Mathematics
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## Research article

# Certain novel estimates within fractional calculus theory on time scales 

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#### Abstract

The key purpose of this study is to suggest a delta Riemann-Liouville (RL) fractional integral operators for deriving certain novel refinements of Pólya-Szegö and Čebyšev type inequalities on time scales. Some new Pólya-Szegö, Čebyšev and extended Čebyšev inequalities via delta-RL fractional integral operator on a time scale that captures some continuous and discrete analogues in the relative literature. New explicit bounds for unknown functions concerned are obtained due to the presented inequalities.


Keywords: Pólya-Szegö type inequality; Čebyšev inequality; Riemann-Liouville fractional integral; time scale
Mathematics Subject Classification: 26D15, 26A33, 26E70

## 1. Introduction

During the past decades, some scientists have shown a great deal of interest in the field of fractional calculus [1-9] which addresses the derivatives and integrals with any order. As a matter of fact, this interest has sprung out by the dint of the substantial results obtained when these scientists used the tools in this calculus in order to study some models from the real world. A variety of results that helped in developing the theory of discrete fractional calculus are given in [10]. Atici and Eloe carefully evoked
the interest in the theory of fractional difference operators [11]. Abdeljawad [12], and Abdeljawad and Atici [13] defined fractional difference with different types of kernel having discrete power law with discrete exponential and generalized Mittag-Leffler functions [14] and discrete exponential and Mittag-Leffler functions on generalized $h \mathbb{Z}$ time scale [15], and kernel containing the product of both power-law and exponential function in [16].

In [17], Hilger proposed the concept of time scales as a study skilled to contain both difference and differential analytic in a systematic manner. The time-scale calculus can be used to unify discrete and continuous approaches to signal processing in one unique setting.

The idea of the fractional-order derivative has been expounded by Bastos [18] via Riemann-Liouville fractional operators on scale versions by considering linear dynamic equations. Interestingly in applications, this type of calculus can describe the anomalous diffusion model in a discrete setting also has the possibility to deal with more complex time domains [19]. One extreme case, covered by the theory of fractional time scales calculus and surprisingly relevant also for the process of signals, appears when one fixes the time scale to be the Cantor set. The numerical discretization of fractional derivative is a challenging work since the treatment will result in great numerical errors quickly. Fractional calculus on time scale provides a new tool which can avoid such problems and some recent works in fractional chaotic maps with application exhibit the new feature. Several researchers had have been paid much attention [3, 10, 12-14].

Since the publications in 2015, several researchers made significant contributions to the history of time scales. Recently, Holm-Hansen and Gao [20] presented the detection of structural defects in a ball bearing using an embedded piezoceramic load sensor and the discrete wavelet transform. Wu et al. [21] considered efficient description dynamics in short time domains having chaos in both continuous and discrete-time cases.

In [3], the authors proposed a discrete fractional logistic map in the left Caputo discrete delta's sense which plays a key role in solving difference equations. Zhu and Wu [22] employed Caputo nabla fractional derivatives in order to find the existence of solutions for Cauchy problems. As certifiable utilities, we refer to the study of calcium ion channels that are impeded with an infusion of calcium-chelator ethylene glycol tetraacetic acid [23]. Actually, the physical utilization of initial value-fractional problems in diverse time scales proliferates [24]. A method of generalizing logistic equations is introduced in [25], by adding general parameters which affect the logistic map greatly. Several studies concerning the discretization of the fractional logistic map and its chaotic behavior are studied in [3, 25].

Here, we broaden fractional integral variants accessible in the literature by presenting increasingly broad ideas on time scales in the frame of delta-Riemann-Liouville fractional integral. At that point, we study the dynamic variants of corresponding generalized fractional-order on time scales. We obtain the inequalities Pólya-Szegö, Čebyšev and several others extended Čebyšev versions using the delta integrals in arbitrary time scales. For $\Delta=1$, the integral will become delta integral and for $\Delta=0$; it advances toward turning out to be nabla integral. The suggested dynamic integral technique is explicit and efficient to acquire new consequences. This technique has more features: it is immediate and brief. In this way, the proposed technique can be prolonged to settle numerous frameworks of nonlinear fractional partial differential equations in mathematical and physical sciences. Also, the analytical solutions can be obtained for the generalized ordinary differential equations to obtain new theorems related to stability and continuous dependence on parameters for dynamic equations on time scales.

The inequalities have been considered by many authors and for such related results, we refer the reader to the recent works in [26-40].

Inspired by the discretization of the Riemann-Liouville, the key aim of this paper is to establish some new inequalities Pólya-Szegö, Čebyšev and extended Čebyšev via delta-RL fractional integral operator on a time scale that captures some continuous and discrete analogues in the relative literature. We presented, in general, two analogous of Čebyšev type inequalities, that can be used to solve some new generalizations with the assumption of time scales analysis have yielded intriguing results. In addition, our findings can offer great opportunities to study the dynamics of such discrete systems powerfully, as well as their chaotic behaviors. Moreover, this work can be helpful in presenting the generalization of the discrete fractional logistic map exploring the effects of the extra general parameters added to the integral inequalities in combination with the extra degree of freedom offered by the fractional order parameter $\beta$.

## 2. Preliminaries

A nonempty closed subsets $\mathbb{R}$ of $\mathcal{T}$ is known as the time scale. The well-known examples of time scales theory are the set of real numbers $\mathbb{R}$ and the integers $\mathcal{Z}$. Throughout the paper, we refer $\mathcal{T}$ as time scale and a time-scaled interval is $\Upsilon_{\mathcal{T}}=[m, n]_{\mathcal{T}}$. We need the concept of jump operators. The forward jump operator is denoted by the symbol $\sigma$ and the backward jump operator is denoted by $\vartheta$, are said through the formulas:

$$
\sigma(t)=\inf \{\varrho \in \mathcal{T}: \varrho>t\} \in \mathcal{T}, \quad \rho(t)=\sup \{\varrho \in \mathcal{T}: \varrho<t\} \in \mathcal{T} .
$$

We accumulate as:

$$
\inf \emptyset:=\sup \mathcal{T}, \quad \sup \emptyset:=\inf \mathcal{T} .
$$

If $\sigma(t)>t$, then the term $t$ is allude to be right-scattered and $t$ is allude to be left-scattered $\varrho(t)<t$. The elements that are most likely all the while appropriate-scattered and scattered are known as isolated. The term $t$ is said to be right dense, if $\sigma(t)=t$, and $t$ is said to be left dense, if $\varrho(t)=t$. The mappings $\mu, v: \mathcal{T} \rightarrow[0,+\infty)$ defined by

$$
\begin{aligned}
\mu(t) & :=\sigma(t)-t, \\
v(t) & :=t-\rho(t)
\end{aligned}
$$

are called the forward and backward graininess functions, respectively.
Definition 2.1. Let $\hbar: \mathcal{T} \rightarrow \mathbb{R}$ be a real-valued function. Then $\hbar$ is said to be $\mathcal{R D}$-continuous on $\mathbb{R}$ if its left limit at any left dense point of $\mathcal{T}$ is finite and it is continuous on every right dense point of $\mathcal{T}$. All $\mathcal{R D}$-continuous functions are denoted by $\mathbb{C}_{\mathcal{R D}}$.

Definition 2.2. Let $\mathcal{P}$ and $Q$ be two integrable functions defined on $[m, \lambda]_{\mathcal{T}}$. If for any $t_{1}, t_{2} \in[m, \lambda]_{\mathcal{T}}$

$$
\left(\mathcal{P}\left(t_{1}\right)-\mathcal{P}\left(t_{2}\right)\right)\left(Q\left(t_{1}\right)-Q\left(t_{2}\right)\right) \geq 0
$$

then $\mathcal{P}$ and $\mathcal{Q}$ are called synchronous functions on $[m, \lambda]_{\mathcal{T}}$.

Definition 2.3. The $\mathcal{R D}$ continuous functions $\hbar_{\beta}: \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}$ is called as generalized polynomials on time scales such that, for all $\varrho, t \in \mathcal{T}$ and $\beta \geq 0$;

$$
\begin{gathered}
\hbar_{0}(t, \varrho)=1, \\
\hbar_{\beta+1}(t, \varrho)=\int_{\varrho}^{t} \hbar_{\beta}(\tau, \varrho) \Delta \tau .
\end{gathered}
$$

Definition 2.4. The Delta-Riemann-Liouville fractional integral operator of order $\beta \geq 1$ on time scales, for a function $\mathcal{P} \in \mathbb{C}_{\mathcal{R} \mathcal{D}}$ is defined as

$$
\begin{gathered}
I_{m}^{\beta} \mathcal{P}(\lambda)=\int_{m}^{\lambda} \hbar_{\beta-1}(\lambda, \sigma(\tau)) \mathcal{P}(\tau) \Delta \tau \\
I_{m}^{0} \mathcal{P}=\mathcal{P}
\end{gathered}
$$

## 3. Pólya-Szegö type inequalities

This section is dedicated to the novel version of Pólya-Szegö type inequalities by employing deltaRL fractional integral operators on a time scale.

Theorem 3.1. For $\beta \geq 1, m \geq 0$ and let there are two positive integrable functions $\mathcal{P}$ and $\boldsymbol{Q}$ defined on $[0, \infty)_{\mathcal{T}}$. Also, assume that there exist four integrable functions $\phi_{1}, \phi_{2}, \psi_{1}$ and $\psi_{2}$ defined on $[0, \infty)_{\mathcal{T}}$ such that

$$
\begin{equation*}
0 \leq \phi_{1}\left(t_{1}\right) \leq \mathcal{P}\left(t_{1}\right) \leq \phi_{1}\left(t_{1}\right), \quad 0 \leq \psi_{1}\left(t_{1}\right) \leq Q\left(t_{1}\right) \leq \psi_{1}\left(t_{1}\right) \tag{3.1}
\end{equation*}
$$

for all $\lambda>m$. Then

$$
\begin{equation*}
\frac{1}{4}\left(\mathcal{I}_{m}^{\beta}\left[\left(\phi_{1} \psi_{1}+\phi_{2} \psi_{2}\right) \mathcal{P} Q\right](\lambda)\right)^{2} \geq \mathcal{I}_{m}^{\beta}\left[\phi_{1} \phi_{2} \mathcal{P}^{2}\right](\lambda) \mathcal{I}_{m}^{\beta}\left[\psi_{1} \psi_{2} Q^{2}\right](\lambda) \tag{3.2}
\end{equation*}
$$

Proof. From (3.1), for $t_{1}>0$, we have

$$
\begin{equation*}
\frac{\phi_{2}\left(t_{1}\right)}{\psi_{1}\left(t_{1}\right)}-\frac{\mathcal{P}\left(t_{1}\right)}{\mathcal{Q}\left(t_{1}\right)} \geq 0 \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\mathcal{P}\left(t_{1}\right)}{Q\left(t_{1}\right)}-\frac{\phi_{1}\left(t_{1}\right)}{\psi_{2}\left(t_{1}\right)} \geq 0 . \tag{3.4}
\end{equation*}
$$

Multiplying (3.3) and (3.4), we get

$$
\begin{equation*}
\left[\phi_{1}\left(t_{1}\right) \psi_{1}\left(t_{1}\right)+\phi_{2}\left(t_{1}\right) \psi_{2}\left(t_{1}\right)\right] \mathcal{P}\left(t_{1}\right) Q\left(t_{1}\right) \geq \psi_{1}\left(t_{1}\right) \psi_{2}\left(t_{1}\right) \mathcal{P}^{2}\left(t_{1}\right)+\phi_{1}\left(t_{1}\right) \phi_{2}\left(t_{1}\right) Q^{2}\left(t_{1}\right) . \tag{3.5}
\end{equation*}
$$

For $t_{1} \in(m, \lambda)$, multiplying both sides of (3.5) by $\hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right)$, and integrating the preceding inequality with respect to $t_{1} \operatorname{over}(m, \lambda)$, we get

$$
\begin{equation*}
\mathcal{I}_{m}^{\beta}\left[\left(\phi_{1} \psi_{1}+\phi_{2} \psi_{2}\right) \mathcal{P} Q\right](\lambda) \geq I_{m}^{\beta}\left[\psi_{1} \psi_{2} \mathcal{P}^{2}\right](\lambda)+I_{m}^{\beta}\left[\phi_{1} \phi_{2} Q^{2}\right](\lambda) . \tag{3.6}
\end{equation*}
$$

Applying the arithmetic-geometric inequality, we have

$$
\mathcal{I}_{m}^{\beta}\left[\left(\phi_{1} \psi_{1}+\phi_{2} \psi_{2}\right) \mathcal{P} Q\right](\lambda) \geq 2 \sqrt{I_{m}^{\beta}\left[\psi_{1} \psi_{2} \mathcal{P}^{2}\right](\lambda)+I_{m}^{\beta}\left[\phi_{1} \phi_{2} Q^{2}\right](\lambda)}
$$

which leads to

$$
\frac{1}{4}\left(\mathcal{I}_{m}^{\beta}\left[\left(\phi_{1} \psi_{1}+\phi_{2} \psi_{2}\right) \mathcal{P} Q\right](\lambda)\right)^{2} \geq \mathcal{I}_{m}^{\beta}\left[\phi_{1} \phi_{2} \mathcal{P}^{2}\right](\lambda) \mathcal{I}_{m}^{\beta}\left[\psi_{1} \psi_{2} Q^{2}\right](\lambda)
$$

the desired inequality (3.1) is obtained.
Theorem 3.2. For $\beta, \gamma \geq 1, m \geq 0$ and let there are two positive integrable functions $\mathcal{P}$ and $\mathcal{Q}$ defined on $[0, \infty)_{\mathcal{T}}$ such that $(3.1)$ holds for all $\lambda>m$. Then

$$
\begin{equation*}
\frac{I_{m}^{\gamma}\left(\psi_{1} \psi_{2}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{P}^{2}\right)(\lambda)+I_{m}^{\gamma}\left(Q^{2}\right)(\lambda) I_{m}^{\beta}\left(\phi_{1} \phi_{2}\right)(\lambda)}{\left(I_{m}^{\gamma}\left(\psi_{1} Q\right)(\lambda) I_{m}^{\beta}\left(\phi_{1} \mathcal{P}\right)(\lambda)+I_{m}^{\gamma}\left(\psi_{2} Q\right)(\lambda) I_{m}^{\beta}\left(\phi_{2} \mathcal{P}\right)(\lambda)\right)^{2}} \leq \frac{1}{4} \tag{3.7}
\end{equation*}
$$

Proof. From (3.1), we observe that

$$
\frac{\phi_{2}\left(t_{1}\right)}{\psi_{1}\left(t_{2}\right)}-\frac{\mathcal{P}\left(t_{1}\right)}{Q\left(t_{2}\right)} \geq 0
$$

and

$$
\frac{\mathcal{P}\left(t_{1}\right)}{Q\left(t_{2}\right)}-\frac{\phi_{1}\left(t_{1}\right)}{\psi_{2}\left(t_{2}\right)} \geq 0
$$

which imply that

$$
\begin{equation*}
\left(\frac{\phi_{1}\left(t_{1}\right)}{\psi_{2}\left(t_{2}\right)}+\frac{\phi_{2}\left(t_{1}\right)}{\psi_{1}\left(t_{2}\right)}\right) \frac{\mathcal{P}\left(t_{1}\right)}{\mathcal{Q}\left(t_{2}\right)} \geq \frac{\mathcal{P}^{2}\left(t_{1}\right)}{Q^{2}\left(t_{2}\right)}+\frac{\phi_{1}\left(t_{1}\right) \phi_{2}\left(t_{1}\right)}{\psi_{1}\left(t_{2}\right) \psi_{2}\left(t_{2}\right)} . \tag{3.8}
\end{equation*}
$$

Multiplying both sides of inequality (3.8) by $\phi_{1}\left(t_{2}\right) \phi_{2}\left(t_{2}\right) Q^{2}\left(t_{2}\right)$, we have

$$
\begin{equation*}
\phi_{1}\left(t_{1}\right) \mathcal{P}\left(t_{1}\right) \psi_{1}\left(t_{2}\right) Q\left(t_{2}\right)+\phi_{2}\left(t_{1}\right) \mathcal{P}\left(t_{1}\right) \psi_{2}\left(t_{2}\right) Q\left(t_{2}\right) \geq \psi_{1}\left(t_{2}\right) \psi_{2}\left(t_{2}\right) \mathcal{P}^{2}\left(t_{1}\right)+\phi_{1}\left(t_{1}\right) \phi_{2}\left(t_{2}\right) Q^{2}\left(t_{2}\right) . \tag{3.9}
\end{equation*}
$$

For $t_{1} \in(m, \lambda)$ and multiplying both sides of (3.9) by $\hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right)$, and integrating integrating the preceding inequality with respect to $t_{1}$ over $(m, \lambda)$, we get

$$
\begin{aligned}
& \psi_{1}\left(t_{2}\right) Q\left(t_{2}\right) \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \phi_{1}\left(t_{1}\right) \mathcal{P}\left(t_{1}\right) \Delta t_{1}+\psi_{2}\left(t_{2}\right) Q\left(t_{2}\right) \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \phi_{2}\left(t_{1}\right) \mathcal{P}\left(t_{1}\right) \Delta t_{1} \\
& \geq \psi_{1}\left(t_{2}\right) \psi_{2}\left(t_{2}\right) \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \mathcal{P}^{2}\left(t_{1}\right) \Delta t_{1}+Q^{2}\left(t_{2}\right) \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \phi_{1}\left(t_{1}\right) \phi_{2}\left(t_{2}\right) \Delta t_{1},
\end{aligned}
$$

which leads to

$$
\begin{align*}
& \psi_{1}\left(t_{2}\right) Q\left(t_{2}\right) I_{m}^{\beta}\left(\phi_{1} \mathcal{P}\right)(\lambda)+\psi_{2}\left(t_{2}\right) Q\left(t_{2}\right) I_{m}^{\beta}\left(\phi_{2} \mathcal{P}\right)(\lambda) \\
& \geq \psi_{1}\left(t_{2}\right) \psi_{2}\left(t_{2}\right) I_{m}^{\beta}\left(\mathcal{P}^{2}\right)(\lambda)+Q^{2}\left(t_{2}\right) I_{m}^{\beta}\left(\phi_{1} \phi_{2}\right)(\lambda) . \tag{3.10}
\end{align*}
$$

For $t_{2} \in(m, \lambda)$ and multiplying both sides of (3.10) by $\hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right)$, and integrating integrating the preceeding inequality with respect to $t_{2}$ over $(m, \lambda)$, one has

$$
I_{m}^{\gamma}\left(\psi_{1} Q\right)(\lambda) I_{m}^{\beta}\left(\phi_{1} \mathcal{P}\right)(\lambda)+I_{m}^{\gamma}\left(\psi_{2} Q\right)(\lambda) I_{m}^{\beta}\left(\phi_{2} \mathcal{P}\right)(\lambda)
$$

$$
\geq I_{m}^{\gamma}\left(\psi_{1} \psi_{2}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{P}^{2}\right)(\lambda)+\mathcal{I}_{m}^{\gamma}\left(Q^{2}\right)(\lambda) I_{m}^{\beta}\left(\phi_{1} \phi_{2}\right)(\lambda)
$$

Making use of the arithmetic-geometric mean inequality, we obtain

$$
\begin{aligned}
& I_{m}^{\gamma}\left(\psi_{1} Q\right)(\lambda) I_{m}^{\beta}\left(\phi_{1} \mathcal{P}\right)(\lambda)+I_{m}^{\gamma}\left(\psi_{2} Q\right)(\lambda) I_{m}^{\beta}\left(\phi_{2} \mathcal{P}\right)(\lambda) \\
\geq & 2 \sqrt{I_{m}^{\gamma}\left(\psi_{1} \psi_{2}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{P}^{2}\right)(\lambda)+I_{m}^{\gamma}\left(Q^{2}\right)(\lambda) I_{m}^{\beta}\left(\phi_{1} \phi_{2}\right)(\lambda)}
\end{aligned}
$$

which leads to the desired inequality (3.7).
Theorem 3.3. For $\beta, \gamma \geq 1, m \geq 0$ and let there are two positive integrable functions $\mathcal{P}$ and $Q$ defined on $[0, \infty)_{\mathcal{T}}$ such that $(3.1)$ holds for all $\lambda>m$. Then

$$
\begin{equation*}
\mathcal{I}_{m}^{\gamma}\left(\frac{\psi_{2} \mathcal{P} Q}{\phi_{1}}\right)(\lambda) \mathcal{I}_{m}^{\beta}\left(\frac{\phi_{2} \mathcal{P} Q}{\psi_{1}}\right)(\lambda) \geq \mathcal{I}_{m}^{\beta} \mathcal{P}^{2}(\lambda) I_{m}^{\gamma} Q^{2}(\lambda) . \tag{3.11}
\end{equation*}
$$

Proof. From (3.1), we observe that

$$
\int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \frac{\phi_{2}\left(t_{1}\right)}{\psi_{1}\left(t_{1}\right)} \mathcal{P}\left(t_{1}\right) Q\left(t_{1}\right) \Delta t_{1} \geq \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \mathcal{P}^{2}\left(t_{1}\right) \Delta t_{1},
$$

which implies

$$
\begin{equation*}
\mathcal{I}_{m}^{\beta}\left(\frac{\phi_{2} \mathcal{P} Q}{\psi_{1}}\right)(\lambda) \geq I_{m}^{\beta} \mathcal{P}^{2}(\lambda) \tag{3.12}
\end{equation*}
$$

Analogously, we obtain

$$
\int_{m}^{\lambda} \hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \frac{\psi_{2}\left(t_{2}\right)}{\phi_{1}\left(t_{2}\right)} \mathcal{P}\left(t_{2}\right) Q\left(t_{2}\right) \Delta t_{2} \geq \int_{m}^{\lambda} \hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right) Q^{2}\left(t_{2}\right) \Delta t_{2},
$$

from which one has

$$
\begin{equation*}
I_{m}^{\gamma}\left(\frac{\psi_{2} \mathcal{P} Q}{\phi_{1}}\right)(\lambda) \geq \mathcal{I}_{m}^{\gamma} Q^{2}(\lambda) . \tag{3.13}
\end{equation*}
$$

Multiplying (3.12) and (3.13) side by side, we get the desired inequality (3.11).

## 4. Čebyšev type inequalities

In this section, we will present some fractional integral inequalities on time scales.
Theorem 4.1. For $\beta, \gamma \geq 1, m \geq 0$ and and let there are two synchronous functions $\mathcal{P}$ and $\mathcal{Q}$ defined on $[0, \infty)_{\mathcal{T}}$ and $\mathcal{S} \geq 0$. Then for all $\lambda>m$, we have

$$
\begin{gather*}
\mathcal{I}_{m}^{\gamma}(\mathcal{P Q S})(\lambda)\left(\hbar_{\beta}(\lambda, m)\right)+I_{m}^{\beta}(\mathcal{P Q S})(\lambda)\left(\hbar_{\gamma}(\lambda, m)\right) \\
\geq I_{m}^{\gamma}(\mathcal{P S})(\lambda) I_{m}^{\beta}(Q)(\lambda)+I_{m}^{\beta}(\mathcal{P})(\lambda) I_{m}^{\gamma}(Q S)(\lambda)+I_{m}^{\gamma}(\mathcal{P})(\lambda) I_{m}^{\beta}(Q S)(\lambda) \\
+I_{m}^{\beta}(\mathcal{P S})(\lambda) I_{m}^{\gamma}(Q)(\lambda)-I_{m}^{\gamma}(\mathcal{P Q})(\lambda) I_{m}^{\beta}(\mathcal{S})(\lambda)-I_{m}^{\beta}(\mathcal{P Q})(\lambda) I_{m}^{\gamma}(\mathcal{S})(\lambda) \tag{4.1}
\end{gather*}
$$

Proof. Under the assumption of Theorem 4.1, we have

$$
\left(\mathcal{P}\left(t_{2}\right)-\mathcal{P}\left(t_{1}\right)\right)\left(Q\left(t_{2}\right)-Q\left(t_{1}\right)\right)\left(\mathcal{S}\left(t_{2}\right)+\mathcal{S}\left(t_{1}\right)\right) \geq 0 .
$$

It follows that

$$
\begin{gather*}
\mathcal{P}\left(t_{2}\right) \mathcal{Q}\left(t_{2}\right) \mathcal{S}\left(t_{2}\right)+\mathcal{P}\left(t_{1}\right) Q\left(t_{1}\right) \mathcal{S}\left(t_{1}\right) \\
\geq \mathcal{P}\left(t_{2}\right) Q\left(t_{1}\right) \mathcal{S}\left(t_{2}\right)+\mathcal{P}\left(t_{1}\right) Q\left(t_{2}\right) \mathcal{S}\left(t_{2}\right)+\mathcal{P}\left(t_{2}\right) Q\left(t_{1}\right) \mathcal{S}\left(t_{1}\right) \\
+\mathcal{P}\left(t_{1}\right) Q\left(t_{2}\right) \mathcal{S}\left(t_{1}\right)-\mathcal{P}\left(t_{2}\right) Q\left(t_{2}\right) \mathcal{S}\left(t_{1}\right)-\mathcal{P}\left(t_{1}\right) Q\left(t_{1}\right) \mathcal{S}\left(t_{2}\right) . \tag{4.2}
\end{gather*}
$$

For $t_{1} \in(m, \lambda)$ and multiplying both sides of (4.2) by $\hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right)$, and integrating integrating the preceding inequality with respect to $t_{1}$ over $(m, \lambda)$, we get

$$
\begin{gather*}
(\mathcal{P Q S})\left(t_{2}\right)\left(\hbar_{\beta}(\lambda, m)\right)+I_{m}^{\beta}(\mathcal{P Q S})(\lambda) \\
\geq \mathcal{P}\left(t_{2}\right) \mathcal{S}\left(t_{2}\right) I_{m}^{\beta}(Q)(\lambda)+I_{m}^{\beta}(\mathcal{P})(\lambda) Q\left(t_{2}\right) \mathcal{S}\left(t_{2}\right)+\mathcal{P}\left(t_{2}\right) I_{m}^{\beta}(Q \mathcal{S})(\lambda) \\
+I_{m}^{\beta}(\mathcal{P S})(\lambda) Q\left(t_{2}\right)-\mathcal{P}\left(t_{2}\right) Q\left(t_{2}\right) I_{m}^{\beta}(\mathcal{S})(\lambda)-I_{m}^{\beta}(\mathcal{P} Q)(\lambda) \mathcal{S}\left(t_{2}\right) \tag{4.3}
\end{gather*}
$$

Again, for $t_{2} \in(m, \lambda)$ and multiplying both sides of (4.2) by $\hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right)$, and integrating integrating the preceding inequality with respect to $t_{2} \operatorname{over}(m, \lambda)$, we get

$$
\begin{gather*}
I_{m}^{\gamma}(\mathcal{P Q S})(\lambda)\left(\hbar_{\beta}(\lambda, m)\right)+I_{m}^{\beta}(\mathcal{P Q S})(\lambda)\left(\hbar_{\gamma}(\lambda, m)\right) \\
\geq I_{m}^{\gamma}(\mathcal{P S})(\lambda) I_{m}^{\beta}(Q)(\lambda)+I_{m}^{\beta}(\mathcal{P})(\lambda) I_{m}^{\gamma}(Q S)(\lambda)+I_{m}^{\gamma}(\mathcal{P})(\lambda) I_{m}^{\beta}(Q \mathcal{S})(\lambda) \\
+I_{m}^{\beta}(\mathcal{P S})(\lambda) I_{m}^{\gamma}(Q)(\lambda)-I_{m}^{\gamma}(\mathcal{P} Q)(\lambda) I_{m}^{\beta}(\mathcal{S})(\lambda)-I_{m}^{\beta}(\mathcal{P Q})(\lambda) I_{m}^{\gamma}(\mathcal{S})(\lambda), \tag{4.4}
\end{gather*}
$$

the required result.
Theorem 4.2. For $\beta, \gamma \geq 1, m \geq 0$ and and let there are three monotonic functions $\mathcal{P}, \mathcal{Q}$ and $\mathcal{S}$ defined on $[0, \infty)_{\mathcal{T}}$ satisfying the following

$$
\left(\mathcal{P}\left(t_{2}\right)-\mathcal{P}\left(t_{1}\right)\right)\left(Q\left(t_{2}\right)-Q\left(t_{1}\right)\right)\left(\mathcal{S}\left(t_{2}\right)-\mathcal{S}\left(t_{1}\right)\right) \geq 0 .
$$

Then for all $\lambda>m$, we have

$$
\begin{gather*}
\mathcal{I}_{m}^{\gamma}(\mathcal{P Q S})(\lambda)\left(\hbar_{\beta}(\lambda, m)\right)-I_{m}^{\beta}(\mathcal{P Q S})(\lambda)\left(\hbar_{\gamma}(\lambda, m)\right) \\
\geq I_{m}^{\gamma}(\mathcal{P S})(\lambda) I_{m}^{\beta}(Q)(\lambda)+I_{m}^{\beta}(\mathcal{P})(\lambda) I_{m}^{\gamma}(Q S)(\lambda)-I_{m}^{\gamma}(\mathcal{P})(\lambda) I_{m}^{\beta}(Q S)(\lambda) \\
-I_{m}^{\beta}(\mathcal{P S})(\lambda) I_{m}^{\gamma}(Q)(\lambda)+I_{m}^{\gamma}(\mathcal{P Q})(\lambda) I_{m}^{\beta}(\mathcal{S})(\lambda)-I_{m}^{\beta}(\mathcal{P Q})(\lambda) I_{m}^{\gamma}(\mathcal{S})(\lambda) . \tag{4.5}
\end{gather*}
$$

Proof. The proof is similar to previous theorem.
Theorem 4.3. For $\beta, \gamma \geq 1, m \geq 0$ and let there are two integrable functions defined on $[0, \infty)_{\mathcal{T}}$. Then for all $\lambda>m$, we have

$$
\mathcal{I}_{m}^{\gamma}\left(\mathcal{P}^{2}\right)(\gamma)\left(\hbar_{\beta}(\lambda, m)\right)+\mathcal{I}_{m}^{\beta}\left(Q^{2}\right)(\lambda)\left(\hbar_{\gamma}(\lambda, m)\right) \geq \mathcal{I}_{m}^{\gamma}(\mathcal{P})(\lambda) I_{m}^{\beta}(Q)(\lambda) .
$$

Proof. Using the elementary result, we have

$$
\left(\mathcal{P}\left(t_{2}\right)-Q\left(t_{1}\right)\right)^{2} \geq 0
$$

Thus

$$
\mathcal{P}^{2}\left(t_{2}\right)+Q^{2}\left(t_{1}\right) \geq 2 \mathcal{P}\left(t_{2}\right) Q\left(t_{1}\right) .
$$

Multiplying both sides of above inequality by $\hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right)$, and integrating integrating the preceding inequality with respect to $t_{1}$ and $t_{2} \operatorname{over}(m, \lambda) \times(m, \lambda)$, we get

$$
I_{m}^{\gamma}\left(\mathcal{P}^{2}\right)(\gamma)\left(\hbar_{\beta}(\lambda, m)\right)+I_{m}^{\beta}\left(Q^{2}\right)(\lambda)\left(\hbar_{\gamma}(\lambda, m)\right) \geq 2 I_{m}^{\gamma}(\mathcal{P})(\lambda) I_{m}^{\beta}(Q)(\lambda),
$$

the required result.
Theorem 4.4. For $\beta, \gamma \geq 1, m \geq 0$ and let there are two integrable functions defined on $[0, \infty)_{\mathcal{T}}$. Then for all $\lambda>m$, we have

$$
I_{m}^{\gamma}\left(\mathcal{P}^{2}\right)(\lambda) I_{m}^{\beta}\left(Q^{2}\right)(\lambda)+I_{m}^{\gamma}\left(Q^{2}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{P}^{2}\right)(\lambda) \geq 2 I_{m}^{\gamma}(\mathcal{P} Q)(\lambda) I_{m}^{\beta}(\mathcal{P} Q)(\lambda) .
$$

Proof. Using the elementary result, we have

$$
\left(\mathcal{P}\left(t_{2}\right) Q\left(t_{1}\right)-\mathcal{P}\left(t_{1}\right) Q\left(t_{2}\right)\right)^{2} \geq 0
$$

Thus

$$
\mathcal{P}^{2}\left(t_{2}\right) Q^{2}\left(t_{1}\right)+\mathcal{P}^{2}\left(t_{1}\right) Q^{2}\left(t_{2}\right) \geq 2 \mathcal{P}\left(t_{2}\right) Q\left(t_{1}\right) \mathcal{P}\left(t_{1}\right) Q\left(t_{2}\right) .
$$

Multiplying both sides of above inequality by $\hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right)$, and integrating integrating the preceding inequality with respect to $t_{1}$ and $t_{2}$ over $(m, \lambda) \times(m, \lambda)$, we get

$$
I_{m}^{\gamma}\left(\mathcal{P}^{2}\right)(\lambda) I_{m}^{\beta}\left(Q^{2}\right)(\lambda)+I_{m}^{\gamma}\left(Q^{2}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{P}^{2}\right)(\lambda) \geq 2 I_{m}^{\gamma}(\mathcal{P} Q)(\lambda) I_{m}^{\beta}(\mathcal{P} Q)(\lambda),
$$

the required result.
Theorem 4.5. For $\beta \geq 1, m \geq 0$, and let two there are two delta-differentiable functions $\mathcal{P}$ and $Q$ defined on $[0, \infty)_{\mathcal{T}}$. Also, assume that there is a positive integrable function $\mathcal{S}_{1}$ defined on $[0, \infty)_{\mathcal{T}}$ such that $\mathcal{P}^{\Delta} \in L_{s}\left([0, \infty)_{\mathcal{T}}\right), Q^{\Delta} \in L_{r}\left([0, \infty)_{\mathcal{T}}\right)$ for $s, r, u>1$ having $s^{-1}+s_{1}{ }^{-1}=1, r^{-1}+r_{1}{ }^{-1}=1$, and $u^{-1}+u_{1}^{-1}=1$. Then the following variants hold for all $\lambda>m$

$$
\begin{gather*}
2\left|\left(\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) \mathcal{I}_{m}^{\beta}(\mathcal{P} Q)(\lambda)-\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} Q\right)(\lambda)\right)\right| \\
\leq\left(\left\|\mathcal{P}^{\Delta}\right\|_{s}^{u} \int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right)\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}} \Delta t_{1} \Delta t_{2}\right)^{\frac{1}{u}} \\
\left(\left\|Q^{\Delta}\right\|_{r}^{u_{1}} \int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right)\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}} \Delta t_{1} \Delta t_{2}\right)^{\frac{1}{u_{1}}} \\
\leq\left\|\mathcal{P}^{\Delta}\right\|_{s}^{u}\left\|Q^{\Delta}\right\|_{r}^{u_{1}}\left(\int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right)\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}} \Delta t_{1} \Delta t_{2}\right) . \tag{4.6}
\end{gather*}
$$

Proof. Let us suppose the function

$$
\begin{equation*}
\mathcal{G}\left(t_{1}, t_{2}\right)=\left(Q\left(t_{1}\right)-Q\left(t_{2}\right)\right)\left(\mathcal{P}\left(t_{1}\right)-\mathcal{P}\left(t_{2}\right)\right) ; \quad t_{1}, t_{2} \in(m, \lambda), \tag{4.7}
\end{equation*}
$$

which can be written as

$$
\begin{equation*}
\mathcal{G}\left(t_{1}, t_{2}\right)=\mathcal{P}\left(t_{1}\right) Q\left(t_{1}\right)-\mathcal{P}\left(t_{1}\right) Q\left(t_{2}\right)-\mathcal{P}\left(t_{2}\right) Q\left(t_{1}\right)-Q\left(t_{2}\right) \mathcal{P}\left(t_{2}\right) . \tag{4.8}
\end{equation*}
$$

Multiplying both sides of (4.8) by $\hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \mathcal{S}_{1}\left(t_{1}\right)$ and then integrating with respect to $t_{1}$ over ( $m, \lambda$ ), we have

$$
\begin{gather*}
\int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{G}\left(t_{1}, t_{2}\right) \Delta t_{1} \\
=\int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{P}\left(t_{1}\right) Q\left(t_{1}\right) \Delta t_{1}-\int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{P}\left(t_{1}\right) Q\left(t_{2}\right) \Delta t_{1} \\
-\int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{P}\left(t_{2}\right) \mathcal{Q}\left(t_{1}\right) \Delta t_{1}-Q\left(t_{2}\right) \mathcal{P}\left(t_{2}\right) \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \Delta t_{1}, \tag{4.9}
\end{gather*}
$$

arrives at

$$
\begin{gather*}
\int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{G}\left(t_{1}, t_{2}\right) \Delta t_{1} \\
=\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P} Q\right)(\lambda)-Q\left(t_{2}\right) \mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda)-\mathcal{P}\left(t_{2}\right) \mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} Q\right)(\lambda)+\mathcal{P}\left(t_{2}\right) Q\left(t_{2}\right) \mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) . \tag{4.10}
\end{gather*}
$$

Further, multiplying both sides of (4.10) $\hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{2}\right)$ and then integrating with respect to $t_{2}$ over ( $m, \lambda$ ), we have

$$
\begin{align*}
& \int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right) \mathcal{G}\left(t_{1}, t_{2}\right) \Delta t_{1} \Delta t_{2} \\
& \quad=2\left(\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) \mathcal{I}_{m}^{\beta}(\mathcal{P} Q)(\lambda)-\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda) \mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} Q\right)(\lambda)\right) . \tag{4.11}
\end{align*}
$$

On contrast, we have

$$
\begin{equation*}
\mathcal{G}\left(t_{1}, t_{2}\right)=\int_{t_{1}}^{t_{2}} \int_{t_{1}}^{t_{2}} \mathcal{P}^{\Delta}(\theta) Q^{\Delta}(\vartheta) \Delta \theta \Delta \vartheta . \tag{4.12}
\end{equation*}
$$

Taking into account the Hölder's inequality, we have

$$
\begin{equation*}
\left|\mathcal{P}\left(t_{1}\right)-\mathcal{P}\left(t_{2}\right)\right| \leq\left.\left.\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}}\left|\int_{t_{1}}^{t_{2}}\right| \mathcal{P}^{\Delta}(\theta)\right|^{s} \Delta \theta\right|^{\frac{1}{s}} \tag{4.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|Q\left(t_{1}\right)-Q\left(t_{2}\right)\right| \leq\left.\left.\left|t_{1}-t_{2}\right|^{\frac{1}{r_{1}}}\left|\int_{t_{1}}^{t_{2}}\right| Q^{\Delta}(\vartheta)\right|^{r} \Delta \vartheta\right|^{\frac{1}{r}} . \tag{4.14}
\end{equation*}
$$

Conducting product between (4.13) and (4.14), we get

$$
\begin{align*}
& \left|\mathcal{G}\left(t_{1}, t_{2}\right)\right| \leq\left|\left(\mathcal{P}\left(t_{1}\right)-\mathcal{P}\left(t_{2}\right)\right)\left(Q\left(t_{1}\right)-Q\left(t_{2}\right)\right)\right| \\
\leq & \left.\left.\left.\left.\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}}\left|\int_{t_{1}}^{t_{2}}\right| \mathcal{P}^{\Delta}(\theta)\right|^{s} \Delta \theta\right|^{\frac{1}{s}}\left|\int_{t_{1}}^{t_{2}}\right| Q^{\Delta}(\vartheta)\right|^{r} \Delta \vartheta\right|^{\frac{1}{r}} . \tag{4.15}
\end{align*}
$$

Thus, from (4.11) and (4.15), we have

$$
\begin{gather*}
2\left|\left(\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) \mathcal{I}_{m}^{\beta}(\mathcal{P} Q)(\lambda)-\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda) \mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{Q}\right)(\lambda)\right)\right| \\
=\int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right)\left|\mathcal{G}\left(t_{1}, t_{2}\right)\right| \Delta t_{1} \Delta t_{2} \\
\leq \int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right) \\
\quad \times\left.\left.\left.\left.\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}}\left|\int_{t_{1}}^{t_{2}}\right| \mathcal{P}^{\Delta}(\theta)\right|^{s} \Delta \theta\right|^{\frac{1}{s}}\left|\int_{t_{1}}^{t_{2}}\right| Q^{\Delta}(\vartheta)\right|^{r} \Delta \vartheta\right|^{\frac{1}{r}} \Delta t_{1} \Delta t_{2} . \tag{4.16}
\end{gather*}
$$

Further, taking into consideration the Hölder's inequality for bivariate integral, we have

$$
\begin{gather*}
2\left|\left(\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) \mathcal{I}_{m}^{\beta}(\mathcal{P} Q)(\lambda)-\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} Q\right)(\lambda)\right)\right| \\
\leq\left(\left.\int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right)\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}} \int_{t_{1}}^{t_{2}}\left|\mathcal{P}^{\Delta}(\theta)\right|^{s} \Delta \theta\right|^{\frac{u}{s}} \Delta t_{1} \Delta t_{2}\right)^{\frac{1}{u}} \\
\times\left(\left.\left.\int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right)\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}}\left|\int_{t_{1}}^{t_{2}}\right| Q^{\Delta}(\vartheta)\right|^{r} \Delta \vartheta\right|^{\frac{u_{1}}{r}} \Delta t_{1} \Delta t_{2}\right)^{\frac{1}{u_{1}}} . \tag{4.17}
\end{gather*}
$$

Now, using the following properties

$$
\begin{equation*}
\left.\left.\left|\int_{t_{1}}^{t_{2}}\right| \mathcal{P}^{\Delta}(\theta)\right|^{s} \Delta \theta\right|^{\frac{1}{s}} \leq\left\|\mathcal{P}^{\Delta}\right\|_{s} \quad \text { and }\left.\left.\quad\left|\int_{t_{1}}^{t_{2}}\right| Q^{\Delta}(\vartheta)\right|^{r} \Delta \vartheta\right|^{\frac{1}{r}} \leq\left\|Q^{\Delta}\right\|_{r} . \tag{4.18}
\end{equation*}
$$

From (4.17), we have

$$
\begin{gathered}
2\left|\left(\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) \mathcal{I}_{m}^{\beta}(\mathcal{P} Q)(\lambda)-\mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{Q}\right)(\lambda)\right)\right| \\
\leq\left(\left\|\mathcal{P}^{\Delta}\right\|_{s}^{u} \int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right)\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}} \Delta t_{1} \Delta t_{2}\right)^{\frac{1}{u}}
\end{gathered}
$$

$$
\begin{equation*}
\times\left(\left\|Q^{\Lambda}\right\|_{r}^{u_{1}} \int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right)\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}} \Delta t_{1} \Delta t_{2}\right)^{\frac{1}{u_{1}}} \tag{4.19}
\end{equation*}
$$

Therefore, we conclude that

$$
\begin{gather*}
2\left|\left(I_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) I_{m}^{\beta}(\mathcal{P} Q)(\lambda)-I_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} Q\right)(\lambda)\right)\right| \\
\leq\left\|\mathcal{P}^{\Delta}\right\|_{s}^{u}\left\|Q^{\Delta}\right\|_{r}^{u_{1}}\left(\int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{1}\left(t_{2}\right)\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}} \Delta t_{1} \Delta t_{2}\right), \tag{4.20}
\end{gather*}
$$

the required result.
Theorem 4.6. For $\beta, \gamma \geq 1, m \geq 0$ and let there are two delta-differentiable functions $\mathcal{P}$ and $Q$ defined on $[0, \infty)_{\mathcal{T}}$. Also, assume that there are two positive integrable function $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ defined on $[0, \infty)_{\mathcal{T}}$ such that $\mathcal{P}^{\Delta} \in L_{s}\left([0, \infty)_{\mathcal{T}}\right), Q^{\Delta} \in L_{r}\left([0, \infty)_{\mathcal{T}}\right)$ for $s, r, u>1$ having $s^{-1}+s_{1}{ }^{-1}=1, r^{-1}+r_{1}{ }^{-1}=1$, and $u^{-1}+u_{1}^{-1}=1$. Then the following variant holds for all $\lambda>m$

$$
\begin{gather*}
\mid \mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{2}\right)(\lambda) \mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P} Q\right)(\lambda)-\mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{1} Q\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda) \\
-\mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{2} \mathcal{P}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} Q\right)(\lambda)+\mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{2} \mathcal{P} Q\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) \mid \\
\leq\left.\left\|\mathcal{P}^{\Delta}\right\|_{s}\left\|Q^{\Delta}\right\|_{r} \int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right)\right|_{1}-\left.t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}} \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{2}\left(t_{2}\right) \Delta t_{1} \Delta t_{2} \tag{4.21}
\end{gather*}
$$

Proof. Multiplying both sides of (4.10) by $\hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{2}\left(t_{2}\right)$ and integrating with respect to $t_{2}$ over ( $m, \lambda$ ), we have

$$
\begin{align*}
& \int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{2}\left(t_{2}\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{G}\left(t_{1}, t_{2}\right) \Delta t_{1} \Delta t_{2} \\
& \quad=\mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{2}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P} Q\right)(\lambda)-\mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{1} Q\right)(\lambda) \mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda) \\
& \quad-\mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{2} \mathcal{P}\right)(\lambda) \mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1} Q\right)(\lambda)+\mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{2} \mathcal{P} Q\right)(\lambda) \mathcal{I}_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) \tag{4.22}
\end{align*}
$$

Taking modulus on both sides of (4.22), one obtains

$$
\begin{gathered}
\mid \mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{2}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P} Q\right)(\lambda)-I_{m}^{\gamma}\left(\mathcal{S}_{1} Q\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} \mathcal{P}\right)(\lambda) \\
-\mathcal{I}_{m}^{\gamma}\left(\mathcal{S}_{2} \mathcal{P}\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1} Q\right)(\lambda)+I_{m}^{\gamma}\left(\mathcal{S}_{2} \mathcal{P} Q\right)(\lambda) I_{m}^{\beta}\left(\mathcal{S}_{1}\right)(\lambda) \mid \\
=\int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{2}\left(t_{2}\right)\left|\mathcal{G}\left(t_{1}, t_{2}\right)\right| \Delta t_{1} \Delta t_{2} \\
\leq \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right)
\end{gathered}
$$

$$
\begin{gather*}
\times\left.\left.\left.\left.\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}}\left|\int_{t_{1}}^{t_{2}}\right| \mathcal{P}^{\Delta}(\theta)\right|^{s} d \theta\right|^{\frac{1}{s}}\left|\int_{t_{1}}^{t_{2}}\right| Q^{\Delta}(\vartheta)\right|^{r} d \vartheta\right|^{\frac{1}{r}} \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{2}\left(t_{2}\right) \Delta t_{1} \Delta t_{2} \\
=\left\|\mathcal{P}^{\Delta}\right\|_{s}\left\|Q^{\Delta}\right\|_{r} \int_{m}^{\lambda} \int_{m}^{\lambda} \hbar_{\beta-1}\left(\lambda, \sigma\left(t_{1}\right)\right) \hbar_{\gamma-1}\left(\lambda, \sigma\left(t_{2}\right)\right) \\
\times\left|t_{1}-t_{2}\right|^{\frac{1}{s_{1}}+\frac{1}{r_{1}}} \mathcal{S}_{1}\left(t_{1}\right) \mathcal{S}_{2}\left(t_{2}\right) \Delta t_{1} \Delta \tag{4.23}
\end{gather*}
$$

## 5. Conclusions

In this work, we have fruitfully applied the $R L$-fractional integral operator on a time scale to derive the Pólya-Szegö and Čebyšev type integral inequalities. Our fractional integral inequalities depends on the graininess function of the time scale. We trust that this possibility can be very useful in applications of signal processing, providing a concept of coarse-graining in time that can be used to model white noise that occurs in signal processing or to obtain generalized entropies and new practical meanings in signal processing. We conclude that the results derived in this paper are general in character and give some contributions to statistical theory, optimization and helpful for finding the existence and uniqueness of the integrodifferential equations.

## Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which led to considerable improvement of the article.

The work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11971142, 11701176).

## Conflict of interest

The authors declare no conflict of interest.

## References

1. Y. Khurshid, M. Adil Khan, Y. M. Chu, Conformable fractional integral inequalities for $G G$ - and GA-convex function, AIMS Math., 5 (2020), 5012-5030.
2. S. Rashid, M. A. Noor, K. I. Noor, et al. Ostrowski type inequalities in the sense of generalized $\mathcal{K}$-fractional integral operator for exponentially convex functions, AIMS Math., 5 (2020), 26292645.
3. S. Rashid, İ. İşcan, D. Baleanu, et al. Generation of new fractional inequalities via n polynomials s-type convexixity with applications, Adv. Differ. Equ., 2020 (2020), 1-20.
4. A. Iqbal, M. Adil Khan, S. Ullah, et al. Some new Hermite-Hadamard-type inequalities associated with conformable fractional integrals and their applications, J. Funct. Space., 2020 (2020), 1-18.
5. S. Rashid, F. Jarad, Y. M. Chu, A note on reverse Minkowski inequality via generalized proportional fractional integral operator with respect to another function, Math. Probl. Eng., 2020 (2020), 1-12.
6. S. Rashid, F. Jarad, H. Kalsoom, et al. On Pólya-Szegö and Ćebyšev type inequalities via generalized $k$-fractional integrals, Adv. Differ. Equ., 2020 (2020), 1-18.
7. M. U. Awan, S. Talib, Y. M. Chu, et al. Some new refinements of Hermite-Hadamard-type inequalities involving $\Psi_{k}$-Riemann-Liouville fractional integrals and applications, Math. Probl. Eng., 2020 (2020), 1-10.
8. S. S. Zhou, S. Rashid, F. Jarad, et al. New estimates considering the generalized proportional Hadamard fractional integral operators, Adv. Differ. Equ., 2020 (2020), 1-15.
9. S. Rafeeq, H. Kalsoom, S. Hussain, et al. Delay dynamic double integral inequalities on time scales with applications, Adv. Differ. Equ., 2020 (2020), 1-32.
10. G. C. Wu, D. Baleanu, Discrete fractional logistic map and its chaos, Nonlinear Dynam., 75 (2014), 283-287.
11. F. M. Atici, P. W. Eloe, Discerete fractional calculus with the nabla operator, Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I, 3 (2009), 1-12.
12. T. Abdeljawad, On delta and nabla Caputo fractional differences and dual identities, Discrete Dyn. Nat. Soc., 2013 (2013), 1-12.
13. T. Abdeljawad, F. M. Atici, On the definitions of nabla fractional operators, Abstr. Appl. Anal., 2012 (2012), 1-13.
14. T. Abdeljawad, Fractional difference operators with discrete generalized Mittag-Leffler kernels, Chaos Solitons Fractals, 126 (2019), 315-324.
15. T. Abdeljawad, Different type kernel h-fractional differences and their fractional h-sums, Chaos Solitons Fractals, 116 (2018), 146-156.
16. T. Abdeljawad, S. Banerjee, G. C. Wu, Discrete tempered fractional calculus for new chaotic systems with short memory and image encryption, Optik, 218 (2020), Article 163698, Available from: https://doi.org/10.1016/j.ijleo.2019.163698.
17. S. Hilger, Ein Mabkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. thesis, Universität Würzburg, 1988.
18. N. R. O. Bastos, Fractional Calculus on Time Scales, Ph.D. thesis, Instituto Politecnico de Viseu (Portugal), 2012.
19. S. S. Haider, M. ur Rehman, On substantial fractional difference operator, Adv. Differ. Equ., 2020 (2020), 1-18.
20. B. T. Holm-Hansen, R. X. Gao, Time-scale analysis adapted for bearing diagnostics, Proc. SPIE 3833, Intelligent Systems in Design and Manufacturing II, (20 August 1999), Available from: https://doi.org/10.1117/12.359515.
21. G. C. Wu, Z. G. Deng, D. Baleanu, et. al. New variable-order fractional chaotic systems for fast image encryption, Chaos, 29 (2019), 1-11.
22. J. Zhu, L. Wu, Fractional Cauchy problem with Caputo nabla derivative on time scales, Abstr. Appl. Anal., 2015 (2015), 1-23.
23. S. L. Gao, Fractional time scale in calcium ion channels model, Int. J. Biomath., 6 (2013), 1-11.
24. J. J. Mohan, Variation of parameters for nabla fractional difference equations, Novi. Sad J. Math., 44 (2014), 149-159.
25. A. G. Radwan, On some generalized discrete logistic maps, J. Adv. Res., 4 (2013), 163-171.
26. S. Khan, M. Adil Khan, Y. M. Chu, Converses of the Jensen inequality derived from the Green functions with applications in information theory, Math. Method. Appl. Sci., 43 (2020), 25772587.
27. M. Adil Khan, J. Pečarić, Y. M. Chu, Refinements of Jensen's and McShane's inequalities with applications, AIMS Math., 5 (2020), 4931-4945.
28. M. A. Latif, S. Rashid, S. S. Dragomir, et al. Hermite-Hadamard type inequalities for co-ordinated convex and qausi-convex functions and their applications, J. Inequal. Appl., 2019 (2019), 1-33.
29. M. U. Awan, N. Akhtar, S. Iftikhar, et al. New Hermite-Hadamard type inequalities for $n$ polynomial harmonically convex functions, J. Inequal. Appl., 2020 (2020), 1-12.
30. M. K. Wang, Z. Y. He, Y. M. Chu, Sharp power mean inequalities for the generalized elliptic integral of the first kind, Comput. Meth. Funct. Th., 20 (2020), 111-124.
31. S. Rashid, R. Ashraf, M. A. Noor, et al. New weighted generalizations for differentiable exponentially convex mapping with application, AIMS Math., 5 (2020), 3525-3546.
32. M. Adil Khan, M. Hanif, Z. A. Khan, et al. Association of Jensen's inequality for s-convex function with Csiszár divergence, J. Inequal. Appl., 2019 (2019), 1-14.
33. T. H. Zhao, M. K. Wang, Y. M. Chu, A sharp double inequality involving generalized complete elliptic integral of the first kind, AIMS Math., 5 (2020), 4512-4528.
34. Z. H. Yang, W. M. Qian, W. Zhang, et al. Notes on the complete elliptic integral of the first kind, Math. Inequal. Appl., 23 (2020), 77-93.
35. M. K. Wang, H. H. Chu, Y. M. Li, et al. Answers to three conjectures on convexity of three functions involving complete elliptic integrals of the first kind, Appl. Anal. Discrete Math., 14 (2020), 255271.
36. W. M. Qian, W. Zhang, Y. M. Chu, Bounding the convex combination of arithmetic and integral means in terms of one-parameter harmonic and geometric means, Miskolc Math. Notes, 20 (2019), 1157-1166.
37. W. M. Qian, Z. Y. He, Y. M. Chu, Approximation for the complete elliptic integral of the first kind, RACSAM, 114 (2020), 1-12.
38. M. Adil Khan, N. Mohammad, E. R. Nwaeze, et al. Quantum Hermite-Hadamard inequality by means of a Green function, Adv. Differ. Equ., 2020 (2020), 1-20.
39. M. U. Awan, N. Akhtar, A. Kashuri, et. al. $2 D$ approximately reciprocal $\rho$-convex functions and associated integral inequalities, AIMS Math., 5 (2020), 4662-4680.
40. S. Zaheer Ullah, M. Adil Khan, Y. M. Chu, A note on generalized convex functions, J. Inequal. Appl., 2019 (2019), 1-10.
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