^{\alpha r l_2}}\Big)^{\frac{1}{r}}$$

$$=\frac{l_2-l_1}{2}\Big(2\int_0^{\frac{1}{2}}\Big(\int_u^{1-u} v^n(1-v)^{\rho-n-1}dv\Big)^s du\Big)^{\frac{1}{s}}\Big(\frac{e^{\alpha r l_2}|\mathcal{Q}'(l_1)|^r+e^{\alpha r l_1}|\mathcal{Q}'(l_2)|^r}{6e^{\alpha r l_1}e^{\alpha r l_2}}\Big)^{\frac{1}{r}}, \tag{4.8}$$

which is the required result. □

## 5. Hermite-Hadamard inequality within the generalized conformble integral operator

This section is devoted to proving some new generalizations for exponentially tgs-convex functions within the generalized conformable integral operator.

**Theorem 5.1.** *For $\rho > 0$ and let $\mathcal{Q} : [l_1, l_2] \subset \mathbb{R} \to \mathbb{R}$ be an exponentially tgs-convex function such that $\mathcal{Q} \in L_1[l_1, l_2]$, then the following inequality holds:*

$$\frac{4}{\delta\rho^\delta}\mathcal{Q}\Big(\frac{l_1+l_2}{2}\Big)\le\frac{\Gamma(\delta)}{(l_2-l_1)^{\rho\delta}}\Big[\mathcal{J}^{\rho,\delta}_{l_1^+}\frac{\mathcal{Q}(l_2)}{e^{\alpha l_2}}+\mathcal{J}^{\rho,\delta}_{l_2^-}\frac{\mathcal{Q}(l_1)}{e^{\alpha l_1}}\Big]$$

$$\le\frac{1}{\rho}\Big[\mathbb{B}\Big(\frac{\rho+1}{\rho},\delta\Big)+\mathbb{B}\Big(\frac{\rho+2}{\rho},\delta\Big)\Big]\Big(\frac{\mathcal{Q}(l_1)}{e^{\alpha l_1}}+\frac{\mathcal{Q}(l_2)}{e^{\alpha l_2}}\Big). \tag{5.1}$$

*Proof.* Taking into account (3.3) and conducting product of (3.3) by $(\frac{1-\vartheta^\rho}{\rho})^{\delta-1}\vartheta^{\rho-1}$ with $\vartheta\in(0,1),\rho>0$ and then integrating the resulting estimate with respect to $\vartheta$ over $[0,1]$, we find

$$4\mathcal{Q}\Big(\frac{l_1+l_2}{2}\Big)\int_0^1\Big(\frac{1-\vartheta^\rho}{\rho}\Big)^{\delta-1}\vartheta^{\rho-1}d\vartheta$$

$$\le\int_0^1\Big(\frac{1-\vartheta^\rho}{\rho}\Big)^{\delta-1}\vartheta^{\rho-1}\frac{\mathcal{Q}(\vartheta l_1+(1-\vartheta)l_2)}{e^{\alpha(\vartheta l_1+(1-\vartheta)l_2)}}d\vartheta$$

$$+\int_0^1\Big(\frac{1-\vartheta^\rho}{\rho}\Big)^{\delta-1}\vartheta^{\rho-1}\frac{\mathcal{Q}(\vartheta l_2+(1-\vartheta)l_1)}{e^{\alpha(\vartheta l_2+(1-\vartheta)l_1)}}d\vartheta$$

$$=R_1+R_2. \tag{5.2}$$

By making change of variable $u=\vartheta l_1+(1-\vartheta)l_2$, we have

$$R_1=\int_0^1\Big(\frac{1-\vartheta^\rho}{\rho}\Big)^{\delta-1}\vartheta^{\rho-1}\frac{\mathcal{Q}(\vartheta l_1+(1-\vartheta)l_2)}{e^{\alpha(\vartheta l_1+(1-\vartheta)l_2)}}d\vartheta$$

$$=\int_{l_2}^{l_1}\Big(\frac{1-(\frac{u-l_2}{l_1-l_2})^\rho}{\rho}\Big)^{\delta-1}\Big(\frac{u-l_2}{l_1-l_2}\Big)^{\rho-1}\frac{\mathcal{Q}(u)}{e^{\alpha u}}\frac{du}{l_1-l_2}$$

$$= \frac{1}{(l_2 - l_1)^{\rho\delta}} \int_{l_1}^{l_2} (\frac{(l_2 - l_1)^\rho - (l_2 - u)^\rho}{\rho})^{\delta-1} (l_2 - u)^{\rho-1} \frac{Q(u)}{e^{\alpha u}} du$$

$$= \frac{\Gamma(\delta}{(l_2 - l_1)^{\rho\delta}} \mathcal{J}^{\rho,\delta}_{l_2^-} \frac{Q(l_1)}{e^{\alpha l_1}}. \tag{5.3}$$

Substituting $v = \vartheta l_2 + (1 - \vartheta) l_1$, we have

$$R_2 = \int_0^1 \Big(\frac{1 - \vartheta^\rho}{\rho}\Big)^{\delta-1} \vartheta^{\rho-1} \frac{Q(\vartheta l_2 + (1 - \vartheta) l_1)}{e^{\alpha(\vartheta l_2 + (1-\vartheta) l_1)}} d\vartheta$$

$$= \int_{l_2}^{l_1} \Big(\frac{1 - (\frac{v - l_1}{l_2 - l_1})^\rho}{\rho}\Big)^{\delta-1} \Big(\frac{v - l_1}{l_2 - l_1}\Big)^{\rho-1} \frac{Q(v)}{e^{\alpha v}} \frac{du}{l_2 - l_1}$$

$$= \frac{1}{(l_2 - l_1)^{\rho\delta}} \int_{l_1}^{l_2} (\frac{(l_2 - l_1)^\rho - (v - l_1)^\rho}{\rho})^{\delta-1} (v - l_1)^{\rho-1} \frac{Q(v)}{e^{\alpha v}} dv$$

$$= \frac{\Gamma(\delta)}{(l_2 - l_1)^{\rho Q}} \mathcal{J}^{\rho,\delta}_{l_2^-} \frac{Q(l_2)}{e^{\alpha l_2}}. \tag{5.4}$$

Thus by using (5.2) and (5.3) in (5.4), we get the first inequality of (5.1).

Consider

$$Q(\vartheta l_1 + (1 - \vartheta) l_2) \leq \vartheta(1 - \vartheta)\Big(\frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}}\Big)$$

and

$$Q(\vartheta l_2 + (1 - \vartheta) l_1) \leq \vartheta(1 - \vartheta)\Big(\frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}}\Big).$$

By adding

$$Q(\vartheta l_1 + (1 - \vartheta) l_2) + Q(\vartheta l_2 + (1 - \vartheta) l_1) \leq 2\vartheta(1 - \vartheta)\Big(\frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}}\Big). \tag{5.5}$$

If we multiply (5.5) by $(\frac{1-\vartheta^\rho}{\rho})^{\delta-1} \vartheta^{\rho-1}$ with $\vartheta \in (0, 1), \rho > 0$ and then integrating the resulting estimate with respect to $\vartheta$ over $[0, 1]$, we get

$$\frac{\Gamma(\delta)}{(l_2 - l_1)^{\rho\delta}} [\mathcal{J}^{\rho,\delta}_{l_1^+} \frac{Q(l_2)}{e^{\alpha l_2}} + \mathcal{J}^{\rho,\delta}_{l_2^-} \frac{Q(l_1)}{e^{\alpha l_1}}]$$

$$\leq \frac{1}{\rho}\Big[\mathbb{B}\Big(\frac{\rho + 1}{\rho}, \delta\Big) + \mathbb{B}\Big(\frac{\rho + 2}{\rho}, \delta\Big)\Big]\Big(\frac{Q(l_1)}{e^{\alpha l_1}} + \frac{Q(l_2)}{e^{\alpha l_2}}\Big), \tag{5.6}$$

the desired inequality is the right hand side of (5.1). □

Our main results depend on the following identity.

**Lemma 5.2.** *For $\rho > 0$ and let $Q : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $(l_1, l_2)$ with $l_1 < l_2$ such that $Q' \in L_1[l_1, l_2]$, then the following identity holds:*

$$\Big(\frac{Q(l_1) + Q(l_2)}{2}\Big) - \frac{\rho^\delta \Gamma(\delta + 1)}{2(l_2 - l_1)^{\rho\delta}} [\mathcal{J}^{\rho,\delta}_{l_1^+} Q(l_2) + \mathcal{J}^{\rho,\delta}_{l_2^+} Q(l_1)]$$

$$= \frac{(l_2 - l_1)\rho^\delta}{2} \int_0^1 \Big[\Big(\frac{1 - \vartheta^\rho}{\rho}\Big)^\delta - \Big(\frac{1 - (1 - \vartheta)^\rho}{\rho}\Big)^\delta\Big] Q'(\vartheta l_1 + (1 - \vartheta) l_2) d\vartheta. \tag{5.7}$$

*Proof.* It suffices that

$$\int_0^1 \Big[\Big(\frac{1-\vartheta^\rho}{\rho}\Big)^\delta - \Big(\frac{1-(1-\vartheta)^\rho}{\rho}\Big)^\delta\Big] Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta$$

$$= \int_0^1 \Big(\frac{1-\vartheta^\rho}{\rho}\Big)^\delta Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta - \Big(\frac{1-(1-\vartheta)^\rho}{\rho}\Big)^\delta Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta$$

$$= M_1 - M_2. \tag{5.8}$$

Using integration by parts and making change of variable technique, we have

$$\begin{aligned} M_1 &= \int_0^1 \Big(\frac{1-\vartheta^\rho}{\rho}\Big)^\delta Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta \\ &= \frac{1}{l_1-l_2}\Big(\frac{1-\vartheta^\rho}{\rho}\Big)^\delta Q(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta\Big|_0^1 \\ &+ \frac{\delta}{l_1-l_2}\int_0^1 \Big(\frac{1-\vartheta^\rho}{\rho}\Big)^{\delta-1} \vartheta^{\rho-1} Q(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta \\ &= \frac{Q(l_2)}{(l_2-l_1)\rho^\delta} - \frac{\delta}{l_2-l_1}\int_0^1 \Big(\frac{1-\vartheta^\rho}{\rho}\Big)^{\delta-1} \vartheta^{\rho-1} Q(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta \\ &= \frac{Q(l_2)}{(l_2-l_1)\rho^\delta} - \frac{\delta\Gamma(\delta)}{(l_2-l_1)^{\rho\delta+1}} \mathcal{J}^{\rho,\delta}_{l_2^-} Q(l_1) \end{aligned}$$

Analogously

$$\begin{aligned} M_2 &= \int_0^1 \Big(\frac{1-(1-\vartheta)^\rho}{\rho}\Big)^\delta Q'(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta \\ &= \frac{1}{l_1-l_2}\Big(\frac{1-(1-\vartheta)^\rho}{\rho}\Big)^\delta Q(\vartheta l_1 + (1-\vartheta)l_2)\Big|_0^1 \\ &- \frac{1}{l_1-l_2}\int_0^1 Q\Big(\frac{1-(1-\vartheta)^\rho}{\rho}\Big)^{\delta-1} (1-\vartheta)^{\rho-1} Q(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta \\ &= \frac{-Q(l_1)}{(l_2-l_1)\rho^\delta} + \frac{\delta}{l_2-l_1}\int_0^1 \Big(\frac{1-(1-\vartheta)^\rho}{\rho}\Big)^{\delta-1} (1-\vartheta)^{\rho-1} Q(\vartheta l_1 + (1-\vartheta)l_2)d\vartheta \\ &= \frac{-Q(l_1)}{(l_2-l_1)\rho^\delta} + \frac{\delta\Gamma(\delta)}{(l_2-l_1)^{\rho\delta+1}} \mathcal{J}^{\rho,\delta}_{l_1^+} Q(l_2). \end{aligned} \tag{5.9}$$

By substituting values of $M_1$ and $M_2$ in (5.8) and then conducting product on both sides by $\frac{(l_2-l_1)\rho^\delta}{2}$, we get the desired result. □

**Theorem 5.3.** *For $\rho > 0$ and let $Q : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ with $l_1 < l_2$ such that $Q' \in L_1([l_1, l_2])$. If $|Q'|^r$, with $r \geq 1$, is an exponentially tgs-convex function, then the following inequality holds*

$$\Big|\Big(\frac{Q(l_1)+Q(l_2)}{2}\Big) - \frac{\rho^\delta\Gamma(\delta+1)}{2(l_2-l_1)^{\rho\delta}}[\mathcal{J}^{\rho,\delta}_{l_1^+}Q(l_2) + \mathcal{J}^{\rho,\delta}_{l_2^+}Q(l_1)]\Big|$$

$$\leq \frac{(l_2-l_1)\rho^\delta}{2}\Big(\frac{1}{\rho^{\delta+1}}\mathbb{B}\Big(\frac{1}{\rho},\delta+1\Big) + \frac{1}{\rho^{\delta+2}}\mathbb{B}\Big(\frac{1}{\rho^2},\delta+1\Big)\Big)^{1-\frac{1}{r}}\Big(\frac{e^{\alpha r l_2}|Q'(l_1)|^r + e^{\alpha r l_1}|Q'(l_2)|^r}{6e^{\alpha r l_1}e^{\alpha r l_2}}\Big)^{\frac{1}{r}}. \tag{5.10}$$

*Proof.* Using exponentially tgs-convexity of $|\mathcal{Q}'|^r$, Lemma 5.2, and the well-known Hölder inequality, we have

$$\begin{aligned}
&\left|\left(\frac{\mathcal{Q}(l_1)+\mathcal{Q}(l_2)}{2}\right)-\frac{\rho^{\delta}\Gamma(\delta+1)}{2(l_2-l_1)^{\rho\delta}}\left[\mathcal{J}^{\rho,\delta}_{l_1^+}\mathcal{Q}(l_2)+\mathcal{J}^{\rho,\delta}_{l_2^+}\mathcal{Q}(l_1)\right]\right| \\
=&\left|\frac{(l_2-l_1)\rho^{\delta}}{2}\int_0^1\left[\left(\frac{1-\vartheta^{\rho}}{\rho}\right)^{\delta}-\left(\frac{1-(1-\vartheta)^{\rho}}{\rho}\right)^{\delta}\right]\mathcal{Q}'(\vartheta l_1+(1-\vartheta)l_2)d\vartheta\right. \\
\leq&\frac{(l_2-l_1)\rho^{\delta}}{2}\left(\int_0^1\left[\left(\frac{1-\vartheta^{\rho}}{\rho}\right)^{\delta}-\left(\frac{1-(1-\vartheta)^{\rho}}{\rho}\right)^{\delta}\right]d\vartheta\right)^{1-\frac{1}{r}} \\
&\times\left(\int_0^1|\mathcal{Q}'(\vartheta l_1+(1-\vartheta)l_2)|^r d\vartheta\right)^{\frac{1}{r}} \\
\leq&\frac{(l_2-l_1)\rho^{\delta}}{2}\left(\int_0^1\left(\frac{1-\vartheta^{\rho}}{\rho}\right)^{\delta}d\vartheta-\int_0^1\left(\frac{1-(1-\vartheta)^{\rho}}{\rho}\right)^{\delta}d\vartheta\right)^{1-\frac{1}{r}} \\
&\times\left(\int_0^1\vartheta(1-\vartheta)\left(\frac{|\mathcal{Q}'(l_1)|^r}{e^{\alpha r l_1}}+\frac{|\mathcal{Q}'(l_2)|^r}{e^{\alpha r l_2}}\right)d\vartheta\right)^{\frac{1}{r}} \\
=&\frac{(l_2-l_1)\rho^{\delta}}{2}\left(\frac{1}{\rho^{\delta+1}}\mathbb{B}\left(\frac{1}{\rho},\delta+1\right)+\frac{1}{\rho^{\delta+2}}\mathbb{B}\left(\frac{1}{\rho^2},\delta+1\right)\right)^{1-\frac{1}{r}}\left(\frac{e^{\alpha r l_2}|\mathcal{Q}'(l_1)|^r+e^{\alpha r l_1}|\mathcal{Q}'(l_2)|^r}{6e^{\alpha r l_1}e^{\alpha r l_2}}\right)^{\frac{1}{r}},
\end{aligned}$$

the required result. □

## 6. Applications

Let $l_1, l_2 > 0$ with $l_1 \neq l_2$. Then the arithmetic mean $\mathcal{A}(l_1, l_2)$, harmonic mean $\mathcal{H}(l_1, l_2)$, logarithmic mean $\mathcal{L}(l_1, l_2)$ and $n$-th generalized logarithmic mean $L_n(l_1, l_2)$ are defined by

$$\mathcal{A}(l_1,l_2)=\frac{l_1+l_2}{2},$$

$$\mathcal{G}(l_1,l_2)=\sqrt{l_1 l_2},$$

$$\mathcal{L}(l_1,l_2)=\frac{l_2-l_1}{\ln l_2-\ln l_1}$$

and

$$\mathcal{L}_n(l_1,l_2)=\left[\frac{l_2^{n+1}-l_1^{n+1}}{(n+1)(l_2-l_1)}\right]^{\frac{1}{n}}\quad(n\neq 0,-1),$$

respectively. Recently, the bivariate means have attracted the attention of many researchers [47–58] due to their are closely related to the special functions.

In this section, we use our obtained results in section 5 to provide several novel inequalities involving the special bivariate means mentioned above.

**Proposition 6.1.** *Let $l_1, l_2 > 0$ with $l_2 > l_1$. Then*

$$\left|\mathcal{A}(l_1^2,l_2^2)-\frac{1}{2}\mathcal{L}_3^3(l_1,l_2)\right|\leq\frac{l_2-l_1}{(6)^{\frac{1}{r}}e^{\alpha(l_1+l_2)}}\left[(e^{\alpha l_2}l_1)^r+(e^{\alpha l_1}l_2)^r\right]^{\frac{1}{r}}.$$

*Proof.* Let $\rho = \delta = 1$ and $Q(z) = z^2$. Then the desired result follows from Theorem 5.3. □

**Proposition 6.2.** *Let $l_1, l_2 > 0$ with $l_2 > l_1$. Then*

$$\Big|\mathcal{H}^{-1}(l_1^2, l_2^2) - \frac{1}{2}\mathcal{L}^{-1}(l_1, l_2)\Big| \leq \frac{l_2 - l_1}{2(6)^{\frac{1}{r}} e^{\alpha(l_1+l_2)}}\Big[\frac{(e^{\alpha l_2} l_2^2)^r + (e^{\alpha l_1} l_1^2)^r}{(l_1 l_2)^{2r}}\Big]^{\frac{1}{r}}.$$

*Proof.* Let $\rho = \delta = 1$ and $Q(z) = \frac{1}{z}$. Then the desired result follows from Theorem 5.3. □

**Proposition 6.3.** *Let $l_1, l_2 > 0$ with $l_2 > l_1$. Then*

$$\Big|\mathcal{A}(l_1^n, l_2^n) - \frac{1}{2}\mathcal{L}_n^n(l_1, l_2)\Big| \leq \frac{(l_2 - l_1)|n|}{2}\Big[\frac{(e^{\alpha l_2} l_1^{n-1})^r + (e^{\alpha l_1} l_2^{n-1})^r}{6e^{\alpha r(l_1+l_2)}}\Big]^{\frac{1}{r}}.$$

*Proof.* Let $\rho = \delta = 1$ and $Q(z) = z^n$. Then the desired result follows from Theorem 5.3. □

## 7. Conclusions

In this paper, we proposed a novel technique with two different approaches for deriving several generalizations for an exponentially $tgs$-convex function that accelerates with a conformable integral operator. We have generalized the Hermite-Hadamard type inequalities for exponentially $tgs$-convex functions. By choosing different parametric values $\rho$ and $\delta$, we analyzed the convergence behavior of our proposed methods in form of corollaries. Another aspect is that to show the effectiveness of our novel generalizations, our results have potential applications in fractional integrodifferential and fractional Schrödinger equations. Numerical applications show that our findings are consistent and efficient. Finally, we remark that the framework of the conformable fractional integral operator, it is of interest to further our results to the framework of Riemann-Liouville, Hadamard and Katugampola fractional integral operators. Our ideas and the approach may lead to a lot of follow-up research.

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## Conflict of interest

The authors declare no conflict of interest.

## References

1. S. Kumar, R. Kumar, C. Cattani, et al. *Chaotic behaviour of fractional predator-prey dynamical system*, Chaos Solutons Fractals, **135** (2020), 1–12.

2. M. A. Akinlar, F. Tchier, M. Inc, *Chaos control and solutions of fractional-order Malkus waterwheel model*, Chaos Solitons Fractals, **135** (2020), 1–8.
3. Y. Khurshid, M. Adil Khan, Y. M. Chu, *Conformable fractional integral inequalities for GG- and GA-convex function*, AIMS Math., **5** (2020), 5012–5030.
4. S. Rafeeq, H. Kalsoom, S. Hussain, et al. *Delay dynamic double integral inequalities on time scales with applications*, Adv. Differ. Equ., **2020** (2020), 1–32.
5. S. Rashid, M. A. Noor, K. I. Noor, et al. *Ostrowski type inequalities in the sense of generalized $\mathcal{K}$-fractional integral operator for exponentially convex functions*, AIMS Math., **5** (2020), 2629–2645.
6. S. Rashid, İ. İşcan, D. Baleanu, et al. *Generation of new fractional inequalities via n polynomials s-type convexixity with applications*, Adv. Differ. Equ., **2020** (2020), 1–20.
7. S. S. Zhou, S. Rashid, F. Jarad, et al. *New estimates considering the generalized proportional Hadamard fractional integral operators*, Adv. Differ. Equ., **2020** (2020), 1–15.
8. A. Iqbal, M. Adil Khan, S. Ullah, et al. *Some new Hermite-Hadamard-type inequalities associated with conformable fractional integrals and their applications*, J. Funct. Space., **2020** (2020), 1–18.
9. S. Rashid, F. Jarad, M. A. Noor, et al. *Inequalities by means of generalized proportional fractional integral operators with respect to another function*, Mathematics, **7** (2019), 1–18.
10. S. Rashid, F. Jarad, Y. M. Chu, *A note on reverse Minkowski inequality via generalized proportional fractional integral operator with respect to another function*, Math. Probl. Eng., **2020** (2020), 1–12.
11. S. Rashid, F. Jarad, H. Kalsoom, et al. *On Pólya-Szegö and Ćebyšev type inequalities via generalized k-fractional integrals*, Adv. Differ. Equ., **2020** (2020), 1–18.
12. M. U. Awan, S. Talib, Y. M. Chu, et al. *Some new refinements of Hermite-Hadamard-type inequalities involving $\Psi_k$-Riemann-Liouville fractional integrals and applications*, Math. Probl. Eng., **2020** (2020), 1–10.
13. D. Baleanu, M. Jleli, S. Kumar, et al. *A fractional derivative with two singular kernels and application to a heat conduction problem*, Advs. Differ. Equ., **2020** (2020), 1–19.
14. J. Singh, D. Kumar, S. Kumar, *An efficient computational method for local fractional transport equation occurring in fractal porous media*, Comput. Appl. Math., **39** ((2020), 1–10.
15. S. Kumar, A. Kumar, Z. Odibat, et al. *A comparison study of two modified analytical approach for the solution of nonlinear fractional shallow water equations in fluid flow*, AIMS Math., **5** (2020), 3035–3055.
16. R. Kumar, S. Kumar, J. Singh, et al. *A comparative study for fractional chemical kinetics and carbon dioxide $Co_2$ absorbed into phenyl glycidyl ether problems*, AIMS Math., **5** (2020), 3201–3222.
17. M. Inc, A. Yusuf, A. I. Aliyu, et al. *Dark and singular optical solitons for the conformable space-time nonlinear Schrödinger equation with Kerr and power law onlinearity*, Optik, **162** (2018), 65–75.
18. Z. Korpinar, M. Inc, *Numerical simulations for fractional variation of* $(1 + 1)$*-dimensional Biswas-Milovic equation*, Optik, **166**(218), 77–85.
19. P. Agarwal, M. Kadakal, İ. İşcan, et al. *Better approaches for n-times differentiable convex functions*, Mathematics, **8** (2020), 1–11.

20. T. H. Zhao, L. Shi, Y. M. Chu, *Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means*, RACSAM, **114** (2020), 1–14.
21. I. Abbas Baloch, Y. M. Chu, *Petrović-type inequalities for harmonic h-convex functions*, J. Funct. Space., **2020** (2020), 1–7.
22. M. K. Wang, Z. Y. He, Y. M. Chu, *Sharp power mean inequalities for the generalized elliptic integral of the first kind*, Comput. Meth. Funct. Th., **20** (2020), 111–124.
23. S. Rashid, R. Ashraf, M. A. Noor, et al. *New weighted generalizations for differentiable exponentially convex mapping with application*, AIMS Math., **5** (2020), 3525–3546.
24. M. Adil Khan, M. Hanif, Z. A. Khan, et al. *Association of Jensen's inequality for s-convex function with Csiszár divergence*, J. Inequal. Appl., **2019** (2019), 1–14.
25. T. H. Zhao, M. K. Wang, Y. M. Chu, *A sharp double inequality involving generalized complete elliptic integral of the first kind*, AIMS Math., **5** (2020), 4512–4528.
26. M. U. Awan, N. Akhtar, A. Kashuri, et. al. 2*D approximately reciprocal ρ-convex functions and associated integral inequalities*, AIMS Math., **5** (2020), 4662–4680.
27. S. Zaheer Ullah, M. Adil Khan, Y. M. Chu, *A note on generalized convex functions*, J. Inequal. Appl., **2019** (2019), 1–10.
28. R. Khalil, M. A. Horani, A. Yousaf, et al. *New definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70.
29. S. Khan, M. Adil Khan, Y. M. Chu, *Converses of the Jensen inequality derived from the Green functions with applications in information theory*, Math. Method. Appl. Sci., **43** (2020), 2577–2587.
30. M. Adil Khan, J. Pečarić, Y. M. Chu, *Refinements of Jensen's and McShane's inequalities with applications*, AIMS Math., **5** (2020), 4931–4945.
31. T. H. Zhao, Y. M. Chu, H. Wang, *Logarithmically complete monotonicity properties relating to the gamma function*, Abstr. Appl. Anal., **2011** (2011), 1-13.
32. Z. H. Yang, W. M. Qian, W. Zhang, et al. *Notes on the complete elliptic integral of the first kind*, Math. Inequal. Appl., **23** (2020), 77–93.
33. M. K. Wang, M. Y. Hong, Y. F. Xu, et al. *Inequalities for generalized trigonometric and hyperbolic functions with one parameter*, J. Math. Inequal., **14** (2020), 1–21.
34. M. K. Wang, H. H. Chu, Y. M. Li, et al. *Answers to three conjectures on convexity of three functions involving complete elliptic integrals of the first kind*, Appl. Anal. Discrete Math., **14** (2020), 255–271.
35. M. K. Wang, Y. M. Chu, Y. P. Jiang, *Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions*, Rocky Mt. J. Math., **46** (2016), 679–691.
36. M. K. Wang, H. H. Chu, Y. M. Chu, *Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals*, J. Math. Anal. Appl., **480** (2019), 1–9.
37. W. M. Qian, Z. Y. He, Y. M. Chu, *Approximation for the complete elliptic integral of the first kind*, RACSAM, **114** (2020), 1–12.
38. M. A. Latif, S. Rashid, S. S. Dragomir, et al. *Hermite-Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications*, J. Inequal. Appl., **2019** (2019), 1–33.
39. M. U. Awan, N. Akhtar, S. Iftikhar, et al. *New Hermite-Hadamard type inequalities for n-polynomial harmonically convex functions*, J. Inequal. Appl., **2020** (2020), 1–12.

40. M. Adil Khan, N. Mohammad, E. R. Nwaeze, et al. *Quantum Hermite-Hadamard inequality by means of a Green function*, Adv. Differ. Equ., **2020** (2020), 1–20.
41. S. Rashid, M. A. Noor, K. I. Noor, et al. *Hermite-Hadamrad type inequalities for the class of convex functions on time scale*, Mathematics, **7** (2019), 1–20.
42. S. Bernstein, *Sur les fonctions absolument monotones*, Acta Math., **52** (1929), 1–66.
43. M. Avriel, *r-convex functions*, Math. Programming, **2** (1972), 309–323.
44. J. Jakšetić, J. Pečarić, *Exponential convexity method*, J. Convex Anal., **20** (2013), 181–197.
45. T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66.
46. F. Jarad, E. Uğurlu, T. Abdeljawad, et al. *On a new class of fractional operators*, Adv. Differ. Equ., **2017** (2017), 1–16.
47. M. U. Awan, M. A. Noor, K. I. Noor, *Hermite-Hadamard inequalitie for exponentially convex function*, Appl. Math. Inf. Sci., **12** (2018), 405–409.
48. Y. M. Chu, Y. F. Qiu, M. K. Wang, *Hölder mean inequalities for the complete elliptic integrals*, Integral Transforms Spec. Funct., **23** (2012), 521–527.
49. G. D. Wang, X. H. Zhang, Y. M. Chu, *A power mean inequality for the Grötzsch ring function*, Math. Inequal. Appl., **14** (2011), 833–837.
50. M. K. Wang, Y. M. Chu, Y. F. Qiu, et al. *An optimal power mean inequality for the complete elliptic integrals*, Appl. Math. Lett., **24** (2011), 887–890.
51. H. Z. Xu, Y. M. Chu, W. M. Qian, *Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means*, J. Inequal. Appl., **2018** (2018), 1–13.
52. B. Wang, C. L. Luo, S. H. Li, et al. *Sharp one-parameter geometric and quadratic means bounds for the Sándor-Yang means*, RACSAM, **114** (2020), 1–10.
53. W. M. Qian, W. Zhang, Y. M. Chu, *Bounding the convex combination of arithmetic and integral means in terms of one-parameter harmonic and geometric means*, Miskolc Math. Notes, **20** (2019), 1157–1166.
54. W. M. Qian, Y. Y. Yang, H. W. Zhang, et al. *Optimal two-parameter geometric and arithmetic mean bounds for the Sándor-Yang mean*, J. Inequal. Appl., **2019** (2019), 1–12.
55. W. M. Qian, Z. Y. He, H. W. Zhang, et al. *Sharp bounds for Neuman means in terms of two-parameter contraharmonic and arithmetic mean*, J. Inequal. Appl., **2019** (2019), 1–13.
56. W. M. Qian, X. H. Zhang, Y. M. Chu, *Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means*, J. Math. Inequal., **11** (2017), 121–127.
57. Y. M. Chu, M. K. Wang, *Optimal Lehmer mean bounds for the Toader mean*, Results Math., **61** (2012), 223–229.
58. Y. M. Chu, M. K. Wang, S. L. Qiu, *Optimal combinations bounds of root-square and arithmetic means for Toader mean*, Proc. Indian Acad. Sci. Math. Sci., **122** (2012), 41–51.