Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 6087–6107.
DOI:10.3934/math.2020391
Received: 11 May 2020
Accepted: 20 July 2020
Published: 28 July 2020

Research article

Revisiting the Hermite-Hadamard fractional integral inequality via a Green function

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Abstract: The Hermite-Hadamard inequality by means of the Riemann-Liouville fractional integral operators is already known in the literature. In this paper, it is our purpose to reconstruct this inequality via a relatively new method called the green function technique. In the process, some identities are established. Using these identities, we obtain loads of new results for functions whose second derivative is convex, monotone and concave in absolute value. We anticipate that the method outlined in this article will stimulate further investigation in this direction.

Keywords: Hermite-Hadamard inequality; Green function; convex function; concave function; Riemann-Liouville fractional integral
Mathematics Subject Classification: 26A51, 26D15, 26E60, 41A55

1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be convex (concave) if the inequality

$$f[\lambda x + (1-\lambda)y] \leq (\geq) \lambda f(x) + (1-\lambda) f(y)$$

holds whenever $x, y \in I$ and $\lambda \in [0, 1]$. It is well-know that the convexity (concavity) has wild applications in pure and applied mathematics [1–5], and many inequalities [6–19] can be found in the literature via the convexity theory. Recently, the generalizations and variants for the convexity have attracted the attention of the researchers, for example, the *GG*- and *GA*-convexity [20], *h*-convexity [21], qausi-convexity [22], ρ-convexity [23], exponential convexity [24], harmonic convexity [25], *s*-convexity [26, 27] and others.

The classical Hermite-Hadamard inequality [28–33] is one of the most famous inequalities in convex function theory, which can be stated as follows:

A real-valued function $\psi : [b_1, b_2] \to \mathbb{R}$ is convex if and only if

$$\psi\left(\frac{b_1+b_2}{2}\right) \leq \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(x)dx \leq \frac{\psi(b_1)+\psi(b_2)}{2}. \tag{1.1}$$

If ψ is concave, then the inequalities in (1.1) remain valid in the reversed direction. The Hermite-Hadamard inequality (1.1) provides both upper and lower estimates of the integral mean of any convex function defined on a closed and bounded interval involving the endpoints and midpoints of the function's domain. Because of the excellent significance of the Hermite-Hadamard inequality, the literature is replete with ample amount of research articles dedicated to the generalizations, refinements, and extensions for the Hermite-Hadamard inequality for various families of convexity.

Besides generalization via convexity, great effort has gone into extending (1.1) by means of fractional integral operators. Most popular of them is the Riemann-Liouville fractional integral operators given in the following definition.

Definition 1. *Let $\alpha > 0$, $b_1, b_2 \in \mathbb{R}$ with $b_1 < b_2$ and $\psi \in L[b_1, b_2]$. Then the left and right Riemann-Liouville fractional integrals $J^{\alpha}_{b_1+}\psi$ and $J^{\alpha}_{b_2-}\psi$ of order α are defined by*

$$J^{\alpha}_{b_1+}\psi(x) = \frac{1}{\Gamma(\alpha)}\int_{b_1}^{x}(x-t)^{\alpha-1}\psi(t)dt \quad (x > b_1)$$

and

$$J^{\alpha}_{b_2-}\psi(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b_2}(t-x)^{\alpha-1}\psi(t)dt \quad (x < b_2)$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$ is the gamma function.

Sarikaya et al. [34] established the following fractional version of the Hermite-Hadamard inequality:

Theorem 2. *Let $0 \leq b_1 < b_2$ and $\psi : [b_1, b_2] \to \mathbb{R}$ be a positive convex function such that $\psi \in L[b_1, b_2]$. Then the fractional integrals inequality*

$$\psi\left(\frac{b_1+b_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\left[J^{\alpha}_{b_1+}\psi(b_2) + J^{\alpha}_{b_2-}\psi(b_1)\right] \leq \frac{\psi(b_1)+\psi(b_2)}{2}$$

holds for $\alpha > 0$.

In view of Theorem 2, it is pertinent to note that the positivity of the function ψ and the numbers b_1 and b_2 is not necessary. From Definition 1, it is clear that b_1 and b_2 are any real numbers such that $b_1 < b_2$.

Our main contribution in this article is to use a similar technique used in [35] to obtain Theorem 2. This time, our main result contains both sided fractional integral operators in the Riemann-Liouville sense. In this method, we use a green function and in the process, we obtain some identities involving the left and right Riemann-Liouville fractional integral operators. These identities are subsequently employed to establish some new results for the class of convex, concave and monotone functions.

2. Main results

Our main result will be anchored on the succeeding lemma.

Lemma 3. *(See [36, 37]) Let G be the green function defined on* $[b_1, b_2] \times [b_1, b_2]$ *by*

$$G(\lambda, \mu) = \begin{cases} b_1 - \mu, & b_1 \leq \mu \leq \lambda; \\ b_1 - \lambda, & \lambda \leq \mu \leq b_2. \end{cases}$$

Then any $\psi \in C^2([b_1, b_2])$ *can be expressed as*

$$\psi(x) = \psi(b_1) + (x - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(x, \mu)\psi''(\mu)d\mu. \tag{2.1}$$

We are now in a position to frame and prove our results.

Theorem 4. *Let* $\psi \in C^2([b_1, b_2])$ *be a convex function. Then, for any* $\alpha > 0$*, the following fractional integral inequalities hold:*

$$\psi\left(\frac{b_1 + b_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b_2 - b_1)^\alpha}\Big[J^\alpha_{b_1+}\psi(b_2) + J^\alpha_{b_2-}\psi(b_1)\Big] \leq \frac{\psi(b_1) + \psi(b_2)}{2}.$$

Proof. Setting $x = \frac{b_1+b_2}{2}$ in (2.1), we get

$$\psi\Big(\frac{b_1 + b_2}{2}\Big) = \psi(b_1) + \Big(\frac{b_1 + b_2}{2} - b_1\Big)\psi'(b_2) + \int_{b_1}^{b_2} G\Big(\frac{b_1 + b_2}{2}, \mu\Big)\psi''(\mu)d\mu.$$

Equivalently,

$$\psi\Big(\frac{b_1 + b_2}{2}\Big) = \psi(b_1) + \Big(\frac{b_2 - b_1}{2}\Big)\psi'(b_2) + \int_{b_1}^{b_2} G\Big(\frac{b_1 + b_2}{2}, \mu\Big)\psi''(\mu)d\mu. \tag{2.2}$$

Using (2.1), we do the following computations:

$$\begin{aligned} J^\alpha_{b_1+}\psi(b_2) &= \frac{1}{\Gamma(\alpha)}\int_{b_1}^{b_2} (b_2 - x)^{\alpha-1}\psi(x)dx \\ &= \frac{1}{\Gamma(\alpha)}\int_{b_1}^{b_2} (b_2 - x)^{\alpha-1}\Big\{\psi(b_1) + (x - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(x, \mu)\psi''(\mu)d\mu\Big\}dx \\ &= \frac{1}{\Gamma(\alpha)}\Big\{\psi(b_1)\int_{b_1}^{b_2} (b_2 - x)^{\alpha-1}dx + \psi'(b_2)\int_{b_1}^{b_2} (b_2 - x)^{\alpha-1}(x - b_1)dx \end{aligned}$$

$$+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big\}$$
$$=\frac{1}{\Gamma(\alpha)}\Big[\psi(b_1)\frac{(b_2-x)^{\alpha}}{-\alpha}\Big|_{b_1}^{b_2}+\psi^{'}(b_2)\Big\{(x-b_1)\frac{(b_2-x)^{\alpha}}{-\alpha}\Big|_{b_1}^{b_2}$$
$$-\int_{b_1}^{b_2}\frac{(b_2-x)^{\alpha}}{-\alpha}dx\Big\}+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big]$$
$$=\frac{1}{\Gamma(\alpha)}\Big[\psi(b_1)\frac{(b_2-b_1)^{\alpha}}{\alpha}+\psi^{'}(b_2)\Big\{0-\frac{(b_2-x)^{\alpha+1}}{\alpha(\alpha+1)}\Big|_{b_1}^{b_2}\Big\}$$
$$+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big].$$

Therefore,

$$J_{b_1+}^{\alpha}\psi(b_2)=\frac{1}{\Gamma(\alpha)}\left[\frac{(b_2-b_1)^{\alpha}}{\alpha}\psi(b_1)+\psi^{'}(b_2)\frac{(b_2-b_1)^{\alpha+1}}{\alpha(\alpha+1)}\right.$$
$$\left.+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\right]. \tag{2.3}$$

Similarly,

$$J_{b_2-}^{\alpha}\psi(b_1)=\frac{1}{\Gamma(\alpha)}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}\psi(x)dx$$
$$=\frac{1}{\Gamma(\alpha)}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}\Big\{\psi(b_1)+(x-b_1)\psi^{'}(b_2)+\int_{b_1}^{b_2}G(x,\mu)\psi^{''}(\mu)d\mu\Big\}dx$$
$$=\frac{1}{\Gamma(\alpha)}\Big\{\psi(b_1)\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}dx+\psi^{'}(b_2)\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}(x-b_1)dx$$
$$+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big\}$$
$$=\frac{1}{\Gamma(\alpha)}\Big[\psi(b_1)\frac{(x-b_1)^{\alpha}}{\alpha}\Big|_{b_1}^{b_2}+\psi^{'}(b_2)\int_{b_1}^{b_2}(x-b_1)^{\alpha}dx$$
$$+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big]$$
$$=\frac{1}{\Gamma(\alpha)}\Big[\psi(b_1)\frac{(b_2-b_1)^{\alpha}}{\alpha}+\psi^{'}(b_2)\frac{(x-b_1)^{\alpha+1}}{\alpha+1}\Big|_{b_1}^{b_2}$$
$$+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big].$$

So,

$$J_{b_2-}^{\alpha}\psi(b_1)=\frac{1}{\Gamma(\alpha)}\left[\frac{(b_2-b_1)^{\alpha}}{\alpha}\psi(b_1)+\psi^{'}(b_2)\frac{(b_2-b_1)^{\alpha+1}}{\alpha+1}\right.$$
$$\left.+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\right]. \tag{2.4}$$

Now, adding (2.3) and (2.4) and then multiplying the resultant sum by $\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}$ to get:

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\Big[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\Big]\\
&=\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\frac{1}{\Gamma(\alpha)}\Big[\frac{(b_2-b_1)^\alpha}{\alpha}\psi(b_1)+\psi^{'}(b_2)\frac{(b_2-b_1)^{\alpha+1}}{\alpha(\alpha+1)}\\
&+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx+\frac{(b_2-b_1)^\alpha}{\alpha}\psi(b_1)\\
&+\psi^{'}(b_2)\frac{(b_2-b_1)^{\alpha+1}}{\alpha+1}+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big]\\
&=\frac{\alpha}{2(b_2-b_1)^\alpha}\Big[\frac{2(b_2-b_1)^\alpha}{\alpha}\psi(b_1)+\psi^{'}(b_2)\frac{(b_2-b_1)^{\alpha+1}}{(\alpha+1)}\Big\{\frac{1}{\alpha}+1\Big\}\\
&+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\\
&+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big]\\
&=\psi(b_1)+\psi^{'}(b_2)\frac{(b_2-b_1)}{2}+\frac{\alpha}{2(b_2-b_1)^\alpha}\Big[\int_{b_1}^{b_2}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\\
&+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big].
\end{aligned}\tag{2.5}$$

Subtracting (2.5) from (2.2), we obtain

$$\begin{aligned}
&\psi\Big(\frac{b_1+b_2}{2}\Big)-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\Big[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\Big]\\
&=\psi(b_1)+\Big(\frac{b_2-b_1}{2}\Big)\psi^{'}(b_2)+\int_{b_1}^{b_2}G\Big(\frac{b_1+b_2}{2},\mu\Big)\psi^{''}(\mu)d\mu-\psi(b_1)\\
&-\psi^{'}(b_2)\frac{(b_2-b_1)}{2}-\frac{\alpha}{2(b_2-b_1)^\alpha}\Big[\int_{b_1}^{b_2}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\\
&+\int_{b_1}^{b_2}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big]\\
&=\int_{b_1}^{b_2}\Big[G\Big(\frac{b_1+b_2}{2},\mu\Big)-\frac{\alpha}{2(b_2-b_1)^\alpha}\Big\{\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)dx\\
&+\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)dx\Big\}\Big]\psi^{''}(\mu)d\mu.
\end{aligned}$$

By the definition of the Green function,

$$G(x,\mu)=\begin{cases} b_1-\mu, & \text{if } b_1\le\mu\le x\\ b_1-x, & \text{if } x\le\mu\le b_2,\end{cases}$$

we obtain

$$\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)dx=\frac{1}{\alpha(\alpha+1)}\Big[(b_2-\mu)^{\alpha+1}-(b_2-b_1)^{\alpha+1}\Big]\tag{2.6}$$

and

$$\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)dx=\frac{1}{\alpha(\alpha+1)}\Big\{(\alpha+1)(b_1-\mu)(b_2-b_1)^\alpha+(\mu-b_1)^{\alpha+1}\Big\}.\tag{2.7}$$

Substituting Eqs (2.6) and (2.7) into (2.6), then we obtain

$$\begin{aligned}&\psi\Big(\frac{b_1+b_2}{2}\Big)-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\Big[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\Big]\\&=\int_{b_1}^{b_2}\Big[G\Big(\frac{b_1+b_2}{2},\mu\Big)-\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{2(\alpha+1)}-\frac{b_1-\mu}{2}\\&\quad-\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}\Big]\psi''(\mu)d\mu.\end{aligned}\tag{2.8}$$

Let

$$\begin{aligned}f(\mu)=G\Big(\frac{b_1+b_2}{2},\mu\Big)&-\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{2(\alpha+1)}-\frac{b_1-\mu}{2}\\&-\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}.\end{aligned}\tag{2.9}$$

Here,

$$G\left(\frac{b_1+b_2}{2},\mu\right)=\begin{cases}b_1-\mu, & b_1\le\mu\le\frac{b_1+b_2}{2};\\ \frac{b_1-b_2}{2}, & \frac{b_1+b_2}{2}\le\mu\le b_2.\end{cases}\tag{2.10}$$

Now, if $b_1\le\mu\le\frac{b_1+b_2}{2}$, then from (2.9) and (2.10), we have

$$f(\mu)=\frac{b_1-\mu}{2}-\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{2(\alpha+1)}-\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}.$$

$$f'(\mu)=-\frac{1}{2}+\frac{(b_2-\mu)^\alpha}{2(b_2-b_1)^\alpha}-\frac{(\mu-b_1)^\alpha}{2(b_2-b_1)^\alpha}\le 0.$$

This shows that f is decreasing and $f(b_1)=0$ then $f(\mu)\le 0$ for all $\mu\in[b_1,\frac{b_1+b_2}{2}]$.

If, on the other hand, $\frac{b_1+b_2}{2}\le\mu\le b_2$, then

$$\begin{aligned}f(\mu)&=\frac{b_1-b_2}{2}-\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{2(\alpha+1)}-\frac{b_1-\mu}{2}-\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}\\&=\frac{\mu-b_2}{2}-\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{2(\alpha+1)}-\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}.\end{aligned}$$

Therefore,

$$\begin{aligned}f'(\mu)&=\frac{1}{2}+\frac{(b_2-\mu)^\alpha}{2(b_2-b_1)^\alpha}-\frac{(\mu-b_1)^\alpha}{2(b_2-b_1)^\alpha},\\f''(\mu)&=-\frac{\alpha(b_2-\mu)^{\alpha-1}}{2(b_2-b_1)^\alpha}-\frac{\alpha(\mu-b_1)^{\alpha-1}}{2(b_2-b_1)^\alpha}\le 0,\end{aligned}$$

which shows that f' is decreasing and $f'(b_2)=0$ and thus $f'(\mu)\ge 0$. Therefore, f is increasing and $f(b_2)=0$. Hence, $f(\mu)\le 0$ for all $\mu\in[\frac{b_1+b_2}{2},b_2]$. Combining the two cases discussed above, we have that

$$f(\mu)\le 0\quad\text{for all}\quad\mu\in[b_1,b_2].$$

Using (2.8), we deduce the first inequality:

$$\psi\Big(\frac{b_1+b_2}{2}\Big) \le \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\Big[J^\alpha_{b_1+}\psi(b_2) + J^\alpha_{b_2-}\psi(b_1)\Big].$$

For the right hand side of the inequality, we recall:

$$\begin{aligned}
\psi(x) &= \psi(b_1) + (x-b_1)\psi^{'}(b_2) + \int_{b_1}^{b_2} G(x,\mu)\psi^{''}(\mu)d\mu, \\
\psi(b_2) &= \psi(b_1) + (b_2-b_1)\psi^{'}(b_2) + \int_{b_1}^{b_2} G(b_2,\mu)\psi^{''}(\mu)d\mu, \\
\psi(b_1)+\psi(b_2) &= 2\psi(b_1) + (b_2-b_1)\psi^{'}(b_2) + \int_{b_1}^{b_2} G(b_2,\mu)\psi^{''}(\mu)d\mu, \\
\frac{\psi(b_1)+\psi(b_2)}{2} &= \psi(b_1) + \frac{(b_2-b_1)}{2}\psi^{'}(b_2) + \frac{1}{2}\int_{b_1}^{b_2} G(b_2,\mu)\psi^{''}(\mu)d\mu. \qquad (2.11)
\end{aligned}$$

Subtracting Eq (2.5) from (2.11), we get:

$$\begin{aligned}
&\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\Big[J^\alpha_{b_1+}\psi(b_2) + J^\alpha_{b_2-}\psi(b_1)\Big] \\
= \;& \psi(b_1) + \frac{(b_2-b_1)}{2}\psi^{'}(b_2) + \frac{1}{2}\int_{b_1}^{b_2} G(b_2,\mu)\psi^{''}(\mu)d\mu - \psi(b_1) - \psi^{'}(b_2)\frac{(b_2-b_1)}{2} \\
& - \frac{\alpha}{2(b_2-b_1)^\alpha}\Big[\int_{b_1}^{b_2}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx \\
& + \int_{b_1}^{b_2}\int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)\psi^{''}(\mu)d\mu dx\Big] \\
= \;& \frac{1}{2}\int_{b_1}^{b_2}\Big[G(b_2,\mu) - \frac{\alpha}{2(b_2-b_1)^\alpha}\int_{b_1}^{b_2}(b_2-x)^{\alpha-1}G(x,\mu)dx \\
& + \int_{b_1}^{b_2}(x-b_1)^{\alpha-1}G(x,\mu)dx\Big]\psi^{''}(\mu)d\mu \\
= \;& \frac{1}{2}\int_{b_1}^{b_2}\Big[G(b_2,\mu) - \frac{(b_2-\mu)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha} + \frac{b_2-b_1}{(\alpha+1)} - b_1 + \mu \\
& - \frac{(\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}\Big]\psi^{''}(\mu)d\mu. \qquad (2.12)
\end{aligned}$$

If we set

$$F(\mu) = G(b_2,\mu) - \frac{(b_2-\mu)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha} + \frac{b_2-b_1}{(\alpha+1)} - b_1 + \mu - \frac{(\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}, \qquad (2.13)$$

then for $b_1 \le \mu \le b_2$,

$$F(\mu) = b_1 - \mu - \frac{(b_2-\mu)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha} + \frac{b_2-b_1}{(\alpha+1)} - b_1 + \mu - \frac{(\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}$$

$$
\begin{aligned}
&= -\frac{(b_2-\mu)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^{\alpha}} + \frac{b_2-b_1}{(\alpha+1)} - \frac{(\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^{\alpha}} \\
&= \frac{(b_2-b_1)^{\alpha+1} - (b_2-\mu)^{\alpha+1} - (\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^{\alpha}}.
\end{aligned}
$$

If $b_1 \le \mu \le \frac{b_1+b_2}{2}$, then

$$
F'(\mu) = \frac{(b_2-\mu)^{\alpha} - (\mu-b_1)^{\alpha}}{(b_2-b_1)^{\alpha}} \ge 0
$$

which proves that F is increasing and $F(b_1) = 0$, and hence $F(\mu) \ge 0$.

Suppose also $\frac{b_1+b_2}{2} \le \mu \le b_2$. Then,

$$
F'(\mu) = \frac{(b_2-\mu)^{\alpha} - (\mu-b_1)^{\alpha}}{(b_2-b_1)^{\alpha}} \le 0.
$$

This implies that F is a decreasing function and $F(b_2) = 0$, and thus $F(\mu) \ge 0$. Also, $\psi''(\mu) \ge 0$ since ψ is convex. Hence, $F(\mu) \ge 0$ for all $\mu \in [b_1, b_2]$ and using (2.12) amounts to:

$$
\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2) + J^{\alpha}_{b_2-}\psi(b_1)\Big] \le \frac{\psi(b_1)+\psi(b_2)}{2}.
$$

That completes the proof. □

Next, we present new Hermite-Hadamard type inequalities for the class of monotone and convex function.

Theorem 5. *Let $\psi \in C^2([b_1, b_2])$ and $\alpha > 0$, Then the following statements are true:*

(i) *If $|\psi''|$ is an increasing function, then*

$$
\left|\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2) + J^{\alpha}_{b_2-}\psi(b_1)\Big]\right| \le \frac{|\psi''(b_2)|\alpha(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}.
$$

(ii) *If $|\psi''|$ is a decreasing function, then*

$$
\left|\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2) + J^{\alpha}_{b_2-}\psi(b_1)\Big]\right| \le \frac{|\psi''(b_1)|\alpha(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}.
$$

(iii) *If $|\psi''|$ is a convex function, then*

$$
\begin{aligned}
&\left|\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2) + J^{\alpha}_{b_2-}\psi(b_1)\Big]\right| \\
\le\ & \frac{\max\{|\psi''(b_1)|, |\psi''(b_2)|\}\alpha(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}.
\end{aligned}
$$

Proof. To prove (i), we use the following identity obtained from (2.12):

$$
\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2) + J^{\alpha}_{b_2-}\psi(b_1)\Big]
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{b_1}^{b_2}\Big[G(b_2,\mu) - \frac{(b_2-\mu)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^{\alpha}} + \frac{b_2-b_1}{(\alpha+1)} - b_1 + \mu \\
&\quad - \frac{(\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^{\alpha}}\Big]\psi''(\mu)d\mu. \qquad (2.14)
\end{aligned}
$$

Taking absolute values on both sides of (2.14) and using the fact that $\left|\psi''\right|$ is an increasing function, we have

$$
\begin{aligned}
&\left|\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2) + J^{\alpha}_{b_2-}\psi(b_1)\Big]\right| \\
\leq\ & \left|\psi''(b_2)\right|\frac{1}{2}\int_{b_1}^{b_2}\Big\{-\frac{(b_2-\mu)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^{\alpha}} + \frac{b_2-b_1}{\alpha+1} - \frac{(\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^{\alpha}}\Big\}d\mu \\
=\ & \left|\psi''(b_2)\right|\frac{1}{2}\Big\{\frac{(b_2-\mu)^{\alpha+2}}{(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\Big|_{b_1}^{b_2} + \frac{b_2-b_1}{\alpha+1}\mu\Big|_{b_1}^{b_2} - \frac{(\mu-b_1)^{\alpha+2}}{(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\Big|_{b_1}^{b_2}\Big\} \\
=\ & \left|\psi''(b_2)\right|\frac{1}{2}\Big\{-\frac{(b_2-b_1)^{\alpha+2}}{(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}} + \frac{(b_2-b_1)^2}{\alpha+1} - \frac{(b_2-b_1)^{\alpha+2}}{(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\Big\} \\
=\ & \left|\psi''(b_2)\right|\frac{1}{2}\Big\{-\frac{2(b_2-b_1)^{\alpha+2}}{(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}} + \frac{(b_2-b_1)^2}{\alpha+1}\Big\} \\
=\ & \frac{\left|\psi''(b_2)\right|\alpha(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}.
\end{aligned}
$$

Therefore,

$$
\left|\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2) + J^{\alpha}_{b_2-}\psi(b_1)\Big]\right| \leq \frac{|\psi''(b_2)|\alpha(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}.
$$

This establishes the inequality in (i).

Part (ii) can be proved in a similar way. For (iii), we make use of (2.13) and the fact that every convex function ψ defined on the interval $[b_1, b_2]$ is bounded above by $\max\left\{\left|\psi(b_1)\right|, \left|\psi(b_2)\right|\right\}$ to get:

$$
\begin{aligned}
&\left|\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2) + J^{\alpha}_{b_2-}\psi(b_1)\Big]\right| \\
\leq\ & \frac{\max\{|\psi''(b_1)|, |\psi''(b_2)|\}}{2(\alpha+1)(b_2-b_1)^{\alpha}}\left|\int_{b_1}^{b_2}\Big[-(b_2-\mu)^{\alpha+1} + (b_2-b_1)^{\alpha+1} - (\mu-b_1)^{\alpha+1}\Big]d\mu\right| \\
=\ & \frac{\max\{|\psi''(b_1)|, |\psi''(b_2)|\}\alpha(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}.
\end{aligned}
$$

□

Remark 6. *By setting $\alpha = 1$ in Theorem 5, we get the following inequalities:*

$$
\left|\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right| \leq \frac{|\psi''(b_2)|(b_2-b_1)^2}{12},
$$

$$
\left|\frac{\psi(b_1)+\psi(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right| \leq \frac{|\psi''(b_1)|(b_2-b_1)^2}{12},
$$

$$\left|\frac{\psi(b_1)+\psi(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right|\leq\frac{\max\left\{|\psi''(b_1)|,|\psi''(b_2)|\right\}(b_2-b_1)^2}{12}.$$

Theorem 7. *Let $\psi\in C^2([b_1,b_2])$ and $\alpha>0$. Then the following statements are true:*

(i) If $|\psi''|$ is an increasing function, then

$$\left|\psi\left(\frac{b_1+b_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\left[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\right]\right|$$
$$\leq\frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)}\left[\left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|+\left|\psi''(b_2)\right|\right].$$

(ii) If $|\psi''|$ is a decreasing function, then

$$\left|\psi\left(\frac{b_1+b_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\left[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\right]\right|$$
$$\leq\frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)}\left[\left|\psi''(b_1)\right|+\left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\right].$$

(iii) If $|\psi''|$ is a convex function, then

$$\left|\psi\left(\frac{b_1+b_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\left[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\right]\right|$$
$$\leq\frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)}\left[\max\left\{\left|\psi''(b_1)\right|,\left|\psi''\Big(\frac{b_1+b_2}{2}\Big)\right|\right\}\right.$$
$$+\left.\left\{\max\left|\psi''\Big(\frac{b_1+b_2}{2}\Big)\right|,\left|\psi''(b_2)\right|\right\}\right].$$

Proof. The inequality in (i) is obtained by using (2.8):

$$\begin{aligned}&\psi\left(\frac{b_1+b_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\left[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\right]\\
=&\int_{b_1}^{\frac{b_1+b_2}{2}}\left[G\Big(\frac{b_1+b_2}{2},\mu\Big)-\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{2(\alpha+1)}-\frac{b_1-\mu}{2}\right.\\
&\left.-\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}\right]\psi''(\mu)d\mu\\
&+\int_{\frac{b_1+b_2}{2}}^{b_2}\left[G\Big(\frac{b_1+b_2}{2},\mu\Big)-\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{2(\alpha+1)}-\frac{b_1-\mu}{2}\right.\\
&\left.-\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^\alpha}\right]\psi''(\mu)d\mu.\end{aligned}$$

Taking absolute values and using triangle inequality, we get

$$\left|\psi\left(\frac{b_1+b_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\left[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\right]\right|$$

$$
\begin{aligned}
\leq\ & \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\left|\int_{b_1}^{\frac{b_1+b_2}{2}}\left\{\frac{b_1-\mu}{2}-\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}+\frac{b_2-b_1}{2(\alpha+1)}\right.\right.\\
& \left.\left.-\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}\right\}d\mu\right|+\left|\psi''(b_2)\right|\left|\int_{\frac{b_1+b_2}{2}}^{b_2}\left\{\frac{\mu-b_2}{2}-\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}\right.\right.\\
& \left.\left.+\frac{b_2-b_1}{2(\alpha+1)}-\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}\right\}d\mu\right|\\
=\ & \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\left|\left\{\frac{2b_1\mu-\mu^2}{4}\right|_{b_1}^{\frac{b_1+b_2}{2}}+\frac{(b_2-\mu)^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\right|_{b_1}^{\frac{b_1+b_2}{2}}\\
& +\frac{(b_2-b_1)}{2(\alpha+1)}\mu\Big|_{b_1}^{\frac{b_1+b_2}{2}}-\frac{(\mu-b_1)^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\Big|_{b_1}^{\frac{b_1+b_2}{2}}\Big\}\Big|\\
& +\left|\psi''(b_2)\right|\left|\left\{\frac{\mu^2-2b_2\mu}{4}\right|_{\frac{b_1+b_2}{2}}^{b_2}+\frac{(b_2-\mu)^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\right|_{\frac{b_1+b_2}{2}}^{b_2}\\
& +\frac{b_2-b_1}{2(\alpha+1)}\mu\Big|_{\frac{b_1+b_2}{2}}^{b_2}-\frac{(\mu-b_1)^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\Big|_{\frac{b_1+b_2}{2}}^{b_2}\Big\}\Big|\\
=\ & \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\left|\frac{2b_1(\frac{b_1+b_2}{2})-(\frac{b_1+b_2}{2})^2}{4}-\frac{2b_1(b_1)-(b_1)^2}{4}+\frac{(b_2-\frac{b_1+b_2}{2})^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\right.\\
& -\frac{(b_2-b_1)^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}+\frac{(b_2-b_1)}{2(\alpha+1)}\left(\frac{b_1+b_2}{2}-b_1\right)\\
& \left.-\frac{(\frac{b_1+b_2}{2}-b_1)^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}+\frac{(b_1-b_1)^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\right|\\
& +|\psi''(b_2)|\left|\frac{(b_2)^2-2(b_2)^2}{4}-\frac{(\frac{b_1+b_2}{2})^2-2b_2(\frac{b_1+b_2}{2})}{4}-\frac{(b_2-\frac{b_1+b_2}{2}))^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\right.\\
& \left.+\frac{b_2-b_1}{2(\alpha+1)}\left(b_2-\frac{b_1+b_2}{2}\right)-\frac{(b_2-b_1)^{\alpha+2}}{2(\alpha+1)(\alpha+2)(b_2-b_1)^{\alpha}}\right|
\end{aligned}
$$

$$
\begin{aligned}
=\ & \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\left|-\frac{(b_2-b_1)^2}{16}+\frac{(b_2-b_1)^2}{2^{\alpha+3}(\alpha+1)(\alpha+2)}-\frac{(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}+\frac{(b_2-b_1)^2}{4(\alpha+1)}\right.\\
& \left.-\frac{(b_2-b_1)^2}{2^{\alpha+3}(\alpha+1)(\alpha+2)}\right|\\
& +\left|\psi''(b_2)\right|\left|-\frac{(b_2-b_1)^2}{16}-\frac{(b_2-b_1)^2}{2^{\alpha+3}(\alpha+1)(\alpha+2)}+\frac{(b_2-b_1)^2}{4(\alpha+1)}-\frac{(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}\right.\\
& \left.+\frac{(b_2-b_1)^2}{2^{\alpha+3}(\alpha+1)(\alpha+2)}\right|\\
=\ & \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\left|-\frac{(b_2-b_1)^2}{16}-\frac{(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}+\frac{(b_2-b_1)^2}{4(\alpha+1)}\right|\\
& +\left|\psi''(b_2)\right|\left|-\frac{(b_2-b_1)^2}{16}+\frac{(b_2-b_1)^2}{4(\alpha+1)}-\frac{(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|(b_2-b_1)^2\left|\frac{-\alpha^2+\alpha-2}{16(\alpha+1)(\alpha+2)}\right| + \left|\psi''(b_2)\right|(b_2-b_1)^2\left|\frac{-\alpha^2+\alpha-2}{16(\alpha+1)(\alpha+2)}\right| \\
&= \left\{\left|\psi''\left(\frac{b_1+b_2}{2}\right)\right| + \left|\psi''(b_2)\right|\right\}(b_2-b_1)^2\left[\frac{\alpha^2-\alpha+2}{16(\alpha+1)(\alpha+2)}\right] \\
&= \left\{|\psi''\left(\frac{b_1+b_2}{2}\right)| + |\psi''(b_2)|\right\}\frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)}.
\end{aligned}
$$

The second part can be deduced by using the same procedure. For Part (iii), we also employ the fact that the convex function ψ is bounded above by $\max\left\{\left|\psi(b_1)\right|, \left|\psi(b_2)\right|\right\}$ since it is defined on the interval $[b_1, b_2]$. That is, we obtain from (2.8):

$$
\begin{aligned}
&\left|\psi\left(\frac{b_1+b_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\left[J^\alpha_{b_1+}(b_2) + J^\alpha_{b_2-}(b_1)\right]\right| \\
\leq\ & \max\left\{\left|\psi''(b_1)\right| + \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|, \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right| + \left|\psi''(b_2)\right|\right\} \\
&\times\frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)} \\
=\ & \left[\max\left\{\left|\psi''(b_1)\right|, \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\right\} + \max\left\{\left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|, \left|\psi''(b_2)\right|\right\}\right] \\
&\times\frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)}
\end{aligned}
$$

which is the desired inequality. □

Remark 8. *In Theorem 7, if we take α=1, then we obtain*

$$
\left|\psi\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right| \leq \frac{(b_2-b_1)^2}{48}\left[\left|\psi''\left(\frac{b_1+b_2}{2}\right)\right| + |\psi''(b_2)|\right],
$$

$$
\left|\psi\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right| \leq \frac{(b_2-b_1)^2}{48}\left[|\psi''(b_1)| + \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\right],
$$

$$
\begin{aligned}
\left|\psi\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right| \leq \frac{(b_2-b_1)^2}{48}\Bigg[&\max\left\{|\psi''(b_1)|, \left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|\right\} \\
&+ \max\left\{\left|\psi''\left(\frac{b_1+b_2}{2}\right)\right|, |\psi''(b_2)|\right\}\Bigg].
\end{aligned}
$$

Theorem 9. *Assume that $\psi \in C^2([b_1, b_2])$ and $|\psi''|$ is a convex function. Then for any $\alpha > 0$ the following inequality holds*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\left[J^\alpha_{b_1+}\psi(b_2) + J^\alpha_{b_2-}\psi(b_1)\right] - \psi\left(\frac{b_1+b_2}{2}\right)\right| \\
\leq\ & \frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)}\left[\left|\psi''(b_1)\right| + \left|\psi''(b_2)\right|\right].
\end{aligned}
$$

Proof. The inequality in (i) is obtained by using (2.8):

$$
\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\left[J^\alpha_{b_1+}\psi(b_2) + J^\alpha_{b_2-}\psi(b_1)\right] - \psi\left(\frac{b_1+b_2}{2}\right)
$$

$$
\begin{aligned}
= & \int_{b_1}^{\frac{b_1+b_2}{2}}\Big[\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}-\frac{b_2-b_1}{2(\alpha+1)}+\frac{\mu-b_1}{2} \\
& +\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}\Big]\psi''(\mu)d\mu \\
& +\int_{\frac{b_1+b_2}{2}}^{b_2}\Big[\frac{(b_2-\mu)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}-\frac{b_2-b_1}{2(\alpha+1)}+\frac{b_2-\mu}{2} \\
& +\frac{(\mu-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}\Big]\psi''(\mu)d\mu.
\end{aligned}
$$

Suppose $\mu=(1-t)b_1+tb_2$ where $d\mu=(b_2-b_1)dt$,

$$
\begin{aligned}
= & \int_{0}^{\frac{1}{2}}\Big[\frac{(b_2-(1-t)b_1-tb_2)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}-\frac{b_2-b_1}{2(\alpha+1)}+\frac{(1-t)b_1+tb_2-b_1}{2} \\
& +\frac{((1-t)b_1+tb_2-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}\Big]\psi''((1-t)b_1+tb_2)(b_2-b_1)dt \\
& +\int_{\frac{1}{2}}^{1}\Big[\frac{(b_2-(1-t)b_1-tb_2)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}-\frac{b_2-b_1}{2(\alpha+1)}+\frac{b_2-(1-t)b_1-tb_2}{2} \\
& +\frac{((1-t)b_1+tb_2-b_1)^{\alpha+1}}{2(\alpha+1)(b_2-b_1)^{\alpha}}\Big]\psi''((1-t)b_1+tb_2)(b_2-b_1)dt \\
= & \frac{(b_2-b_1)^2}{2(\alpha+1)}\Big[\int_{0}^{\frac{1}{2}}\Big[(1-t)^{\alpha+1}-1+(\alpha+1)t+t^{\alpha+1}\Big]\psi''((1-t)b_1+tb_2)dt \\
& +\int_{\frac{1}{2}}^{1}\Big[(1-t)^{\alpha+1}+\alpha(1-t)-t+t^{\alpha+1}\Big]\psi''((1-t)b_1+tb_2)dt\Big].
\end{aligned}
\tag{2.15}
$$

Taking absolute and triangular inequality,

$$
\begin{aligned}
\leq & \frac{(b_2-b_1)^2}{2(\alpha+1)}\Big[\int_{0}^{\frac{1}{2}}\Big[(1-t)^{\alpha+1}-1+(\alpha+1)t+t^{\alpha+1}\Big](1-t)\Big|\psi''(b_1)\Big|dt \\
& +\int_{0}^{\frac{1}{2}}\Big[(1-t)^{\alpha+1}-1+(\alpha+1)t+t^{\alpha+1}\Big]t\Big|\psi''(b_2)\Big|dt \\
& +\int_{\frac{1}{2}}^{1}\Big[(1-t)^{\alpha+1}+\alpha(1-t)-t+t^{\alpha+1}\Big](1-t)\Big|\psi''(b_1)\Big|dt \\
& +\int_{\frac{1}{2}}^{1}\Big[(1-t)^{\alpha+1}+\alpha(1-t)-t+t^{\alpha+1}\Big]t\Big|\psi''(b_2)\Big|\Big]dt\Big] \\
= & \frac{(b_2-b_1)^2}{2(\alpha+1)}\Big[\Big|\psi''(b_1)\Big|\Big\{\int_{0}^{\frac{1}{2}}\Big((1-t)^{\alpha+2}-(1-t)+(\alpha+1)(t-t^2)+t^{\alpha+1}-t^{\alpha+2}\Big)dt \\
& +\int_{\frac{1}{2}}^{1}\Big((1-t)^{\alpha+2}+\alpha(1-t)^2-(t-t^2)+t^{\alpha+1}-t^{\alpha+2}\Big)dt\Big\} \\
& +\Big|\psi''(b_2)\Big|\Big\{\int_{0}^{\frac{1}{2}}\Big(t(1-t)^{\alpha+1}-t+(\alpha+1)t^2+t^{\alpha+2}\Big)dt
\end{aligned}
$$

$$+\int_{\frac{1}{2}}^{1}\left(t(1-t)^{\alpha+1}+\alpha(t-t^2)-t^2+t^{\alpha+2}\right)dt\Big\}\Bigg]$$
$$= \frac{(b_2-b_1)^2}{2(\alpha+1)}\Bigg[\left|\psi''(b_1)\right|\Big\{I_1\Big\}+\left|\psi''(b_2)\right|\Big\{I_2\Big\}\Bigg]. \tag{2.16}$$

Putting the values of I_1 and I_2 in above (2.16), we get:

$$= \frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)}\Bigg[\left|\psi''(b_1)\right|+\left|\psi''(b_2)\right|\Bigg].$$

□

Remark 10. *In Theorem 9, if we take $\alpha = 1$, then we obtain*

$$\left|\psi\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right| \le \frac{(b_2-b_1)^2}{48}\Bigg[\left|\psi''(b_1)\right|+\left|\psi''(b_2)\right|\Bigg].$$

Theorem 11. *Assume that $\psi \in C^2([b_1, b_2])$ and $|\psi''|$ is a convex function. Then for any $\alpha > 0$ the following inequality holds*

$$\left|\frac{\psi(b_1)+\psi(b_2)}{2}-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^\alpha}\Big[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\Big]\right|$$
$$\le \frac{\alpha(b_2-b_1)^2}{4(\alpha+1)(\alpha+2)}\Bigg[\left|\psi''(b_1)\right|+\left|\psi''(b_2)\right|\Bigg].$$

Proof. We start by recalling the following identity from (2.12):

$$\frac{\psi(b_1)+\psi(b_2)}{2}-\frac{\Gamma(\alpha+1)}{(b_2-b_1)^\alpha}\Big[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\Big]$$
$$= \frac{1}{2}\int_{b_1}^{b_2}\Bigg[G(b_2,\mu)-\frac{(b_2-\mu)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{(\alpha+1)}-b_1+\mu$$
$$-\frac{(\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}\Bigg]\psi''(\mu)d\mu$$
$$= \frac{1}{2}\int_{b_1}^{b_2}\Bigg[b_1-\mu-\frac{(b_2-\mu)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{(\alpha+1)}-b_1+\mu$$
$$-\frac{(\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}\Bigg]\psi''(\mu)d\mu$$
$$= \frac{1}{2}\int_{b_1}^{b_2}\Bigg[-\frac{(b_2-\mu)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{(\alpha+1)}-\frac{(\mu-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}\Bigg]\psi''(\mu)d\mu.$$

If $\mu = (1-t)b_1 + tb_2$ with $t \in [0, 1]$, then we obtain

$$\frac{\psi(b_1)+\psi(b_2)}{2}-\frac{\Gamma(\alpha+1)}{(b_2-b_1)^\alpha}\Big[J^\alpha_{b_1+}\psi(b_2)+J^\alpha_{b_2-}\psi(b_1)\Big]$$
$$= \frac{1}{2}\int_{0}^{1}\Bigg[-\frac{(b_2-(1-t)b_1-tb_2)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}+\frac{b_2-b_1}{(\alpha+1)}-\frac{((1-t)b_1+tb_2-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^\alpha}\Bigg]$$

$$\times\psi''((1-t)b_1+tb_2)(b_2-b_1)dt$$
$$= \frac{1}{2}\int_0^1\Big[-\frac{(b_2-b_1)^{\alpha+1}(1-t)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^{\alpha}}+\frac{b_2-b_1}{(\alpha+1)}-\frac{t^{\alpha+1}(b_2-b_1)^{\alpha+1}}{(\alpha+1)(b_2-b_1)^{\alpha}}\Big]$$
$$\times\psi''((1-t)b_1+tb_2)(b_2-b_1)dt$$
$$\leq \frac{(b_2-b_1)^2}{2(\alpha+1)}\int_0^1\Big[-(1-t)^{\alpha+1}+1-t^{\alpha+1}\Big]\Big[(1-t)\big|\psi''(b_1)\big|+t\big|\psi''b_2)\big|\Big]dt$$
$$= \frac{(b_2-b_1)^2}{2(\alpha+1)}\Big[\big|\psi''(b_1)\big|\int_0^1\Big\{-(1-t)^{\alpha+2}+1-t-t^{\alpha+1}+t^{\alpha+2}\Big\}dt$$
$$+\big|\psi''b_2)\big|\int_0^1\Big\{-t(1-t)^{\alpha+1}+t-t^{\alpha+2}\Big\}dt\Big].$$

Taking absolute values and applying the triangle inequality amounts to the intended result. □

Remark 12. *Let $\alpha=1$. Then the inequality in Theorem 11 becomes:*

$$\left|\frac{\psi(b_1)+\psi(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right|\leq\frac{(b_2-b_1)^2}{24}\Big[\big|\psi''(b_1)\big|+\big|\psi''(b_2)\big|\Big].$$

Next, we present results associated with the concave functions.

Theorem 13. *Let $\psi\in C^2([b_1,b_2])$ and $\big|\psi''\big|$ be a concave function. Then, for any $\alpha>0$,*

$$\left|\psi\Big(\frac{b_1+b_2}{2}\Big)-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2)+J^{\alpha}_{b_2-}\psi(b_1)\Big]\right|$$
$$\leq \frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)}\Bigg[$$
$$\times\left|\psi''\left(\frac{b_1\Big(\frac{2+2^{\alpha+3}(\alpha+2)}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{2\alpha-7}{24}\Big)+b_2\Big(\frac{2^{\alpha+3}-2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{\alpha-2}{24}\Big)}{\frac{\alpha^2-\alpha+2}{8(\alpha+2)}}\right)\right|$$
$$+\left|\psi''\left(\frac{b_1\Big(\frac{\alpha-2}{24}+\frac{2^{\alpha+3}-2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\Big)+b_2\Big(\frac{2\alpha-7}{24}+\frac{2^{\alpha+3}(\alpha+2)+2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\Big)}{\frac{\alpha^2-\alpha+2}{8(\alpha+2)}}\right)\right|\Bigg].$$

Proof. From (2.15), we have:

$$\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2)+J^{\alpha}_{b_2-}\psi(b_1)\Big]-\psi\left(\frac{b_1+b_2}{2}\right)$$
$$= \frac{(b_2-b_1)^2}{2(\alpha+1)}\Bigg[\int_0^{\frac{1}{2}}\Big[(1-t)^{\alpha+1}-1+(\alpha+1)t+t^{\alpha+1}\Big]\psi''((1-t)b_1+tb_2)dt$$
$$+\int_{\frac{1}{2}}^1\Big[(1-t)^{\alpha+1}+\alpha(1-t)-t+t^{\alpha+1}\Big]\psi''((1-t)b_1+tb_2)dt\Bigg].$$

Now, we use Jensen's integral inequality to get:

$$
\begin{aligned}
&\leq \frac{(b_2-b_1)^2}{2(\alpha+1)}\Bigg[\int_0^{\frac{1}{2}}\Big[(1-t)^{\alpha+1}-1+(\alpha+1)t+t^{\alpha+1}\Big]dt \\
&\qquad \times\left|\psi''\left(\frac{\int_0^{\frac{1}{2}}\Big((1-t)^{\alpha+1}-1+(\alpha+1)t+t^{\alpha+1}\Big)((1-t)b_1+tb_2)dt}{\int_0^{\frac{1}{2}}\Big(1-t)^{\alpha+1}-1+(\alpha+1)t+t^{\alpha+1}\Big)dt}\right)\right| \\
&\quad +\int_{\frac{1}{2}}^1\Big((1-t)^{\alpha+1}+\alpha(1-t)-t+t^{\alpha+1}\Big)dt \\
&\qquad \times\left|\psi''\left(\frac{\int_{\frac{1}{2}}^1\Big((1-t)^{\alpha+1}+\alpha(1-t)-t+t^{\alpha+1}\Big)((1-t)b_1+tb_2)dt}{\int_{\frac{1}{2}}^1\Big((1-t)^{\alpha+1}+\alpha(1-t)-t+t^{\alpha+1}\Big)dt}\right)\right|\Bigg]. \\
&= \frac{(b_2-b_1)^2}{2(\alpha+1)}\Bigg[\int_0^{\frac{1}{2}}\Big[I_1\Big]dt\left|\psi''\left(\frac{\int_0^{\frac{1}{2}}\Big(I_2\Big)dt}{\int_0^{\frac{1}{2}}\Big(I_1\Big)dt}\right)\right|+\int_{\frac{1}{2}}^1\Big(A_1\Big)dt\left|\psi''\left(\frac{\int_{\frac{1}{2}}^1\Big(A_2\Big)dt}{\int_{\frac{1}{2}}^1\Big(A_1\Big)dt}\right)\right|\Bigg].
\end{aligned} \tag{2.17}
$$

For this we calculate:

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}}\Big[(1-t)^{\alpha+1}-1+(\alpha+1)t+t^{\alpha+1}\Big]dt \\
&= -\frac{(1-t)^{\alpha+2}}{\alpha+2}\Big|_0^{\frac{1}{2}}-t\Big|_0^{\frac{1}{2}}+(\alpha+1)\frac{t^2}{2}\Big|_0^{\frac{1}{2}}+\frac{t^{\alpha+2}}{\alpha+2}\Big|_0^{\frac{1}{2}} \\
&= -\frac{1}{2^{\alpha+2}(\alpha+2)}+\frac{1}{\alpha+2}-\frac{1}{2}+\frac{\alpha+1}{8}+\frac{1}{2^{\alpha+2}(\alpha+2)} \\
&= \frac{\alpha^2-\alpha+2}{8(\alpha+2)},
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \int_0^{\frac{1}{2}}\Big((1-t)^{\alpha+1}-1+(\alpha+1)t+t^{\alpha+1}\Big)((1-t)b_1+tb_2)dt \\
&= b_1\int_0^{\frac{1}{2}}\Big((1-t)^{\alpha+2}-(1-t)+(\alpha+1)(t-t^2)+t^{\alpha+1}-t^{\alpha+2}\Big)dt \\
&\quad +b_2\int_0^{\frac{1}{2}}\Big(t(1-t)^{\alpha+1}-t+(\alpha+1)t^2+t^{\alpha+2}\Big)dt \\
&= b_1\Big(-\frac{(1-t)^{\alpha+3}}{\alpha+3}-t+\frac{t^2}{2}+(\alpha+1)(\frac{t^2}{2}-\frac{t^3}{3})+\frac{t^{\alpha+2}}{\alpha+2}-\frac{t^{\alpha+3}}{\alpha+3}\Big)\Big|_0^{\frac{1}{2}} \\
&\quad +b_2\Big(-t\frac{(1-t)^{\alpha+2}}{\alpha+2}-\int_0^{\frac{1}{2}}\frac{-(1-t)^{\alpha+2}}{\alpha+2}dt-\frac{t^2}{2}+(\alpha+1)\frac{t^3}{3}+\frac{t^{\alpha+3}}{\alpha+3}\Big)\Big|_0^{\frac{1}{2}} \\
&= b_1\Big(\frac{2+2^{\alpha+3}(\alpha+2)}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{2\alpha-7}{24}\Big)+b_2\Big(\frac{2^{\alpha+3}-2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{\alpha-2}{24}\Big),
\end{aligned}
$$

$$\begin{aligned}
A_1 &= \int_{\frac{1}{2}}^{1}\Big((1-t)^{\alpha+1}+\alpha(1-t)-t+t^{\alpha+1}\Big)dt\\
&= -\frac{(1-t)^{\alpha+2}}{\alpha+2}\Big|_{\frac{1}{2}}^{1}+\alpha\Big(t-\frac{t^2}{2}\Big)\Big|_{\frac{1}{2}}^{1}-\frac{t^2}{2}\Big|_{\frac{1}{2}}^{1}+\frac{t^{\alpha+2}}{\alpha+2}\Big|_{\frac{1}{2}}^{1}\\
&= \frac{\alpha^2-\alpha+2}{8(\alpha+2)}
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= \int_{\frac{1}{2}}^{1}\Big((1-t)^{\alpha+1}+\alpha(1-t)-t+t^{\alpha+1}\Big)((1-t)b_1+tb_2)dt\\
&= b_1\int_{\frac{1}{2}}^{1}\Big((1-t)^{\alpha+2}+\alpha(1-t)^2-t+t^2+t^{\alpha+1}-t^{\alpha+2}\Big)dt\\
&\quad +b_2\int_{\frac{1}{2}}^{1}\Big(t(1-t)^{\alpha+1}+\alpha(t-t^2)-t^2+t^{\alpha+2}\Big)dt\\
&= b_1\Big[-\frac{(1-t)^{\alpha+3}}{\alpha+3}\Big|_{\frac{1}{2}}^{1}-\alpha\frac{(1-t)^3}{3}\Big|_{\frac{1}{2}}^{1}-\frac{t^2}{2}\Big|_{\frac{1}{2}}^{1}+\frac{t^3}{3}\Big|_{\frac{1}{2}}^{1}+\frac{t^{\alpha+2}}{\alpha+2}\Big|_{\frac{1}{2}}^{1}-\frac{t^{\alpha+3}}{\alpha+3}\Big|_{\frac{1}{2}}^{1}\Big]\\
&\quad +b_2\Big[-t\frac{(1-t)^{\alpha+2}}{\alpha+2}-\int_{\frac{1}{2}}^{1}\frac{(1-t)^{\alpha+2}}{-(\alpha+2)}dt+\alpha\Big(\frac{t^2}{2}-\frac{t^3}{3}\Big)-\frac{t^3}{3}+\frac{t^{\alpha+3}}{\alpha+3}\Big]_{\frac{1}{2}}^{1}\\
&= b_1\Big[\frac{\alpha-2}{24}+\frac{2^{\alpha+3}-2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\Big]+b_2\Big[\frac{2\alpha-7}{24}+\frac{2^{\alpha+3}(\alpha+2)+2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\Big].
\end{aligned}$$

Putting the values of I_1, I_2, A_1 and A_2 in (2.17), we obtain:

$$\begin{aligned}
&= \frac{(b_2-b_1)^2}{2(\alpha+1)}\Bigg[\frac{\alpha^2-\alpha+2}{8(\alpha+2)}\Bigg|\psi''\Bigg(\frac{b_1\Big(\frac{2+2^{\alpha+3}(\alpha+2)}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{2\alpha-7}{24}\Big)+b_2\Big(\frac{2^{\alpha+3}-2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{\alpha-2}{24}\Big)}{\frac{\alpha^2-\alpha+2}{8(\alpha+2)}}\Bigg)\Bigg|\\
&\quad +\frac{\alpha^2-\alpha+2}{8(\alpha+2)}\Bigg|\psi''\Bigg(\frac{b_1\Big(\frac{\alpha-2}{24}+\frac{2^{\alpha+3}-2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\Big)+b_2\Big(\frac{2\alpha-7}{24}+\frac{2^{\alpha+3}(\alpha+2)+2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\Big)}{\frac{\alpha^2-\alpha+2}{8(\alpha+2)}}\Bigg)\Bigg|\Bigg]\\
&= \frac{(b_2-b_1)^2(\alpha^2-\alpha+2)}{16(\alpha+1)(\alpha+2)}\\
&\quad \times\Bigg[\Bigg|\psi''\Bigg(\frac{b_1\Big(\frac{2+2^{\alpha+3}(\alpha+2)}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{2\alpha-7}{24}\Big)+b_2\Big(\frac{2^{\alpha+3}-2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{\alpha-2}{24}\Big)}{\frac{\alpha^2-\alpha+2}{8(\alpha+2)}}\Bigg)\Bigg|\\
&\quad +\Bigg|\psi''\Bigg(\frac{b_1\Big(\frac{\alpha-2}{24}+\frac{2^{\alpha+3}-2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\Big)+b_2\Big(\frac{2\alpha-7}{24}+\frac{2^{\alpha+3}(\alpha+2)+2}{2^{\alpha+3}(\alpha+2)(\alpha+3)}\Big)}{\frac{\alpha^2-\alpha+2}{8(\alpha+2)}}\Bigg)\Bigg|\Bigg].
\end{aligned}$$

□

Remark 14. *If we take $\alpha = 1$ in Theorem 13, then we get:*

$$\begin{aligned}&\left|\psi\Big(\frac{b_1+b_2}{2}\Big)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right|\\ \leq\ & \frac{(b_2-b_1)^2}{48}\Bigg[\left|\psi''\Big(\frac{5b_1+3b_2}{8}\Big)\right|+\left|\psi''\Big(\frac{3b_1+5b_2}{8}\Big)\right|\Bigg].\end{aligned}$$

Theorem 15. *Let $\psi \in C^2([b_1, b_2])$ and $\left|\psi''\right|$ be a concave function. Then, for any $\alpha > 0$,*

$$\begin{aligned}&\left|\frac{\psi(b_1)+\psi(b_2)}{2}-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2)+J^{\alpha}_{b_2-}\psi(b_1)\Big]\right|\\ \leq\ & \frac{\alpha(b_2-b_1)^2}{2(\alpha+1)(\alpha+2)}\left|\psi''\Big(\frac{b_1+b_2}{2}\Big)\right|.\end{aligned}$$

Proof. For this, we use (2.14) to obtain the following inequality:

$$\begin{aligned}&\left|\frac{\psi(b_1)+\psi(b_2)}{2}-\frac{\Gamma(\alpha+1)}{2(b_2-b_1)^{\alpha}}\Big[J^{\alpha}_{b_1+}\psi(b_2)+J^{\alpha}_{b_2-}\psi(b_1)\Big]\right|\\ =\ & \frac{(b_2-b_1)^2}{2(\alpha+1)}\int_0^1\Big[-(1-t)^{\alpha+1}+1-t^{\alpha+1}\Big]\Big[\psi''((1-t)b_1+tb_2)\Big]dt.\end{aligned}$$

We use Jensen's integral inequality to get:

$$\begin{aligned}\leq\ & \frac{(b_2-b_1)^2}{2(\alpha+1)}\Bigg\{\int_0^1\Big(-(1-t)^{\alpha+1}+1-t^{\alpha+1}\Big)dt\\ &\times\left|\psi''\left(\frac{\int_0^1\Big(-(1-t)^{\alpha+1}+1-t^{\alpha+1}\Big)((1-t)b_1+tb_2)dt}{\int_0^1\Big(-(1-t)^{\alpha+1}+1-t^{\alpha+1}\Big)dt}\right)\right|\Bigg\}\\ =\ & \frac{(b_2-b_1)^2}{2(\alpha+1)}\Bigg\{\frac{\alpha}{\alpha+2}\left|\psi''\left(\frac{b_1\Big(\frac{\alpha}{2(\alpha+2)}\Big)+b_2\Big(\frac{\alpha}{2(\alpha+2)}\Big)}{\frac{\alpha}{\alpha+2}}\right)\right|\Bigg\}\\ =\ & \frac{(b_2-b_1)^2}{2(\alpha+1)}\Bigg\{\frac{\alpha}{\alpha+2}\left|\psi''\Big(\frac{b_1+b_2}{2}\Big)\right|\Bigg\}.\end{aligned}$$

□

Remark 16. *If we set $\alpha = 1$, then Theorem 15 amounts to:*

$$\left|\frac{\psi(b_1)+\psi(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\psi(t)dt\right|\leq\frac{(b_2-b_1)^2}{12}\left|\psi''\Big(\frac{b_1+b_2}{2}\Big)\right|.$$

3. Conclusion

By means of a green function, we outlined a new method of proving the Hermite-Hadamard inequality involving the Riemann-Liouville fractional integral operators. In the process of doing this, some new identities were obtained. We applied these identities to prove more results in this direction. Results established in this paper can be recast by using the green function $G_2(\lambda,\mu)$ defined on $[b_1,b_2]\times[b_1,b_2]$ by

$$G_2(\lambda,\mu)=\begin{cases}\lambda-b_1, & b_1\leq\mu\leq\lambda;\\ \mu-b_1, & \lambda\leq\mu\leq b_2.\end{cases}$$

Acknowledgments

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions. The work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485).

Conflict of interest

There is no conflict of interest to report.

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