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Research article

On the Ulam stability of fuzzy differential equations

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Abstract: Ulam stability problems have received considerable attention in the field of differential equations. However, how to effectively build the fuzzy model for Ulam stability problems is less attractive due to varies of differentiabilities requirements. The paper discusses the Ulam stability of fuzzy differential equations in Banach spaces. After introducing the new definitions of differentiabilities for fuzzy number-valued mappings, we give some important properties about these differentiabilities. On these bases, with different differentiabilities and conditions, we prove the Ulam stability of three kinds of fuzzy differential equations. The obtained conclusions generalize the existing results.

Keywords: fuzzy sets; differential equations; Ulam stability; Banach spaces **Mathematics Subject Classification:** 03E72, 34D99, 34G99

1. Introduction

Stability is one of the most important characteristics of dynamic systems. One of the methods to study stability is introduced by Ulam [1], in which the problem of functional equations concerning the stability of group homomorphisms was first discussed. Ulam method-based stability analysis was then investigated by many researchers such as Hyers [2], Rassias [3], and so on. Then, Obloza [4] explored the Ulam stability of differential equations. Afterwards, Alsina and Ger [5] proved the differential equation y' = y satisfied Hyers-Ulam stability. Miura et al. [6,7] and Takahasi et al. [8] discussed the Ulam stability of the differential equation $y' = \lambda y$ in some abstract spaces.

As is known to all, fuzzy differential equations play an important role in practical applications, such as power systems and artificial intelligence. Kaleva [9] proved the existence and uniqueness of a solution to a fuzzy differential equation x'(t) = f(t, x(t)) provided f satisfied a Lipschitz condition. After that, Kaleva [10] and Kloeden [11] discussed the peano theorem of fuzzy differential equations one after another. In the following year, Tomasiello and Macías-Díaz [12] extended a Picard-like

approach to fuzzy fractional differential equations. Rashid et al. [13] established a relation between the solutions and approximate solutions of complex fuzzy differential equations with complex membership grades. Nowadays, the study of fuzzy differential equations is still active, See [14–22].

It is worth noting that the stability of fuzzy differential equations has got scholars' much attentions in recent years. Diamond [23] studied the Lyapunov stability. Song et al. [24] studied the stability properties of the trivial fuzzy solution of fuzzy differential equations. After that, Yakar et al. [25] considered the Lagrange stability. Other results about the stability of fuzzy differential equations can be found in Refs. [26–28]. Recently, the Ulam stability of fuzzy differential equations is developed. For example, Shen and Wang [29] discussed the Ulam stability of fuzzy differential equations under generalized differentiability by a fixed point approach. Shen [30] explored the Ulam stability of the following three linear fuzzy differential equations in the real number space:

$$u'(t) + \delta(t)u(t) = \sigma(t), \tag{1.1}$$

$$u'(t) = \delta(t)u(t) + \sigma(t), \tag{1.2}$$

$$u'(t) + \sigma(t) = \delta(t)u(t). \tag{1.3}$$

where u and σ are fuzzy number-valued functions (mappings), δ is a real valued function and equality symbol means identity of membership functions on each side. Ren et al. [31] introduced a fuzzy Mellin transform method for solving Hermite fuzzy differential equations and give some Hyers-Ulam stability results of Hermite fuzzy differential equations. Wang and Sun [32] investigated the existence and uniqueness of solution to Cauchy problems for a class of nonlinear fuzzy fractional differential equations with the Riemann-Liouville H-derivative.

In this paper, we further the discussions of Ulam stability for Eqs (1.1)–(1.3) under the different differentiability and domain space from those in Ref. [30]. Here, the concept of differentiability is the generalization of that introduced in [33]. Meanwhile, the domain space is a Banach space instead of the real number space. We first give some preliminaries in section 2, and then, investigate the Ulam stability of Eqs (1.1)–(1.3) in sections 3–5, respectively.

In this paper, *R* denotes the set of all real numbers, $R_+ = (0, \infty)$, I = [0, 1], $I_0 = (0, 1)$, T = [a, b] and $T_0 = (a, b)$, where $a, b \in R$ with a < b. *X* is a Banach space, $P_{kc}(X)$ denotes the set of all non-empty compact convex subsets of *X*.

2. Preliminaries

In this section, we review some necessary notions and fundamental results which are useful in this paper.

If a function $u: X \rightarrow [0, 1]$ satisfies the following conditions:

(i) *u* is normal, i.e., there exists $x_0 \in X$ such that $u(x_0) = 1$;

(ii) $[u]^{\alpha} = \{x \in X : u(x) \ge \alpha\} \in P_{kc}(X), \forall \alpha \in (0, 1];$

(iii) *u* is upper-semicontinuous;;

(iii) the support set of $u : [u]^0 = \text{supp}(u) = \text{cl}\{x : u(x) > 0\}$ is a compact set, where "cl" means the closure operator.

Then u is called a fuzzy number on X. The set of all fuzzy numbers on X is denoted by X_F .

A mapping $D: X_F \times X_F \to R_+ \cup \{0\}$ is defined by $D(u, v) = \sup_{\alpha \in I} d_H([u]^{\alpha}, [v]^{\alpha})$, where d_H is Hausdorff metric, then (X_F, D) is a complete metric space. For $u, v \in X_F$, $\lambda \in R$, the addition u + v and scalar multiplication $\lambda \cdot u$ are introduced by Zadeh extension principle. If there exists $w \in X_F$ such that u = v + w, then w is called the H-difference of u and v, and denoted by u - v. Obviously, u = v + (u - v).

The details about the fuzzy number space can be found in [9,22,34] and so on. On these bases, we obtain the following result easily:

Theorem 2.1. Let $u, v, w, \gamma \in X_F$, and the H-difference has the following properties:

- (H1) For $u, v, w \in X_F$, if u w exists, then (u + v) w and (u + v) w = (u w) + v exist;
- (H2) For $u, v, w, \gamma \in X_F$, if u v exists, then $(\gamma + u) (\gamma + v)$ and $(\gamma + u) (\gamma + v) = u v$ exist.

The following definition extends the corresponding concept in [33] from the real number space to a Banach space.

Definition 2.2. Let $F : T_0 \to X_F$ be a fuzzy number-valued mapping, $t_0 \in T_0$, $F'(t_0) \in X_F$. If for all h > 0 sufficiently small,

(i) the H-differences $F(t_0 + h) - F(t_0)$ and $F(t_0) - F(t_0 - h)$ exist, and $\lim_{h \to 0} \frac{F(t_0 + h) - F(t_0)}{h} = \lim_{h \to 0} \frac{F(t_0) - F(t_0 - h)}{h} = F'(t_0);$

or

(ii) the H-differences $F(t_0) - F(t_0 + h)$ and $F(t_0 - h) - F(t_0)$ exist, and $\lim_{h \to 0} \frac{F(t_0) - F(t_0 + h)}{(-h)} = \lim_{h \to 0} \frac{F(t_0 - h) - F(t_0)}{(-h)} = F'(t_0)$

or

(iii) the H-differences $F(t_0 + h) - F(t_0)$ and $F(t_0 - h) - F(t_0)$ exist, and

$$\lim_{h \to 0} \frac{F(t_0 + h) - F(t_0)}{h} = \lim_{h \to 0} \frac{F(t_0 - h) - F(t_0)}{(-h)} = F'(t_0)$$

or

(iv) the H-differences $F(t_0) - F(t_0 + h)$ and $F(t_0) - F(t_0 - h)$ and exist, and

$$\lim_{h \to 0} \frac{F(t_0) - F(t_0 + h)}{(-h)} = \lim_{h \to 0} \frac{F(t_0) - F(t_0 - h)}{h} = F'(t_0)$$

(*h* and -h at denominators denote $\frac{1}{h}$ and $-\frac{1}{h}$, respectively), then *F* is said to be (i)-differentiable or (ii)-differentiable or (iv)-differentiable at t_0 , respectively. If *F* satisfies all of (i)-(iv), then *F* is said to be differentiable at t_0 . If *F* is (i)-differentiable ((ii)-differentiable, (iii)-differentiable, differentiable) at every point of T_0 , then *F* is said to be (i)-differentiable, (iv)-differentiable, (iv)-differentiable) on T_0 .

In this paper, we develop our discussion in the cases of (iii) and (iv) differentiabilities.

Theorem 2.3. Let $F, G : T_0 \to X_F$ be two fuzzy mappings. Suppose the H-difference F(t) - G(t) exists for $t \in T_0$.

(1) If F is (iii)-differentiable and G is (iv)-differentiable on T_0 , then F - G is (iii)-differentiable on T_0 ;

(2) If F is (iv)-differentiable and G is (iii)-differentiable on T_0 , then F - G is (iv)-differentiable on T_0 .

Moreover, in the case (1) or (2), for all $t \in T_0$, we have $(F - G)'(t) = F'(t) + (-1) \cdot G'(t)$. *Proof.* We only prove (1), the method used here is similar to Theorem 9 in [33].

Since *F* is (iii)-differentiable and *G* is (iv)-differentiable, for any $t \in T_0$ and h > 0, there exist $u_1(t,h), u_2(t,h), v_1(t,h), v_2(t,h) \in X_F$ such that

$$F(t+h) = F(t) + u_1(t,h), \qquad (2.1)$$

$$F(t-h) = F(t) + u_2(t,h), \qquad (2.2)$$

$$G(t) = G(t+h) + v_1(t,h), \qquad (2.3)$$

$$G(t) = G(t - h) + v_2(t, h), \qquad (2.4)$$

and

$$\lim_{h \to 0} \frac{u_1(t,h)}{h} = \lim_{h \to 0} \frac{u_2(t,h)}{h} = F'(t),$$
(2.5)

$$\lim_{h \to 0} \frac{v_1(t,h)}{(-h)} = \lim_{h \to 0} \frac{v_2(t,h)}{(-h)} = G'(t),$$
(2.6)

From (2.1) and (2.3), we get

$$F(t+h) + G(t) = F(t) + G(t+h) + u_1(t,h) + v_1(t,h), \qquad (2.7)$$

Since the H-differences F(t) - G(t) and F(t + h) - G(t + h) exist, from (2.7) and Theorem 2.1 we have

$$F(t+h) - G(t+h) = F(t) - G(t) + u_1(t,h) + v_1(t,h)$$

which implies that $(F(t + h) - G(t + h)) - (F(t) - G(t)) = u_1(t, h) + v_1(t, h)$. From (2.5) and (2.6), we get

$$\lim_{h \to 0} \frac{(F(t+h) - G(t+h)) - (F(t) - G(t))}{h} = F'(t) + (-1)G'(t).$$

Similarly, from (2.2), (2.4)–(2.6) and Theorem 2.1, we obtain

$$\lim_{h \to 0} \frac{(F(t-h) - G(t-h)) - (F(t) - G(t))}{-h} = F'(t) + (-1)G'(t).$$

Hence, F - G is (iii)-differentiable on T_0 and $(F - G)'(t) = F'(t) + (-1) \cdot G'(t)$. The proof ends. For convenience, we introduce the following conditions:

(C1) For a given $t \in T_0$, if F(t + h) - F(t) and F(t - h) - F(t) exist for sufficiently small h > 0;

(C2) For a given
$$t \in T_0$$
, if $F(t) - F(t+h)$ and $F(t) - F(t-h)$ exist for sufficiently small $h > 0$;

- (C3) For a given $t \in T_0$, if F(t + h) F(t) and F(t) F(t h) exist for sufficiently small h > 0;
- (C4) For a given $t \in T_0$, if F(t) F(t+h) and F(t-h) F(t) exist for sufficiently small h > 0.

Based on Theorem 2.5 in [30], we can verify the following theorem.

Theorem 2.4. Let $f : T_0 \to R$ be differentiable, and $G : T_0 \to X_F$ be (iii)-differentiable or (iv)-differentiable on T_0 ,

(i) if $f(t) \cdot f'(t) > 0$ and G is (iii)-differentiable on T_0 , then $f \cdot G$ is (iii)-differentiable on T_0 and

$$(f \cdot G)'(t) = f'(t) \cdot G(t) + f(t) \cdot G'(t);$$

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(ii) if $f(t) \cdot f'(t) < 0$ and G is (iii)-differentiable on T_0 , then $f \cdot G$ is (iii)-differentiable on T_0 and

$$(f \cdot G)'(t) = f(t) \cdot G'(t) - (-f'(t)) \cdot G(t);$$

(iii) if $f(t) \cdot f'(t) < 0$ and G is (iii)-differentiable on T_0 and $f \cdot G$ satisfies condition (C2), then $f \cdot G$ is (iv)-differentiable on T_0 and

$$(f \cdot G)'(t) = f'(t) \cdot G(t) - (-f(t)) \cdot G'(t);$$

(iv) if $f(t) \cdot f'(t) > 0$ and G is (iv)-differentiable on T_0 and $f \cdot G$ satisfies condition (C1), then $f \cdot G$ is (iii)-differentiable on T_0 and

$$(f \cdot G)'(t) = f'(t) \cdot G(t) - (-f(t)) \cdot G'(t);$$

(v) if $f(t) \cdot f'(t) > 0$ and G is (iv)-differentiable on T_0 , then $f \cdot G$ is (iv)-differentiable on T_0 and

$$(f \cdot G)'(t) = f(t) \cdot G'(t) - (-f'(t)) \cdot G(t);$$

(vi) if $f(t) \cdot f'(t) < 0$ and G is (iv)-differentiable on T_0 , then $f \cdot G$ is (iv)-differentiable on T_0 and

$$(f \cdot G)'(t) = f'(t) \cdot G(t) + f(t) \cdot G'(t).$$

Proof. The proofs of them is similar, we only show (vi). Suppose that f(t) < 0, f'(t) > 0 and *G* is (iv)-differentiable on T_0 , then for any $t \in T_0$ and any sufficiently small h > 0, there exists $u(t,h) \in X_F$ such that

$$G(t) = G(t+h) + u(t,h)$$
 and $\lim_{h \to 0} \frac{u(t,h)}{-h} = G'(t)$.

We also have f(t) = f(t+h) + v(t,h), where $\lim_{h \to 0} \frac{v(t,h)}{(-h)} = f'(t)$. Then

$$G(t) \cdot f(t) = G(t+h) \cdot f(t+h) + u(t,h) \cdot f(t+h) + G(t+h) \cdot v(t,h) + u(t,h) \cdot v(t,h),$$

so

$$G(t) \cdot f(t) - G(t+h) \cdot f(t+h) = u(t,h) \cdot f(t+h) + G(t+h) \cdot v(t,h) + u(t,h) \cdot v(t,h).$$

Hence,

$$\lim_{h \to 0} \frac{G(t) \cdot f(t) - G(t+h) \cdot f(t+h)}{(-h)} \\ = \lim_{h \to 0} \frac{u(t,h)}{(-h)} \cdot f(t+h) + \lim_{h \to 0} \frac{v(t,h)}{(-h)} \cdot G(t+h) + \lim_{h \to 0} \frac{u(t,h)}{(-h)} \cdot v(t,h) \\ = f(t) \cdot G'(t) + f'(t) \cdot G(t).$$

Similarly, we have

$$\lim_{h \to 0} \frac{G(t) \cdot f(t) - G(t-h) \cdot f(t-h)}{h} = f(t) \cdot G'(t) + f'(t) \cdot G(t).$$

The proof ends.

Theorem 2.5. Let $F: T \to X_F$ be (iii)-differentiable or (iv)-differentiable and $t_1, t_2 \in T$, then

(1) $F(t_2) - F(t_1)$ exists;

(2) there exists a positive real number M such that $D(F(t_2), F(t_1)) \le M |t_2 - t_1|$.

Proof. We only show the case that *F* is (iii)-differentiable.

Case 1: If $t_1 = t_2$, we have $F(t_2) - F(t_1) = \theta$ and $D(F(t_2), F(t_1)) = 0$ directly.

Case 2: If $t_1 < t_2$. For all $s \in [t_1, t_2]$, by the assumption there exists $\delta(s) > 0$ such that F(s + h) - F(s) exists and $D\left(\frac{F(s+h)-F(s)}{h}, F'(s)\right) < 1$ or $D\left(F(s+h) - F(s), hF'(s)\right) < h$ for $h \in (0, \delta(s))$, hence

$$D(F(s+h), F(s)) = D(F(s+h) - F(s), \theta)$$

$$\leq D(F(s+h) - F(s), hF'(s)) + D(hF'(s), \theta)$$

$$< h + hD(F'(s), \theta)$$

$$= h + hD_{F'}(s),$$

where $D_{F'}(s) = D(F'(s), \theta)$.

So we can find a finite sequence: $t_1 = s_1 < s_2 \cdots < s_n = t_2$ such that the interval set $\{I_i : I_i = (s_i - \delta(s_i), s_i + \delta(s_i)), i = 1, 2, \cdots, n\}$ cover $[t_1, t_2]$ and $I_i \cap I_{i+1} \neq \phi$ $(i = 1, 2, 3 \cdots, n-1)$. Let $v_i \in I_i \cap I_{i+1}$, such that $s_i < v_i < s_{i+1}$ $(i = 1, 2, 3 \cdots, n-1)$, then there exist $\gamma_1^{(i)}, \gamma_2^{(i)} \in X_F$, such that $F(s_{i+1}) = F(v_i) + \gamma_1^{(i)} = F(s_i) + \gamma_2^{(i)} = F(s_i) + \eta_i$ and

$$D(F(s_{i+1}), F(s_i)) \le D(F(s_{i+1}), F(v_i)) + D(F(v_i), F(s_i))$$

$$< (s_{i+1} - v_i) + (s_{i+1} - v_i)D_{F'}(v_i) + (v_i - s_i) + (v_i - s_i)D_{F'}(s_i)$$

$$= (s_{i+1} - s_i) + (s_{i+1} - s_i)D_{F'}$$

where $D_{F'} = \max_{1 \le i \le n-1} \{ D_{F'}(s_i), D_{F'}(v_i) \}, \eta_i = \gamma_2^{(i)} + \gamma_1^{(i)} \in X_F \ (i = 1, 2 \cdots, n-1).$ Let $\sum_{i=1}^{n-1} \eta_i = u \in X_F$, then $F(t_2) = F(t_1) + u$ and $D(F(t_2), F(t_1)) < (t_2 - t_1)(1 + D_{F'}).$

Let $M_1 = (1 + D_{F'}) > 0$, then $D(F(t_2), F(t_1)) < M_1(t_2 - t_1)$.

Case 3: $t_1 > t_2$. By the similar discussion in Case 2, we can prove there exists $M_2 > 0$ such that $D(F(t_2), F(t_1)) < M_2(t_1 - t_2)$. The proof ends..

The following contents about the integral of a fuzzy mapping are generalizations of these in [9]. **Definition 2.6.** A mapping $F : T \to X_F$ is measurable if for all $\alpha \in (0, 1]$ the set-valued mapping $F_{\alpha} : T \to P_{kc}(X)$ defined by

$$F_{\alpha}(t) = [F(t)]^{\alpha}$$

is measurable. The set of all measurable selections of $F_{\alpha}(t)$ is denoted by $S_{F_{\alpha}}$. **Definition 2.7.** Let $F: T \to X_F$ be measurable. If there exists $u \in X_F$ such that

$$\int_T F_\alpha(t)dt = [u]^{\alpha}$$

for all $\alpha \in (0, 1]$, then *F* is said to be integrable over *T*, *u* is the integral of *F* over *T* and denoted by $\int_T F(t)dt$ or $\int_a^b F(t)dt$, where

$$\int_{T} F_{\alpha}(t)dt = \left\{ \int_{T} f(t)dt | f \in S_{F_{\alpha}} \right\}$$

Definition 2.8. A mapping $F : T \to X_F$ is called integrably bounded if there exists an integrable mapping $h : [a, b] \to X$ such that $|x| \le h(t)$ for all $x \in [F(t)]^0$.

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It is well known that if a measurable mapping $F : T \to X_F$ is integrably bounded, then it is integrable.

Theorem 2.9. If $F : [a, b] \to X_F$ is integrable and $c \in [a, b]$, then

$$\int_{a}^{b} F(t)dt = \int_{a}^{c} F(t)dt + \int_{c}^{b} F(t)dt.$$
 (2.8)

Remark. Let $\int_a^b F(t)dt = \left(-\int_b^a F(t)dt\right)$ when a > b, we can prove that (2.8) still holds for a > c > b. The set of all continuous mapping $F: T \to (X_F, D)$ is denoted by $C[T, X_F]$.

The following theorem is the generalization of Corollary 4.1 in [9] from R^n to a Banach space.

Theorem 2.10. If $F : T \to X_F$ be a continuous fuzzy mapping, then it is integrable.

Simillar to Theorems 2.3 and 2.4 in [30], we can show the following theorem.

Theorem 2.11. Let $F : T \to X_F$ be a continuous fuzzy mapping. Then, for any $x \in T$,

(1) the mapping $H(x) = \int_{a}^{x} F(t)dt$ is (i)-differentiable, and H(x) is Lipschitz continuous. Moreover, H'(x) = F(x);

(2) $H(x) = \gamma - \int_a^x ((-1) \cdot F(t)) dt$ is (ii)-differentiable, and H'(x) = F(x).

3. Ulam stability for Eq (1.1)

In this section, we mainly discuss the Ulam stability for the linear fuzzy differential Eq (1.1) in two cases: (1) *u* is (iii)-differentiable and $\delta(t) > 0$; (2) *u* is (iv)-differentiable and $\delta(t) < 0$.

Theorem 3.1. Let $\sigma \in C[T, X_F]$ and $\delta : T \to R_+$ be a continuous function. Suppose that $u : T_0 \to X_F$ is continuous and (iii)-differentiable, and the H-difference $\exp\left(\int_a^t \delta(\tau)d\tau\right)u(t) - \int_a^t \exp\left(\int_a^v \delta(\tau)d\tau\right)\sigma(v)dv$ exists for each $t \in T_0$.

(1) there exists a $u_0 \in X_F$ such that the mapping: $\forall t \in T_0$,

$$\tilde{u}(t) = \exp\left(-\int_{a}^{t} \delta(\tau) d\tau\right) \left(u_{0} + \int_{a}^{t} \exp\left(\int_{a}^{v} \delta(\tau) d\tau\right) \sigma(v) dv\right)$$
(3.1)

satisfies

$$D(u(t), \tilde{u}(t)) \le \exp\left(-\int_{a}^{t} \delta(\tau) d\tau\right) (M+M') (b-t),$$
(3.2)

where M, M' > 0.

(2) If $\tilde{u}(t)$ satisfies the condition (C3), then $\tilde{u}(t)$ is the (i)-differentiable solution of Eq (1.1). *Proof.* (1) For convenience, we set

$$G(t) = \exp\left(\int_a^t \delta(\tau) d\tau\right) u(t) \text{ and } w(t) = G(t) - \int_a^t \exp\left(\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv$$

for each $t \in T_0$. Then for arbitrary $s, t \in T_0$ with t < s, we have

$$D(w(t), w(s)) = D\left(G(t) + \int_t^s \exp\left(\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv, G(s)\right).$$

From (i) of Theorem 2.4, we know that G(t) is (iii)-differentiable on T_0 , which together with Theorem 2.5 implies that there exists M > 0 such that

$$D(w(t), w(s)) = D\left(G(t) + \int_{t}^{s} \exp\left(\int_{a}^{v} \delta(\tau) d\tau\right) \sigma(v) dv, G(s)\right)$$

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Since σ and δ are continuous functions and by Theorem 2.10, we know that $\exp\left(\int_a^t \delta(\tau) d\tau\right) \sigma(t)$ is integrable on *T*. Then from (1) of Theorem 2.11, we obtain

$$D\left(\theta, \int_{t}^{s} \exp\left(\int_{a}^{v} \delta(\tau) d\tau\right) \sigma(v) dv\right) \leq M'(s-t)$$

where M' > 0. Thus

$$D(w(t), w(s)) \le (M + M')(s - t),$$
 (3.3)

which means that $\{w(s)\}_{s \in T_0}$ is a Cauchy net in X_F . The completeness of the metric space (X_F, D) implies that there exists a $u_0 \in X_F$ such that w(s) converges to u_0 as $s \to b^-$. Based on the above arguments, we obtain

$$\begin{split} [D(u(t), \tilde{u}(t)) &= D\left(u(t), \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)\left(u_{0} + \int_{a}^{t} \exp\left(\int_{a}^{v} \delta(\tau)d\tau\right)\sigma(v)dv\right) \\ &= \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)D\left(G(t), u_{0} + \int_{a}^{t} \exp\left(\int_{a}^{v} \delta(\tau)d\tau\right)\sigma(v)dv\right) \\ &= \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)D\left(G(t) - \int_{a}^{t} \exp\left(\int_{a}^{v} \delta(\tau)d\tau\right)\sigma(v)dv, u_{0}\right) \\ &= \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)D(w(t), u_{0}) \\ &\leq \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)\left(D(w(t), w(s)) + D(w(s), u_{0})\right) \\ &\leq \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)\left((M + M')\left(s - t\right) + D\left(w\left(s\right), u_{0}\right)\right). \end{split}$$
(3.4)

It follows from (3.3) that the expression (3.4) tends to $\exp\left(-\int_a^t \delta(\tau)d\tau\right)(M+M')(b-t)$ as $s \to b^-$. That is, the inequality (6) holds.

Now we prove the uniqueness of u_0 . Assume that there exists $u_1 \in X_F$ such that inequality (3.2) holds where u_0 in (3.1) is replaced by u_1 . It follows from (3.2) that

$$\begin{split} &\exp\left(-\int_{a}^{t}\delta(\tau)d\tau\right)D(u_{0},u_{1})\\ &=\exp\left(-\int_{a}^{t}\delta(\tau)d\tau\right)D\left(u_{0}+\int_{a}^{t}\exp\left(\int_{a}^{v}\delta(\tau)d\tau\right)\sigma(v)dv,u_{1}+\int_{a}^{t}\exp\left(\int_{a}^{v}\delta(\tau)d\tau\right)\sigma(v)dv\right)\\ &\leq D\left(u(t),\exp\left(-\int_{a}^{t}\delta(\tau)d\tau\right)\left(u_{0}+\int_{a}^{t}\exp\left(\int_{a}^{v}\delta(\tau)d\tau\right)\sigma(v)dv\right)\right)\\ &+D\left(u(t),\exp\left(-\int_{a}^{t}\delta(\tau)d\tau\right)\left(u_{1}+\int_{a}^{t}\exp\left(\int_{a}^{v}\delta(\tau)d\tau\right)\sigma(v)dv\right)\right) \end{split}$$

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$$\leq 2 \exp\left(-\int_{a}^{t} \delta(\tau) d\tau\right) (M+M') (b-t)$$

for each $t \in T_0$. So

$$D(u_0, u_1) \le 2(M + M')(b - t)$$

for each $t \in T_0$. Since 2(M+M')(b-t) tends to 0 as $t \to b^-$, we get $u_0 \equiv u_1$. This completes the proof. (2) Next, we shall consider the differentiability of \tilde{u} given by (3.2). Setting

$$f(t) = \exp\left(-\int_a^t \delta(\tau)d\tau\right)$$
 and $H(t) = u_0 + \int_a^t \exp\left(\int_a^v \delta(\tau)d\tau\right)\sigma(v)dv$

From (1) of Theorem 2.11, it is easy to see that H(t) is (i)-differentiable on T_0 . If $f(t) \cdot H(t)$ satisfies the condition (C3) on T_0 , by (ii) of Theorem 5 in [30], we obtain that $\tilde{u}(t) = f(t) \cdot H(t)$ is (i)-differentiable on T_0 , since $f(t) \cdot f'(t) < 0$ holds for $t \in T_0$. Moreover, we have

$$\tilde{u}'(t) = (f(t) \cdot H(t))' = f(t) \cdot H'(t) - (-f'(t)) \cdot H(t) = \sigma(t) - \delta(t)\tilde{u}(t)$$

Thus $\tilde{u}'(t) + \delta(t)\tilde{u}(t) = \sigma(t)$. That is to say, $\tilde{u}(t)$ is a (i)-differentiable solution of the fuzzy differential Eq (1.1).

Theorem 3.2. Let $\sigma \in C[T, X_F]$ and $\delta : T \to R_-$ be a continuous function. Suppose $u : T_0 \to X_F$ is continuous and (iv)-differentiable.

(1) Then there exists a $u_0 \in X_F$ such that

$$\exp\left(\int_{a}^{t} \delta(\tau)d\tau\right)u(t) + \int_{a}^{t} - \exp\left(\int_{a}^{v} \delta(\tau)d\tau\right)\sigma(v)dv \to u_{0}$$
(3.5)

as $t \to b^-$.

(2) If the H-difference $u_0 - \int_a^t - \exp\left(\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv$ exists on T_0 , then the mapping: $\forall t \in T_0$,

$$\tilde{u}(t) = \exp\left(-\int_{a}^{t} \delta(\tau) d\tau\right) \left(u_{0} - \int_{a}^{t} -\exp\left(\int_{a}^{v} \delta(\tau) d\tau\right) \sigma(v) dv\right)$$
(3.6)

satisfies

$$D(u(t), \tilde{u}(t)) \le \exp\left(-\int_{a}^{t} \delta(\tau) d\tau\right) (M+M')(b-t)$$
(3.7)

where M, M' > 0.

(3) If $\tilde{u}(t)$ satisfies the condition (C4), then $\tilde{u}(t)$ is the (ii)-differentiable solution of Eq (1.1). *Proof.* (1) For simplicity, let

$$G(t) = \exp\left(\int_a^t \delta(\tau) d\tau\right) u(t) \text{ and } w(t) = G(t) + \int_a^t -\exp\left(\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv$$

for each $t \in T_0$. By (vi) of Theorem 2.4, we obtain G(t) is (iv)-differentiable. Then, for any $s, t \in T_0$ with t < s, we can infer from Theorem 2.5 that there exists M > 0 such that

$$D(w(t), w(s)) = D\left(G(t) + \int_{a}^{t} -\exp\left(\int_{a}^{v} \delta(\tau) d\tau\right) \sigma(v) dv, G(s) + \int_{a}^{s} -\exp\left(\int_{a}^{v} \delta(\tau) d\tau\right) \sigma(v) dv\right)$$

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$$\leq D(G(t), G(s)) + D\left(\theta, \int_{t}^{s} -\exp\left(\int_{a}^{v} \delta(\tau)d\tau\right)\sigma(v)dv\right)$$

$$\leq M(s-t) + D\left(\theta, \int_{t}^{s} -\exp\left(\int_{a}^{v} \delta(\tau)d\tau\right)\sigma(v)dv\right).$$

Since σ and δ are continuous functions and by Theorem 2.10, we get $\exp\left(\int_a^t \delta(\tau) d\tau \sigma(t)\right)$ is integrable on *T*, which together with (1) of Theorem 2.11, we have

$$D\left(\theta, \int_{t}^{s} -\exp\left(\int_{a}^{v} \delta(\tau)d\tau\right)\sigma(v)dv\right) \leq M'(s-t),$$

where M' > 0. Then

$$D(w(t), w(s)) \le M(s-t) + M'(s-t) = (M+M')(s-t)$$

From the proof of Theorem 3.1, we obtain that w(s) converges to u_0 as $s \to b^-$. That is to say, the expression (3.5) holds.

(2) If the H-difference $u_0 - \int_a^t - \exp\left(\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv = u_{0H}(t)$ exists on T_0 , then

$$D(u(t), \tilde{u}(t)) = D\left(u(t), \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)u_{0H}(t)\right)$$

$$= \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)D\left(\exp\left(\int_{a}^{t} \delta(\tau)d\tau\right)u(t), u_{0H}(t)\right)$$

$$= \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)D(w(t), u_{0})$$

$$\leq \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)(D(w(t), w(s)) + D(w(s), u_{0}))$$

$$\leq \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right)((M+M')(s-t) + D(w(s), u_{0}))$$
(3.8)

for any $t \in T_0$. It is easy to see that the expression (3.8) tends to $\exp\left(-\int_a^t \delta(\tau)d\tau\right)(M+M')(b-t)$ as $s \to b^-$. Thus, the inequality (3.7) holds.

Now we prove the uniqueness of u_0 . Assume that there exists $u_1 \in X_F$ such that the H-difference

$$u_1 - \int_a^t -\exp\left(\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv = u_{1H}(t)$$

exists on T_0 and inequality (3.7) holds if u_0 in (3.6) is replaced by u_1 . Then, we have

$$\exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right) D(u_{0}, u_{1})$$

$$= \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right) D\left(u_{0H}(t), u_{1H}(t)\right)$$

$$\leq D\left(u(t), \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right) u_{0H}(t)\right) + D\left(u(t), \exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right) u_{1H}(t)\right)$$

$$\leq 2\exp\left(-\int_{a}^{t} \delta(\tau)d\tau\right) (M+M') (b-t)$$

for each $t \in T_0$. Therefore for each $D(u_0, u_1) \le 2(M+M')(b-t)$. Since 2(M+M')(b-t) tends to 0 as $t \to b^-$, we get $u_0 \equiv u_1$. This completes the proof.

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(3) Next, we shall consider the differentiability of the fuzzy number-valued mapping $\tilde{u}(t)$ given by (3.6). Setting

$$f(t) = \exp\left(-\int_a^t \delta(\tau)d\tau\right)$$
 and $H(t) = u_0 - \int_a^t -\exp\left(\int_a^v \delta(\tau)d\tau\right)\sigma(v)dv$.

By (2) of Theorem 2.11, it is easy to see that H(t) is (ii)-differentiable and $H'(t) = \exp\left(\int_a^t \delta(\tau) d\tau\right) \sigma(t)$. Moreover, $f(t) \cdot f'(t) > 0$. From (v) of Theorem 5 in [30] and the assumption that $\tilde{u}(t) = f(t) \cdot H(t)$ satisfies the condition (C4) on T_0 , we know that $\tilde{u}(t)$ is (ii)-differentiable on T_0 and fulfills

$$\tilde{u}'(t) = (f(t) \cdot H(t))' = f(t) \cdot H'(t) - (-f'(t)) \cdot H(t) = \sigma(t) - \delta(t)\tilde{u}(t)$$

which implies that $\tilde{u}'(t) + \delta(t)\tilde{u}(t) = \sigma(t)$, i.e., $\tilde{u}(t)$ is a (ii)-differentiable solution of the fuzzy differential Eq (1.1).

4. Ulam stability for Eqs (1.2) and (1.3)

In this section, with the similar proof method of Theorems 3.1 and 3.2, we can prove the Ulam stability for the linear fuzzy differential Eqs (1.2) and (1.3) in two cases: (1) *u* is (iii)-differentiable and $\delta(t) > 0$; (2) *u* is (iv)-differentiable and $\delta(t) < 0$.

Theorem 4.1. Let $\sigma \in C[T, X_F]$ and $\delta : T \to R_+$ be a continuous function. Suppose $u : T_0 \to X_F$ is continuous and (iii)-differentiable, $exp\left(-\int_a^t \delta(\tau)d\tau\right)u(t)$ satisfies the condition (C1) on T_0 . If the H-difference $exp\left(-\int_a^t \delta(\tau)d\tau\right)u(t) - \int_a^t exp\left(-\int_a^v \delta(\tau)d\tau\right)\sigma(v)dv$ exists for any $t \in T_0$, then

(1) there exists a $u_0 \in X_F$ such that the mapping: $\forall t \in T_0$,

$$\tilde{u}(t) = \exp\left(\int_a^t \delta(\tau) d\tau\right) \left(u_0 + \int_a^t \exp\left(-\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv\right)$$

satisfies

$$D(u(t), \tilde{u}(t)) \le \exp\left(\int_a^t \delta(\tau) d\tau\right) (M + M') (b - t),$$

where M, M' > 0.

(2) $\tilde{u}(t)$ is the (i)-differentiable solution of Eq (1.2).

Theorem 4.2. Let $\sigma \in C[T, X_F]$ and $\delta : T \to R_-$ be a continuous function. Suppose $u : T_0 \to X_F$ is continuous and (iv)-differentiable, $\exp\left(-\int_a^t \delta(\tau)d\tau\right)u(t)$ satisfies the condition (C2) on T_0 .

(1) Then there exists a $u_0 \in X_F$ such that the mapping

$$w(t) = \exp\left(-\int_a^t \delta(\tau)d\tau\right)u(t) + \int_a^t -\exp\left(-\int_a^v \delta(\tau)d\tau\right)\sigma(v)dv \to u_0$$

as $t \to b^-$.

(2) If the H-difference $u_0 - \int_a^t -\exp\left(-\int_a^v \delta(\tau)d\tau\right)\sigma(v)dv$ exists on T_0 , then u_0 is unique such that the mapping: $\forall t \in T_0$,

$$\tilde{u}(t) = \exp\left(\int_a^t \delta(\tau) d\tau\right) \left(u_0 - \int_a^t - \exp\left(-\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv\right)$$

satisfies

$$D(u(t), \tilde{u}(t)) \le \exp\left(\int_{a}^{t} \delta(\tau) d\tau\right) (M + M') (b - t)$$

where M, M' > 0.

(3) $\tilde{u}(t)$ is the (ii)-differentiable solution of Eq (1.2).

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Theorem 4.3. Let $\sigma \in C[T, X_F]$ and $\delta : T \to R_+$ be a continuous function. Suppose $u : T_0 \to X_F$ is continuous and (iii)-differentiable, $\exp\left(-\int_a^t \delta(\tau) d\tau\right) u(t)$ satisfies the condition (C1) on T_0 .

(1) Then there exists a $u_0 \in X_F$ such that the mapping

$$w(t) = \exp\left(-\int_a^t \delta(\tau) d\tau\right) u(t) + \int_a^t \exp\left(-\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv \to u_0$$

as $t \to b^-$.

(2) If the H-difference $u_0 - \int_a^t \exp\left(-\int_a^v \delta(\tau)d\tau\right)\sigma(v)dv$ exists for each $t \in T_0$, then the mapping: $\forall t \in T_0$,

$$\tilde{u}(t) = \exp\left(\int_{a}^{t} \delta(\tau) d\tau\right) \left(u_{0} - \int_{a}^{t} \exp\left(-\int_{a}^{v} \delta(\tau) d\tau\right) \sigma(v) dv\right)$$

satisfies

$$D(u(t), \tilde{u}(t)) \le \exp\left(\int_a^t \delta(\tau) d\tau\right) (M + M') (b - t),$$

where M, M' > 0.

(3) If $\tilde{u}(t)$ satisfies the condition (C3), then $\tilde{u}(t)$ is the (i)-differentiable solution of Eq (1.3).

Theorem 4.4. Let $\sigma \in C[T, X_F]$ and $\delta : T \to R_-$ be a continuous function. Suppose $u : T_0 \to X_F$ is continuous and (iv)-differentiable, $\exp\left(-\int_a^t \delta(\tau)d\tau\right)u(t)$ satisfies the condition (C2) on T_0 and the H-difference $\exp\left(-\int_a^t \delta(\tau)d\tau\right)u(t) - \int_a^t -\exp\left(-\int_a^v \delta(\tau)d\tau\right)\sigma(v)dv$ exists for each $t \in T_0$.

(1) Then there exists a $u_0 \in X_F$ such that the mapping: $\forall t \in T_0$,

$$\tilde{u}(t) = \exp\left(\int_a^t \delta(\tau) d\tau\right) \left(u_0 + \int_a^t - \exp\left(-\int_a^v \delta(\tau) d\tau\right) \sigma(v) dv\right)$$

satisfies

$$D(u(t), \tilde{u}(t)) \le \exp\left(\int_{a}^{t} \delta(\tau) d\tau\right) (M + M') (b - t)$$

where M, M' > 0.

(2) If $\tilde{u}(t)$ satisfies the condition (C4), then $\tilde{u}(t)$ is the (ii)-differentiable solution of Eq (1.3).

5. Conclusion

The paper introduces some new definitions of differentiabilities for fuzzy number-valued mappings in Banach spaces and gives their important properties. Under suitable conditions, the paper shows the solutions of some fuzzy differential equations are stable in the sense of Ulam. By the techniques introduced in this paper, more results about Ulam stability theorems for linear fuzzy differential equations are prospective. They will play important roles in the applications of fuzzy differential equations.

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Conflict of interest

The authors declare that there is no conflicts of interest in this paper.

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