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Research article

Multistep hybrid viscosity method for split monotone variational inclusion and fixed point problems in Hilbert spaces

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Abstract: In this paper, we present a multi-step hybrid iterative method. It is proven that under appropriate assumptions, the proposed iterative method converges strongly to a common element of fixed point of a finite family of nonexpansive mappings, the solution set of split monotone variational inclusion problem and the solution set of triple hierarchical variational inequality problem ($THVI$) in real Hilbert spaces. In addition, we give a numerical example of a triple hierarchical system derived from our generalization.

Keywords: split monotone variational inclusion; fixed point problem; variational inequality problem; Hilbert spaces; triple hierarchical variational inequality problem
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1. Introduction

It is known that variational inequality, as a very important tool, has already been studied for a wide class of unilateral, obstacle, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Many numerical methods have been developed for solving variational inequalities and some related optimization problems; see [1–6] and the references therein.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty, closed and convex subset of H and $A : C \to H$ be a nonlinear mapping. The *variational inequality problem (VIP)* associated with the set C and the mapping A is stated as follows:

$$\text{find } x^* \in C \text{ such that } \langle Ax^*, x - x^* \rangle \geq 0, \forall x \in C. \tag{1.1}$$

In particular, the VIP (1.1) in the case C is the set $Fix(T)$ of fixed points of a nonexpansive self-mapping T of C and A is of the form $A = I - S$, with S another nonexpansive self-mapping of C. In other words, VIP is of the form

$$\text{find } x^* \in Fix(T) \text{ such that } \langle x^* - Sx^*, x - x^* \rangle \geq 0, \forall x \in Fix(T). \tag{1.2}$$

This problem, introduced by Mainge and Moudafi [8] , is called *hierarchical fixed point problem (HFPP)*.

Subsequently, Moudafi and Mainge [7] studied the explicit scheme for computing a solution of VIP (1.2) by introducing the following iterative algorithm:

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)(\alpha_n S x_n + (1 - \alpha_n)T x_n), \tag{1.3}$$

where $f : C \to C$ and $\{\alpha_n\}, \{\lambda_n\} \subset (0, 1)$. They also proved the strong convergence of the sequence $\{x_n\}$ generalized by (1.3) to a solution of VIP (1.2).

Yao et al. [9] introduced and analyzed the following two-step iterative algorithm that generates a sequence $\{x_n\}$ by the following explicit scheme:

$$\begin{cases} y_n = \beta_n S x_n + (1 - \beta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T y_n, \ n \geq 1. \end{cases} \tag{1.4}$$

Under appropriate conditions, the above iterative sequence $\{x_n\}$ converges strongly to some fixed point of T where T is nonexpansive mapping and $\{x_n\}$ is solves VIP (1.2).

Marino et al. [10] introduced a multistep iterative method that generalizes the two-step method studied in [9] from two nonexpansive mappings to a finite family of nonexpansive mappings that generates a sequence $\{x_n\}$ by the following iterative scheme:

$$\begin{cases} F(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle, \forall y \in C, \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i})y_{n,i-1}, i = 2, \dots, N, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T y_{n,N}], \ n \geq 1. \end{cases} \tag{1.5}$$

They prove that strong convergence of the method to a common fixed point of a finite number of nonexpansive mappings that also solves a suitable equilibrium problem.

On the other hand, by combining the regularization method, the hybrid steepest descent method, and the projection method, Ceng et al. [11] proposed an iterative algorithm that generates a sequence via the explicit scheme and proved that this sequence converges strongly to a unique solution of the following problem.

Problem 1.1 Let $F : C \to H$ be k-Lipschitzian and η-strongly monotone on the nonempty, closed and convex subset C of H, where k and η are positive constants, that is,

$$\|Fx - Fy\| \leq k\|x - y\| \text{ and } \langle Fx - Fy, x - y \rangle \geq \eta\|x - y\|^2, \forall x, y \in C. \tag{1.6}$$

Let $f : C \to H$ be a ρ-contraction with a coefficient $\rho \in [0, 1)$ and $S, T : C \to C$ be two nonexpansive mappings with $Fix(T) \neq \emptyset$. Let $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Consider the following *triple hierarchical variational inequality problem (THVI)*: find $x^* \in \Xi$ such that

$$\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, \forall x \in \Xi, \tag{1.7}$$

where Ξ denotes the solution set of the following *hierarchical variational inequality problem (HVIP)*: find $z^* \in Fix(T)$ such that

$$\langle(\mu F - \gamma S)z^*, z - z^*\rangle \geq 0, \forall z \in Fix(T), \quad (1.8)$$

where the solution set Ξ is assumed to be nonempty.

Since Problem 1.1 has a triple hierarchical structure, in contrast to bilevel programming problems [12, 13], that is, a variational inequality problem with a variational inequality constraint over the fixed point set $Fix(T)$, we also call (1.8) a *triple hierarchical variational inequality problem (THVIP)*, which is a generalization of *the triple hierarchical constrained optimization problem (THCOP)* considered by [14, 15].

Recently, many authors introduced the split monotone variational inequality inclusion problem, which is the core of the modeling of many inverse problems arising in phase retrieval and other real-world problems. It has been widely studied in sensor networks, intensity-modulated radiation therapy treatment planning, data compression, and computerized tomography in recent years; see, e.g., [18, 19, 21, 26, 27] and the references therein.

The *split monotone variational inclusion problem* (*SMVIP*) was first introduced by Moudafi [20] as follows: find $x^* \in H_1$ such that

$$\begin{cases} 0 \in f_1 x^* + B_1 x^*, \\ y^* = Ax^* \in H_2 : 0 \in f_2 y^* + B_2 y^*, \end{cases} \quad (1.9)$$

where $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are two given single-valued mappings, $A : H_1 \to H_2$ is a bounded linear operator, and $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multivalued maximal monotone mappings.

If $f_1 = f_2 \equiv 0$, then (1.9) reduces to the following *split variational inclusion problem* (*SVIP*): find $x^* \in H_1$ such that

$$\begin{cases} 0 \in B_1 x^*, \\ y^* = Ax^* \in H_2 : 0 \in B_2 y^*. \end{cases} \quad (1.10)$$

Additionally, if $f_1 \equiv 0$, then (1.9) reduces to the following *split monotone variational inclusion problem* ($SMVIP$): find $x^* \in H_1$ such that

$$\begin{cases} 0 \in B_1 x^*, \\ y^* = Ax^* \in H_2 : 0 \in f y^* + B_2 y^*. \end{cases} \quad (1.11)$$

We denote the solution sets of variational inclusion $0 \in B_1 x^*$ and $0 \in f y^* + B_2 y^*$ by $SOLVIP(B_1)$ and $SOLVIP(f + B_2)$, respectively. Thus, the solution set of (1.11) can denoted by $\Gamma = \{x^* \in H_1 : x^* \in SOLVIP(B_1), Ax^* \in SOLVIP(f + B_2)\}$.

In 2012, Byrne et al. [21] studied the following iterative scheme for $SVIP$ (1.10): for a given $x_0 \in H_1$ and $\lambda > 0$,

$$x_{n+1} = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]. \quad (1.12)$$

In 2014, Kazmi and Rizvi [22] introduced a new iterative scheme for $SVIP$ (1.10) and the fixed point problem of a nonexpansive mapping:

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \end{cases} \quad (1.13)$$

where A is a bounded linear operator, A^* is the adjoint of A, f is a contraction on H_1, and T is a nonexpansive mapping of H_1. They obtained a strong convergence theorem under some mild restrictions on the parameters.

Jitsupa et al. [1] modified algorithm (1.13) for $SVIP$ (1.10) and the fixed point problem of a family of strict pseudo-contractions:

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n], \\ y_n = \beta_n u_n + (1-\beta_n)\sum_{n=1}^{\infty} \eta_i^{(n)} T_i u_n, \\ x_{n+1} = \alpha_n \tau f(x_n) + (I - \alpha_n D)y_n, n \geq 1, \end{cases} \tag{1.14}$$

where A is a bounded linear operator, A^* is the adjoint of A, $\{T_i\}_{i=1}^N$ is a family of k_i-strictly pseudo-contractions, f is a contraction, and D is a strong positive linear bounded operator. In [1], they prove under certain appropriate assumptions on the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\eta_i^{(n)}\}_{i=1}^N$ that $\{x_n\}$, defined by (1.14), converges strongly to a common solution of $SVIP$ (1.10) and a fixed point of a finite family of k_i-strictly pseudo-contractions, which solve a variational inequality problem (1.1).

In this paper, we consider the following system of variational inequalities defined over a set consisting of the set of solutions of split monotone variational inclusion, the set of common fixed points of nonexpansive mappings, and the set of fixed points of a mapping.

Problem 1.2 Let $F : C \to H$ be k-Lipschitzian and η-strongly monotone on the nonempty closed and convex subset C of H, $\psi : C \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1)$ and $S_i, S, T : C \to C$ be nonexpansive mappings for all $i \in \{1, \ldots, N\}$. Let $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \xi \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Then, the objective is to find $x^* \in \Omega$ such that

$$\begin{cases} \langle (\mu F - \xi\psi)x^*, x - x^* \rangle \geq 0, \forall x \in \Omega, \\ \langle (\mu F - \xi S)x^*, y - x^* \rangle \geq 0, \forall y \in \Omega, \end{cases} \tag{1.15}$$

where $\Omega = Fix(T) \cap (\bigcap_i Fix(S_i)) \cap \Gamma \neq \emptyset$.

Motivated and inspired by the Moudafi and Mainge [7], Marino et al. [10], Ceng et al. [11] and Kazmi and Rizvi [22], in this paper, we consider a multistep which difference from (1.5). It is proven that under appropriate assumptions the proposed iterative method, the sequence $\{x_n\}$ converges strongly to a unique solution to Problem 1.2 and which is solve $THVI$(1.7). Finally, we give some example and numerical results to illustrate our main results.

2. Preliminaries

In this section, we collect some notations and lemmas. Let C be a nonempty closed convex subset of a real Hilbert space H. We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is also well known [24] that the Hilbert space H satisfies *Opail's condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{2.1}$$

holds for every $y \in H$ with $y \neq x$.

In the sequel, given a sequence $\{z_n\}$, we denote with $\omega_w(z_n)$ the set of cluster points of $\{z_n\}$ with respect to the weak topology, that is,

$$\omega_w(z_n) = \{z \in H : \text{ there exists } n_k \to \infty \text{ for which } z_{n_k} \rightharpoonup z\}.$$

Analogously, we denote by $\omega_s(z_n)$ the set of cluster points of $\{z_n\}$ with respect to the norm topology, that is,

$$\omega_s(z_n) = \{z \in H : \text{ there exists } n_k \to \infty \text{ for which } z_{n_k} \to z\}.$$

Lemma 2.1. *In a real Hilbert space H, the following inequalities hold:*

(1) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \forall x, y \in H;$
(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H;$
(3) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \forall \lambda \in [0, 1], \forall x, y \in H;$

An element $x \in C$ is called a *fixed point* of S if $x \in Sx$. The set of all fixed point of S is denoted by $Fix(S)$, that is, $Fix(S) = \{x \in C : x \in Sx\}$.

Recall the following definitions. Moreover, $S : H_1 \to H_1$ is called

(1) a *nonexpansive mapping* if
$$\|Sx - Sy\| \leq \|x - y\|, \ \forall x, y \in H_1. \tag{2.2}$$
A nonexpansive mapping with $k = 1$ can be strengthened to a *firmly nonexpansive* mapping in H_1 if the following holds:
$$\|Sx - Sy\|^2 \leq \langle x - y, Sx - Sy\rangle, \ \forall x, y \in H_1. \tag{2.3}$$
We note that every nonexpansive operator $S : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality
$$\langle (x - Sx) - (y - Sy), Sy - Sx\rangle \leq \frac{1}{2}\|(Sx - x) - (Sy - y)\|^2, \tag{2.4}$$
and therefore, we obtain, for all $(x, y) \in H_1 \times Fix(S)$,
$$\langle x - Sx, y - Sx\rangle \leq \frac{1}{2}\|Sx - x\|^2 \tag{2.5}$$
(see, e.g., Theorem 3 in [16] and Theorem 1 in [17]).

(2) a *contractive* if there exists a constant $\alpha \in (0, 1)$ such that
$$\|Sx - Sy\| \leq \alpha\|x - y\|, \ \forall x, y \in H_1. \tag{2.6}$$

(3) an *L-Lipschitzian* if there exists a positive constant L such that
$$\|Sx - Sy\| \leq L\|x - y\|, \ \forall x, y \in H_1. \tag{2.7}$$

(4) an *η-strongly monotone* if there exists a positive constant η such that
$$\langle Sx - Sy, x - y\rangle \geq \eta\|x - y\|^2, \ \forall x, y \in H_1. \tag{2.8}$$

(5) an *β-inverse strongly monotone* ($\beta - ism$) if there exists a positive constant β such that

$$\langle Sx - Sy, x - y\rangle \geq \beta\|Sx - Sy\|^2,\ \forall x, y \in H_1. \tag{2.9}$$

(6) *averaged* if it can be expressed as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$S := (1-\alpha)I + \alpha T, \tag{2.10}$$

where $\alpha \in (0,1)$, I is the identity operator on H_1 and $T : H_1 \to H_1$ is nonexpansive.
It is easily seen that averaged mappings are nonexpansive. In the meantime, firmly nonexpansive mappings are averaged.

(7) A linear operator D is said to be a *strongly positive bounded linear operator* on H_1 if there exists a positive constant $\bar{\tau} > 0$ such that

$$\langle Dx, x\rangle \geq \bar{\tau}\|x\|^2, \forall x \in H_1. \tag{2.11}$$

From the definition above, we easily find that a strongly positive bounded linear operator D is $\bar{\tau}$-strongly monotone and $\|D\|$-Lipschitzian.

(8) A multivalued mapping $M : D(M) \subseteq H_1 \to 2^{H_1}$ is called *monotone* if for all $x, y \in D(M), u \in Mx$ and $v \in My$,

$$\langle x - y, u - v\rangle \geq 0. \tag{2.12}$$

A monotone mapping M is maximal if the $Graph(M)$ is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if for $x \in D(M), u \in H_1, \langle x - y, u - v\rangle \geq 0$ for each $(y, v) \in Graph(M), u \in Mx$.

(9) Let $M : D(M) \subseteq H_1 \to 2^{H_1}$ be a multivalued maximal monotone mapping. Then, the resolvent operator $J_\lambda^M : H_1 \to D(M)$ is defined by

$$J_\lambda^M x := (I + \lambda M)^{-1}(x), \forall x \in H_1, \tag{2.13}$$

for $\forall \lambda > 0$, where I stands for the identity operator on H_1. We observe that J_λ^M is single-valued, nonexpansive, and firmly nonexpansive.

We recall some concepts and results that are needed in the sequel. A mapping P_C is said to be a *metric projection* of H_1 onto C if for every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|,\ \ \forall y \in C. \tag{2.14}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle,\ \ \forall x, y \in H_1. \tag{2.15}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0,\ \ \forall x \in H_1, y \in C, \tag{2.16}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2,\ \ \forall x \in H_1, y \in C, \tag{2.17}$$

and

$$\|(x - y) - (P_C x - P_C y)\|^2 \geq \|x - y\|^2 - \|P_C x - P_C y\|^2,\ \ \forall x, y \in H_1. \tag{2.18}$$

Proposition 2.2. *[20]*

(1) If $T = (1-\alpha)S + \alpha V$, where $S : H_1 \to H_1$ is averaged, $V : H_1 \to H_1$ is nonexpansive, and if $\alpha \in [0,1]$, then T is averaged.
(2) The composite of finitely many averaged mappings is averaged.
(3) If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a nonempty common fixed point, then

$$\bigcap_{i=1}^{N} F(T_i) = F(T_1 \circ T_2 \circ \ldots \circ T_N). \tag{2.19}$$

(4) If T is a $v-ism$, then for $\gamma > 0, \gamma T$ is a $\frac{v}{\gamma} - ism$.
(5) T is averaged if and only if its complement $I - T$ is a $v - ism$ for some $v > \frac{1}{2}$.

Proposition 2.3. *[20] Let $\lambda > 0, h$ be an $\alpha - ism$ operator, and B be a maximal monotone operator. If $\lambda \in (0, 2\alpha)$, then it is easy to see that the operator $J_\lambda^B(I - \lambda h)$ is averaged.*

Proposition 2.4. *[20] Let $\lambda > 0$ and B_1 be a maximal monotone operator. Then,*

$$x^* \text{ solves } (1.9) \Leftrightarrow x^* = J_\lambda^{B_1}(I - \lambda f_1)x^* \text{ and } Ax^* = J_\lambda^{B_2}(I - \lambda f_2)Ax^*. \tag{2.20}$$

Lemma 2.5. *[23] Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$s_{n+1} \le (1 - \gamma_n)s_n + \gamma_n \delta_n, n \ge 1,$$

where $\{\gamma_n\}, \{\delta_n\}$ are the sequences of real numbers such that

(i) $\{\gamma_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$, or equivalently,

$$\prod_{n=1}^{\infty}(1 - \gamma_n) := \lim_{n\to\infty} \prod_{k=1}^{\infty}(1 - \gamma_k) = 0;$$

(ii) $\limsup_{n\to\infty} \delta_n \le 0$, or
(iii) $\sum_{n=1}^{\infty} \gamma_n \delta_n$ is convergent.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6. *[23] Let λ be a number $(0,1]$, and let $\mu > 0$. Let $F : C \to H$ be an operator on C such that for some constant $k, \eta > 0, F$ is k-Lipschitzian and η-strongly monotone. Associating with a nonexpansive mapping $T : C \to C$, we define the following the mapping $T^\lambda : C \to H$ by*

$$T^\lambda x := Tx - \lambda \mu F(Tx), \ \forall x \in C. \tag{2.21}$$

Then, T^λ is a contraction provided $\mu < \frac{2\eta}{k^2}$, that is,

$$\|T^\lambda x - T^\lambda y\| \le (1 - \lambda\tau)\|x - y\|, \ \forall x, y \in C, \tag{2.22}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0,1]$.

Lemma 2.7. *[25] Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers with $\limsup_{n\to\infty} \alpha_n < \infty$ and $\{\beta_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \beta_n \le 0$. Then, $\limsup_{n\to\infty} \alpha_n\beta_n \le 0$.*

Lemma 2.8. *[28] Assume that T is nonexpansive self-mapping of a closed convex subset C of a Hilbert space H_1. If T has a fixed point, then $I-T$ is demiclosed, i.e., whenever $\{x_n\}$ weakly converges to some x and $\{(I-T)x_n\}$ converges strongly to y, it follows that $(I-T)x = y$. Here, I is the identity mapping on H_1.*

3. Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A, and r be the spectral radius of the operator A^*A. Let $f : H_2 \to H_2$ be a ς-inverse strongly monotone operator, $B_1 : C \to 2^{H_1}, B_2 : H_2 \to 2^{H_2}$ be two multivalued maximal monotone operators, and $F : C \to H_1$ be k-Lipschitzian and η-strongly monotone. Let $\psi : C \to H_1$ be a ρ-contraction with a coefficient $\rho \in [0,1)$ and $S_i, S, T : C \to C$ be nonexpansive mappings for all $i \in \{1,\ldots,N\}$. Let $\{\lambda_n\}, \{\alpha_n\}, \{\beta_{n,i}\}, i = 1,\ldots,N$ be sequences in $(0,1)$ such that $\beta_{n,i} \to \beta_i \in (0,1)$ as $n \to \infty$ for all $i \in \{1,\ldots,N\}$, $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \xi \le \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu k^2)}$. Then, the sequence $\{x_n\}$ is generated from an arbitrary initial point $x_1 \in C$ by the following:*

$$\begin{cases} u_n = J_{\lambda_1}^{B_1}[x_n + \gamma A^*(J_{\lambda_2}^{B_2}(I-\lambda_2 f) - I)Ax_n], \\ y_{n,1} = \beta_{n,1}S_1u_n + (1-\beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}S_iu_n + (1-\beta_{n,i})y_{n,i-1}, i = 2,\ldots,N, \\ x_{n+1} = P_C[\lambda_n\xi(\alpha_n\psi(x_n) + (1-\alpha_n)Sx_n) + (I-\lambda_n\mu F)Ty_{n,N}],\ n \ge 1. \end{cases} \tag{3.1}$$

Assume that Problem 1.2 has a solution. Suppose that the following conditions are satisfied:

(C1) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(C2) $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(C3) $\sum_{n=2}^{\infty} |\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}|}{\lambda_n} = 0$;

(C4) $\sum_{n=2}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} = 0$;

(C5) $\sum_{n=2}^{\infty} |\beta_{n,i} - \beta_{n-1,i}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\lambda_n} = 0$ *for all* $i \in \{1,\ldots,N\}$;

(C6) $\lambda_1 > 0, 0 < \lambda_2 < 2\varsigma, 0 < \gamma < \frac{1}{r}$.

Then, $\{x_n\}$ converges strongly to a unique solution $x^* \in \Omega$ of Problem 1.2.

Proof. Let $\{x_n\}$ be a sequence generated by scheme (4.1). First, note that $0 < \xi \le \tau$ and

$$\begin{aligned} \mu\eta \ge \tau \quad &\Leftrightarrow \quad \mu\eta \ge 1 - \sqrt{1-\mu(2\eta-\mu k^2)} \\ &\Leftrightarrow \quad \sqrt{1-\mu(2\eta-\mu k^2)} \ge 1 - \mu\eta \\ &\Leftrightarrow \quad 1 - 2\mu\eta + \mu^2k^2 \ge 1 - 2\mu\eta + \mu^2\eta^2 \\ &\Leftrightarrow \quad k^2 \ge \eta^2 \end{aligned}$$

$$\Leftrightarrow \quad k \geq \eta.$$

Then, it follows from the ρ-contractiveness of ψ that

$$\langle(\mu F - \xi\psi)x - (\mu F - \xi\psi)y, x - y\rangle \geq (\mu\eta - \xi\rho)\|x - y\|^2, \forall x, y \in C.$$

Hence, from $\xi\rho < \xi \leq \tau \leq \mu\eta$, we deduce that $\mu F - \xi\psi$ is $(\mu\eta - \xi\rho)$-strongly monotone. Since it is clear that $\mu F - \xi\psi$ is Lipshitz continuous, there exists a unique solution to the VIP:

$$\text{find } x^* \in \Omega \text{ such that } \langle(\mu F - \xi\psi)x^*, x - x^*\rangle \geq 0, \forall x \in \Omega.$$

Additionally, since Problem 1.2 has a solution, it is easy to see that Problem 1.2 has a unique solution. In addition, taking into account condition $(C1)$, without loss of generality, we may assume that $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$.

Let $\mathcal{U} := J_{\lambda_2}^{B_2}(I - \lambda_2 f)$; the iterative scheme (4.1) can be rewritten as

$$\begin{cases} u_n = J_{\lambda_1}^{B_1}[x_n + \gamma A^*(\mathcal{U} - I)Ax_n], \\ y_{n,1} = \beta_{n,1}S_1 u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}S_i u_n + (1 - \beta_{n,i})y_{n,i-1}, i = 2, \dots, N, \\ x_{n+1} = P_C[\lambda_n\xi(\alpha_n\psi(x_n) + (1 - \alpha_n)Sx_n) + (I - \lambda_n\mu F)Ty_{n,N}], \ n \geq 1. \end{cases} \tag{3.2}$$

The rest of the proof is divided into several steps.

Step 1. We show that the sequences $\{x_n\}, \{y_{n,i}\}$ for all i, $\{u_n\}$ are bounded.

Indeed, take a point $p \in \Omega$ arbitrarily. Then, $J_{\lambda_1}^{B_1}p = p, \mathcal{U}(Ap) = Ap$, and it is easily seen that $Wp = p$, where $W := I + \gamma A^*(\mathcal{U} - I)A$. From the definition of firm nonexpansion and Proposition 2.3, we have that $J_{\lambda_1}^{B_1}$ and $\mathcal{U}$ are averaged. Likewise, W is also averaged because it is a $\frac{v}{r} - ism$ for some $v > \frac{1}{2}$. Actually, by Proposition 2.2 (5), we know that $I - \mathcal{U}$ is a $v - ism$ with $v > \frac{1}{2}$. Hence, we have

$$\begin{aligned} \langle A^*(I - \mathcal{U})Ax - A^*(I - \mathcal{U})Ay, x - y\rangle &= \langle(I - \mathcal{U})Ax - (I - \mathcal{U})Ay, Ax - Ay\rangle \\ &\geq v\|(I - \mathcal{U})Ax - (I - \mathcal{U})Ay\|^2 \\ &\geq \frac{v}{r}\|A^*(I - \mathcal{U})Ax - A^*(I - \mathcal{U})Ay\|^2. \end{aligned}$$

Thus, $\gamma A^*(I - \mathcal{U})A$ is a $\frac{v}{\gamma r} - ism$. Due to the condition $0 < \gamma < \frac{1}{r}$, the complement $I - \gamma A^*(I - \mathcal{U})A$ is averaged, as well as $M := J_{\lambda_1}^{B_1}[I + \gamma A^*(\mathcal{U} - I)A]$. Therefore, $J_{\lambda_1}^{B_1}, \mathcal{U}, W$, and M are nonexpansive mappings.

From (3.2), we estimate

$$\begin{aligned} \|u_n - p\|^2 &= \|J_{\lambda_1}^{B_1}[x_n + \gamma A^*(\mathcal{U} - I)Ax_n] - J_{\lambda_1}^{B_1}p\|^2 \\ &\leq \|x_n + \gamma A^*(\mathcal{U} - I)Ax_n - p\|^2 \\ &= \|x_n - p\|^2 + \gamma^2\|A^*(\mathcal{U} - I)Ax_n\|^2 + 2\gamma\langle x_n - p, A^*(\mathcal{U} - I)Ax_n\rangle. \end{aligned} \tag{3.3}$$

Thus, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \gamma^2\langle(\mathcal{U} - I)Ax_n, AA^*(\mathcal{U} - I)Ax_n\rangle + 2\gamma\langle x_n - p, A^*(\mathcal{U} - I)Ax_n\rangle. \tag{3.4}$$

Next, setting $\vartheta_1 := \gamma^2\langle(\mathcal{U} - I)Ax_n, AA^*(\mathcal{U} - I)Ax_n\rangle$, we estimate

$$\begin{aligned}\vartheta_1 &= \gamma^2\langle(\mathcal{U} - I)Ax_n, AA^*(\mathcal{U} - I)Ax_n\rangle \\ &\leq r\gamma^2\langle(\mathcal{U} - I)Ax_n, (\mathcal{U} - I)Ax_n\rangle \\ &= r\gamma^2\|(\mathcal{U} - I)Ax_n\|^2. \end{aligned} \tag{3.5}$$

Setting $\vartheta_2 := 2\gamma\langle x_n - p, A^*(\mathcal{U} - I)Ax_n\rangle$, we obtain from (2.5) the following:

$$\begin{aligned}\vartheta_2 &= 2\gamma\langle x_n - p, A^*(\mathcal{U} - I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - p), (\mathcal{U} - I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - p) + (\mathcal{U} - I)Ax_n - (\mathcal{U} - I)Ax_n, (\mathcal{U} - I)Ax_n\rangle \\ &= 2\gamma(\langle \mathcal{U}Ax_n - Ap, (\mathcal{U} - I)Ax_n\rangle - \|(\mathcal{U} - I)Ax_n\|^2) \\ &\leq 2\gamma\Big(\frac{1}{2}\|(\mathcal{U} - I)Ax_n\|^2 - \|(\mathcal{U} - I)Ax_n\|^2\Big) \\ &\leq -\gamma\|(\mathcal{U} - I)Ax_n\|^2. \end{aligned} \tag{3.6}$$

In view of (3.4)-(3.6), we have

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \gamma(r\gamma - 1)\|(\mathcal{U} - I)Ax_n\|^2. \tag{3.7}$$

From $0 < \gamma < \frac{1}{r}$, we obtain

$$\|u_n - p\| \leq \|x_n - p\|. \tag{3.8}$$

Thus, we have from (3.2) and (3.8) that

$$\|y_{n,1} - p\| \leq \beta_{n,1}\|S_1u_n - p\| + (1 - \beta_{n,1})\|u_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.9}$$

For all i from $i = 2$ to $i = N$, by induction, one proves that

$$\|y_{n,i} - p\| \leq \beta_{n,i}\|u_n - p\| + (1 - \beta_{n,i})\|y_{n,i-1} - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.10}$$

Hence, we obtain that for all $i \in \{1, \dots, N\}$,

$$\|y_{n,i} - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.11}$$

In addition, utilizing Lemma 2.6 and (3.2), we have

$$\begin{aligned}&\|x_{n+1} - p\| \\ =\ & \|P_C[\lambda_n\xi(\alpha_n\psi(x_n) + (1 - \alpha_n)Sx_n) + (I - \lambda_n\mu F)Ty_{n,N}] - P_Cp\| \\ \leq\ & \|\lambda_n\xi(\alpha_n\psi(x_n) + (1 - \alpha_n)Sx_n) + (I - \lambda_n\mu F)Ty_{n,N} - p\| \\ =\ & \|\lambda_n\xi(\alpha_n\psi(x_n) + (1 - \alpha_n)Sx_n) - \lambda_n\mu FTp \\ & +(I - \lambda_n\mu F)Ty_{n,N} - (I - \lambda_n\mu F)Tp\| \\ \leq\ & \|\lambda_n\xi(\alpha_n\psi(x_n) + (1 - \alpha_n)Sx_n) - \lambda_n\mu FTp\| \\ & +\|(I - \lambda_n\mu F)Ty_{n,N} - (I - \lambda_n\mu F)Tp\| \\ =\ & \lambda_n\|\alpha_n(\xi\psi(x_n) - \mu Fp) + (1 - \alpha_n)(\xi Sx_n - \mu Fp)\|\end{aligned}$$

$$
\begin{aligned}
& +\|(I-\lambda_n\mu F)Ty_{n,N}-(I-\lambda_n\mu F)Tp\| \\
\le\ & \lambda_n[\alpha_n\|\xi\psi(x_n)-\mu Fp\|+(1-\alpha_n)\|\xi Sx_n-\mu Fp\|] \\
& +(1-\lambda_n\tau)\|y_{n,N}-p\| \\
\le\ & \lambda_n[\alpha_n(\|\xi\psi(x_n)-\xi\psi(p)\|+\|\xi\psi(p)-\mu Fp\|) \\
& +(1-\alpha_n)(\|\xi Sx_n-\xi Sp\|+\|\xi Sp-\mu Fp\|)] \\
& +(1-\lambda_n\tau)\|y_{n,N}-p\| \\
\le\ & \lambda_n[\alpha_n\xi\rho\|x_n-p\|+\alpha_n\|\xi\psi(p)-\mu Fp\|+(1-\alpha_n)\xi\|x_n-p\| \\
& +(1-\alpha_n)\|\xi Sp-\mu Fp\|]+(1-\lambda_n\tau)\|x_n-p\| \\
\le\ & \lambda_n[\xi(1-\alpha_n(1-\rho))\|x_n-p\|+\max\{\|\xi\psi(p)-\mu Fp\|,\|\xi Sp-\mu Fp\|\}] \\
& +(1-\lambda_n\tau)\|x_n-p\| \\
\le\ & (1-\lambda_n\xi\alpha_n(1-\rho))\|x_n-p\|+\lambda_n\max\{\|\xi\psi(p)-\mu Fp\|,\|\xi Sp-\mu Fp\|\} \\
\le\ & (1-\lambda_n\xi a(1-\rho))\|x_n-p\|+\lambda_n\max\{\|\xi\psi(p)-\mu Fp\|,\|\xi Sp-\mu Fp\|\}, \qquad (3.12)
\end{aligned}
$$

due to $0<\xi\le\tau$. Thus, calling

$$
M=\max\left\{\|x_1-p\|,\frac{\|\xi\psi(p)-\mu Fp\|}{\xi a(1-\rho)},\frac{\|\xi Sp-\mu Fp\|}{\xi a(1-\rho)}\right\},
$$

by induction, we derive $\|x_n-p\|\le M$ for all $n\ge 1$. We thus obtain the claim.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Indeed, for each $n\ge 1$, we set

$$
z_n=\lambda_n\xi\Big(\alpha_n\psi(x_n)+(1-\alpha_n)Sx_n\Big)+(I-\lambda_n\mu F)Ty_{n,N}.
$$

Then, we observe that

$$
\begin{aligned}
z_n-z_{n-1} =\ & \alpha_n\lambda_n\xi[\psi(x_n)-\psi(x_{n-1})]+\lambda_n(1-\alpha_n)\xi(Sx_n-Sx_{n-1}) \\
& +[(I-\lambda_n\mu F)Ty_{n,N}-(I-\lambda_n\mu F)Ty_{n-1,N}] \\
& +(\alpha_n\lambda_n-\alpha_{n-1}\lambda_{n-1})\xi[\psi(x_{n-1})-Sx_{n-1}] \\
& +(\lambda_n-\lambda_{n-1})(\xi Sx_{n-1}-\mu FTy_{n-1,N}). \qquad (3.13)
\end{aligned}
$$

Let $M_0>0$ be a constant such that

$$
\sup_{n\ge1}\Big\{\xi\|\psi(x_n)-Sx_n\|+\|\xi Sx_n-\mu FTy_{n,N}\|\Big\}\le M_0.
$$

It follows from (3.2) and (3.13) that

$$
\begin{aligned}
& \|x_{n+1}-x_n\| \\
=\ & \|P_Cz_n-P_Cz_{n-1}\| \\
\le\ & \|z_n-z_{n-1}\| \\
\le\ & \alpha_n\lambda_n\xi\|\psi(x_n)-\psi(x_{n-1})\|+\lambda_n(1-\alpha_n)\xi\|Sx_n-Sx_{n-1}\| \\
& +\|(I-\lambda_n\mu F)Ty_{n,N}-(I-\lambda_n\mu F)Ty_{n-1,N}\|
\end{aligned}
$$

$$
\begin{aligned}
&+|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}|\xi\|\psi(x_{n-1}) - Sx_{n-1}\| \\
&+|\lambda_n - \lambda_{n-1}|\|\xi Sx_{n-1} - \mu FTy_{n-1,N}\| \\
\leq\ & \alpha_n\lambda_n\xi\rho\|x_n - x_{n-1}\| + \lambda_n(1-\alpha_n)\xi\|x_n - x_{n-1}\| \\
&+(1-\lambda_n\tau)\|y_{n,N} - y_{n-1,N}\| \\
&+|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}|M_0 + |\lambda_n - \lambda_{n-1}|M_0 \\
=\ & \lambda_n(1-\alpha_n(1-\rho))\xi\|x_n - x_{n-1}\| + (1-\lambda_n\tau)\|y_{n,N} - y_{n-1,N}\| \\
&+[|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|]M_0 \\
\leq\ & \lambda_n\xi(1-a(1-\rho))\|x_n - x_{n-1}\| + (1-\lambda_n\tau)\|y_{n,N} - y_{n-1,N}\| \\
&+[|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|]M_0. \qquad (3.14)
\end{aligned}
$$

By the definition of $y_{n,i}$, we obtain that for all $i = N, \dots, 2$,

$$
\begin{aligned}
\|y_{n,i} - y_{n-1,i}\| \leq\ & \beta_{n,i}\|u_n - u_{n-1}\| + \|S_iu_{n-1} - y_{n-1,i-1}\||\beta_{n,i} - \beta_{n-1,i}| \\
&+(1-\beta_{n,i})\|y_{n,i-1} - y_{n-1,i-1}\|. \qquad (3.15)
\end{aligned}
$$

In this case $i = 1$, we have

$$
\begin{aligned}
\|y_{n,1} - y_{n-1,1}\| \leq\ & \beta_{n,1}\|u_n - u_{n-1}\| \\
&+\|S_1u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}| + (1-\beta_{n,1})\|u_n - u_{n-1}\| \\
=\ & \|u_n - u_{n-1}\| + \|S_1u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|. \qquad (3.16)
\end{aligned}
$$

Substituting (3.16) in all (3.15)-type inequalities, we find that for $i = 2, \dots, N$,

$$
\begin{aligned}
\|y_{n,i} - y_{n-1,i}\| \leq\ & \|u_n - u_{n-1}\| + \sum_{k=2}^{N}\|S_ku_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\
&+\|S_1u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|.
\end{aligned}
$$

Thus, we conclude that

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
\leq\ & \lambda_n\xi(1-a(1-\rho))\|x_n - x_{n-1}\| + (1-\lambda_n\tau)\|y_{n,N} - y_{n-1,N}\| \\
&+[|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|]M_0 \\
\leq\ & \lambda_n\xi(1-a(1-\rho))\|x_n - x_{n-1}\| + [|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|]M_0 \\
&+(1-\lambda_n\tau)\|u_n - u_{n-1}\| + \sum_{k=2}^{N}\|S_ku_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\
&+\|S_1u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|. \qquad (3.17)
\end{aligned}
$$

Since $J_{\lambda_1}^{B_1}[I + \gamma A^*(\mathcal{U} - I)A]$ is nonexpansive, we obtain

$$
\begin{aligned}
\|u_n - u_{n-1}\| =\ & \|J_{\lambda_1}^{B_1}[I + \gamma A^*(\mathcal{U} - I)A]x_n - J_{\lambda_1}^{B_1}[I + \gamma A^*(\mathcal{U} - I)A]x_{n-1}\| \\
\leq\ & \|x_n - x_{n-1}\|. \qquad (3.18)
\end{aligned}
$$

Substituting (3.18) into (3.17), we have

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
\leq\ & \lambda_n\xi(1-a(1-\rho))\|x_n-x_{n-1}\| + [|\alpha_n\lambda_n-\alpha_{n-1}\lambda_{n-1}| + |\lambda_n-\lambda_{n-1}|]M_0 \\
&+(1-\lambda_n\tau)\|x_n-x_{n-1}\| + \sum_{k=2}^{N}\|S_k u_{n-1}-y_{n-1,k-1}\||\beta_{n,k}-\beta_{n-1,k}| \\
&+\|S_1u_{n-1}-u_{n-1}\||\beta_{n,1}-\beta_{n-1,1}|. \qquad (3.19)
\end{aligned}$$

If we call $M_1 := \max\left\{M_0, \sup_{n\geq 2, i=2,\ldots,N}\|S_iu_{n-1}-y_{n-1,i-1}\|, \sup_{n\geq 2}\|S_1u_{n-1}-u_{n-1}\|\right\}$, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| \leq\ & (1-\lambda_n\xi a(1-\rho))\|x_n-x_{n-1}\| \\
&+M_1\Big[|\alpha_n\lambda_n-\alpha_{n-1}\lambda_{n-1}| + |\lambda_n-\lambda_{n-1}| \\
&+\sum_{k=2}^{N}|\beta_{n,k}-\beta_{n-1,k}|\Big], \qquad (3.20)
\end{aligned}$$

due to $0<\xi<\tau$. By condition $(C2)-(C5)$ and Lemma 2.5, we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \qquad (3.21)$$

Step 3. We show that $\lim_{n\to\infty}\|x_n-u_n\| = 0$.

From (3.2) and (3.7), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & \|\lambda_n\xi(\alpha_n\psi(x_n)+(1-\alpha_n)Sx_n)+(I-\lambda_n\mu F)Ty_{n,N}-p\|^2 \\
=\ & \|\lambda_n\xi(\alpha_n\psi(x_n)+(1-\alpha_n)Sx_n)-\lambda_n\mu FTp \\
&+(I-\lambda_n\mu F)Ty_{n,N}-(I-\lambda_n\mu F)Tp\|^2 \\
\leq\ & \{\|\lambda_n\xi(\alpha_n\psi(x_n)+(1-\alpha_n)Sx_n)-\lambda_n\mu FTp\| \\
&+\|(I-\lambda_n\mu F)Ty_{n,N}-(I-\lambda_n\mu F)Tp\|\}^2 \\
\leq\ & \{\lambda_n\|\alpha_n(\xi\psi(x_n)-\mu Fp)+(1-\alpha_n)(\xi Sx_n-\mu Fp)\| \\
&+(1-\lambda_n\tau)\|y_{n,N}-p\|\}^2 \\
\leq\ & \lambda_n\frac{1}{\tau}[\alpha_n\|\xi\psi(x_n)-\mu Fp\|+(1-\alpha_n)\|\xi Sx_n-\mu Fp\|]^2 \\
&+(1-\lambda_n\tau)\|y_{n,N}-p\|^2 \\
\leq\ & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2 \\
&+(1-\lambda_n\tau)\|u_n-p\|^2 \\
\leq\ & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2 \\
&+(1-\lambda_n\tau)[\|x_n-p\|^2+\gamma(r\gamma-1)\|(\mathcal{U}-I)Ax_n\|^2] \\
=\ & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2 \\
&+(1-\lambda_n\tau)\|x_n-p\|^2-\gamma(1-r\gamma)(1-\lambda_n\tau)\|(\mathcal{U}-I)Ax_n\|^2, \qquad (3.22)
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\lambda_n\tau)\gamma(1-r\gamma)\|(\mathcal{U}-I)Ax_n\|^2\\
\le\ & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2+(1-\lambda_n\tau)\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
\le\ & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
\le\ & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2+\|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}p\|). \qquad (3.23)
\end{aligned}$$

Since $\gamma(1-r\gamma)>0, \|x_{n+1}-x_n\|\to 0, \lambda_n\to 0$ as $n\to\infty$, and by the boundedness of $\{x_n\}$, we conclude that

$$\lim_{n\to\infty}\|(\mathcal{U}-I)Ax_n\|=0. \qquad (3.24)$$

In addition, by the firm nonexpansion of $J_{\lambda_1}^{B_1}$, (3.3), (3.7), and $\gamma\in(0,\frac{1}{r})$, we estimate

$$\begin{aligned}
\|u_n-p\|^2 &= \|J_{\lambda_1}^{B_1}[x_n+\gamma A^*(\mathcal{U}-I)Ax_n]-J_{\lambda_1}^{B_1}p\|^2\\
&\le \langle J_{\lambda_1}^{B_1}[x_n+\gamma A^*(\mathcal{U}-I)Ax_n]-J_{\lambda_1}^{B_1}p, x_n+\gamma A^*(\mathcal{U}-I)Ax_n-p\rangle\\
&= \langle u_n-p, x_n+\gamma A^*(\mathcal{U}-I)Ax_n-p\rangle\\
&= \frac{1}{2}(\|u_n-p\|^2+\|x_n+\gamma A^*(\mathcal{U}-I)Ax_n-p\|^2\\
&\quad -\|(u_n-p)-[x_n+\gamma A^*(\mathcal{U}-I)Ax_n-p]\|^2)\\
&\le \frac{1}{2}[\|u_n-p\|^2+\|x_n-p\|^2+\gamma(r\gamma-1)\|(\mathcal{U}-I)Ax_n\|^2\\
&\quad -\|u_n-x_n-\gamma A^*(\mathcal{U}-I)Ax_n\|^2]\\
&\le \frac{1}{2}[\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n-\gamma A^*(\mathcal{U}-I)Ax_n\|^2]\\
&= \frac{1}{2}[\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2-\gamma^2\|A^*(\mathcal{U}-I)Ax_n\|^2\\
&\quad +2\gamma\langle u_n-x_n, A^*(\mathcal{U}-I)Ax_n\rangle]\\
&\le \frac{1}{2}[\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\langle u_n-x_n, A^*(\mathcal{U}-I)Ax_n\rangle]\\
&= \frac{1}{2}[\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\langle A(u_n-x_n), (\mathcal{U}-I)Ax_n\rangle]\\
&\le \frac{1}{2}[\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n-x_n)\|\|(\mathcal{U}-I)Ax_n\|],
\end{aligned}$$

and hence,

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n-x_n)\|\|(\mathcal{U}-I)Ax_n\|. \qquad (3.25)$$

In view of (3.22) and (3.25),

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
\le\ & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2\\
&+(1-\lambda_n\tau)\|u_n-p\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 \\
&\quad +(1-\lambda_n\tau)[\|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\|A(u_n - x_n)\|\|(\mathcal{U} - I)Ax_n\|] \\
&= \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 \\
&\quad +(1-\lambda_n\tau)\|x_n - p\|^2 - (1-\lambda_n\tau)\|u_n - x_n\|^2 + 2\gamma(1-\lambda_n\tau)\|A(u_n - x_n)\|\|(\mathcal{U} - I)Ax_n\|, \quad (3.26)
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\lambda_n\tau)\|u_n - x_n\|^2 \\
\leq\ & \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 \\
&+(1-\lambda_n\tau)\|x_n - p\|^2 + 2\gamma(1-\lambda_n\tau)\|A(u_n - x_n)\|\|(\mathcal{U} - I)Ax_n\| - \|x_{n+1} - p\|^2 \\
\leq\ & \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 \\
&+2\gamma(1-\lambda_n\tau)\|A(u_n - x_n)\|\|(\mathcal{U} - I)Ax_n\| + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
\leq\ & \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 \\
&+2\gamma(1-\lambda_n\tau)\|A(u_n - x_n)\|\|(\mathcal{U} - I)Ax_n\| + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|). \quad (3.27)
\end{aligned}$$

Since $\|x_{n+1} - x_n\| \to 0, \|(\mathcal{U} - I)Ax_n\| \to 0$, and $\lambda_n \to 0$ as $n \to \infty$, and owing to the boundedness of $\{x_n\}$, we conclude that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \quad (3.28)$$

Step 4. We show that $\lim_{n\to\infty}\|S_i u_n - u_n\| = 0$ for $i \in \{1, \ldots, N\}$.

Take a point $p \in \Omega$ arbitrarily. When $i = N$, utilizing Lemma 2.8 and (3.2), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|\lambda_n\xi(\alpha_n\psi(x_n) + (1-\alpha_n)Sx_n) + (I - \lambda_n\mu F)Ty_{n,N} - p\|^2 \\
&= \|\lambda_n\xi(\alpha_n\psi(x_n) + (1-\alpha_n)Sx_n) - \lambda_n\mu FTp \\
&\quad +(I - \lambda_n\mu F)Ty_{n,N} - (I - \lambda_n\mu F)Tp\|^2 \\
&\leq \{\|\lambda_n\xi(\alpha_n\psi(x_n) + (1-\alpha_n)Sx_n) - \lambda_n\mu FTp\| \\
&\quad +\|(I - \lambda_n\mu F)Ty_{n,N} - (I - \lambda_n\mu F)Tp\|\}^2 \\
&\leq \{\lambda_n\|\alpha_n(\xi\psi(x_n) - \mu Fp) + (1-\alpha_n)(\xi Sx_n - \mu Fp)\| \\
&\quad +(1-\lambda_n\tau)\|y_{n,N} - p\|\}^2 \\
&\leq \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 \\
&\quad +(1-\lambda_n\tau)\|y_{n,N} - p\|^2 \\
&= \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 \\
&\quad +(1-\lambda_n\tau)\beta_{n,N}\|S_N u_N - p\|^2 \\
&\quad +(1-\lambda_n\tau)(1-\beta_{n,N})\|y_{n,N-1} - p\|^2 \\
&\quad -(1-\lambda_n\tau)(1-\beta_{n,N})\beta_{n,N}\|S_N u_n - y_{n,N-1}\|^2 \\
&\leq \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2
\end{aligned}$$

$$
\begin{aligned}
& +(1-\lambda_n\tau)\|u_n-p\|^2 \\
& -(1-\lambda_n\tau)(1-\beta_{n,N})\beta_{n,N}\|S_Nu_n-y_{n,N-1}\|^2 \\
\leq\; & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2+\|x_n-p\|^2 \\
& -(1-\lambda_n\tau)(1-\beta_{n,N})\beta_{n,N}\|S_Nu_n-y_{n,N-1}\|^2. \qquad (3.29)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& (1-\lambda_n\tau)(1-\beta_{n,N})\beta_{n,N}\|S_Nu_n-y_{n,N-1}\|^2 \\
\leq\; & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
\leq\; & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2+\|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|). \qquad (3.30)
\end{aligned}
$$

Since $\beta_{n,N}\to\beta_N\in(0,1)$, $\|x_{n+1}-x_n\|\to 0$ and $\lambda_n\to 0$ as $n\to\infty$, by the boundedness of $\{x_n\}$, we conclude that

$$
\lim_{n\to\infty}\|S_Nu_n-y_{n,N-1}\|=0. \qquad (3.31)
$$

Take $i\in\{1,\ldots,N-1\}$ arbitrarily. Then, we obtain

$$
\begin{aligned}
& \|x_{n+1}-p\|^2 \\
\leq\; & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2+(1-\lambda_n\tau)\|y_{n,N}-p\|^2 \\
\leq\; & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2 \\
& +(1-\lambda_n\tau)[\beta_{n,N}\|S_Nu_n-p\|^2+(1-\beta_{n,N})\|y_{n,N-1}-p\|^2] \\
\leq\; & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2 \\
& +(1-\lambda_n\tau)\beta_{n,N}\|x_n-p\|^2+(1-\lambda_n\tau)(1-\beta_{n,N})\|y_{n,N-1}-p\|^2 \\
\leq\; & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2+(1-\lambda_n\tau)\beta_{n,N}\|x_n-p\|^2 \\
& +(1-\lambda_n\tau)(1-\beta_{n,N})[\beta_{n,N-1}\|S_{N-1}u_n-p\|^2+(1-\beta_{n,N-1})\|y_{n,N-2}-p\|^2] \\
\leq\; & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2 \\
& +(1-\lambda_n\tau)(\beta_{n,N}+(1-\beta_{n,N})\beta_{n,N-1})\|x_n-p\|^2 \\
& +(1-\lambda_n\tau)\prod_{k=N-1}^{N}(1-\beta_{n,k})\|y_{n,N-2}-p\|^2. \qquad (3.32)
\end{aligned}
$$

Hence, after $(N-i+1)$-iterations,

$$
\begin{aligned}
& \|x_{n+1}-p\|^2 \\
\leq\; & \lambda_n\frac{1}{\tau}[\|\xi\psi(x_n)-\mu Fp\|+\|\xi Sx_n-\mu Fp\|]^2 \\
& +(1-\lambda_n\tau)\Big(\beta_{n,N}+\sum_{j=i+2}^{N}\Big(\prod_{p=j}^{N}(1-\beta_{n,p}\Big)\beta_{n,j-1}\Big)
\end{aligned}
$$

$$
\begin{aligned}
& \times\|x_n - p\|^2 + (1-\lambda_n\tau)\prod_{k=i+1}^{N}(1-\beta_{n,k})\|y_{n,i} - p\|^2 \\
\leq\ & \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 \\
& +(1-\lambda_n\tau)\Big(\beta_{n,N} + \sum_{j=i+2}^{N}\Big(\prod_{p=j}^{N}(1-\beta_{n,p})\Big)\beta_{n,j-1}\Big) \\
& \times\|x_n - p\|^2 + (1-\lambda_n\tau)\prod_{k=i+1}^{N}(1-\beta_{n,k}) \\
& \times[\beta_{n,i}\|S_i u_n - p\|^2 + (1-\beta_{n,i})\|y_{n,i-1} - p\|^2 \\
& -\beta_{n,i}(1-\beta_{n,i})\|S_i u_n - y_{n,i-1}\|^2] \\
\leq\ & \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 + (1-\lambda_n\tau)\|x_n - p\|^2 \\
& -\beta_{n,i}(1-\lambda_n\tau)\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i u_n - y_{n,i-1}\|^2. \qquad (3.33)
\end{aligned}
$$

Again, we obtain

$$
\begin{aligned}
& \beta_{n,i}(1-\lambda_n\tau)\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i u_n - y_{n,i-1}\|^2 \\
\leq\ & \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
\leq\ & \lambda_n \frac{1}{\tau}[\|\xi\psi(x_n) - \mu F p\| + \|\xi S x_n - \mu F p\|]^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|). \qquad (3.34)
\end{aligned}
$$

Since for all $k \in \{1, \ldots, N\}, \beta_{n,k} \to \beta_k \in (0,1), \|x_{n+1} - x_n\| \to 0$ and $\lambda_n \to 0$ as $n \to \infty$, by the boundedness of $\{x_n\}$, we conclude that

$$
\lim_{n\to\infty}\|S_i u_n - y_{n,i-1}\| = 0. \qquad (3.35)
$$

Obviously, for $i = 1$, we have $\lim_{n\to\infty}\|S_1 u_n - u_n\| = 0$. To conclude, we have that

$$
\|S_2 u_n - u_n\| \leq \|S_2 u_n - y_{n,1}\| + \|y_{n,1} - u_n\| = \|S_2 u_n - y_{n,1}\| + \beta_{n,1}\|S_1 u_n - u_n\|, \qquad (3.36)
$$

which implies that $\lim_{n\to\infty}\|S_2 u_n - u_n\| = 0$. Consequently, by induction, we obtain

$$
\lim_{n\to\infty}\|S_i u_n - u_n\| = 0 \ \text{ for all } i = 2, \ldots, N. \qquad (3.37)
$$

It is enough to observe that

$$
\begin{aligned}
\|S_i u_n - u_n\| &\leq \|S_i u_n - y_{n,i-1}\| + \|y_{n,i-1} - S_{i-1} u_n\| + \|S_{i-1} u_n - u_n\| \\
&\leq \|S_i u_n - y_{n,i-1}\| + (1-\beta_{n,i-1})\|S_{i-1} u_n - y_{n,i-2}\| + \|S_{i-1} u_n - u_n\|. \qquad (3.38)
\end{aligned}
$$

Step 5. We show that $\lim_{n\to\infty}\|y_{n,N} - x_n\| = \lim_{n\to\infty}\|x_n - T x_n\| = 0$ and $\omega_w(x_n) \subset \Omega$.

Indeed, since $\|x_n - u_n\| \to 0$ as $n \to \infty$, we have $\omega_w(x_n) = \omega_w(u_n)$ and $\omega_s(x_n) = \omega_s(u_n)$. Now, we observe that

$$\|x_n - y_{n,1}\| \le \|x_n - u_n\| + \|y_{n,1} - u_n\| = \|x_n - u_n\| + \beta_{n,1}\|S_1 u_n - u_n\|. \tag{3.39}$$

By Step 4, $\|S_1 u_n - u_n\| \to 0$ as $n \to \infty$. Hence, we obtain

$$\lim_{n\to\infty} \|x_n - y_{n,1}\| = 0. \tag{3.40}$$

This implies that $\omega_w(x_n) = \omega_w(y_{n,1})$ and $\omega_s(x_n) = \omega_s(y_{n,1})$.

Take a point $q \in \omega_w(x_n)$ arbitrarily. Since $q \in \omega_w(u_n)$, by Step 4 and the demiclosedness principle, we have $q \in Fix(S_i)$ for all $i \in \{1, \dots, N\}$, that is, $q \in \bigcap_i Fix(S_i)$. Moreover, note that

$$\|y_{n,N} - x_n\| \le \sum_{k=2}^{N} \|y_{n,k} - y_{n,k-1}\| + \|y_{n,1} - x_n\| = \sum_{k=2}^{N} \beta_{n,k}\|S_k u_n - y_{n,k-1}\| + \|y_{n,1} - x_n\|; \tag{3.41}$$

hence,

$$\begin{aligned}
& \|x_n - T x_n\| \\
\le\ & \|x_n - x_{n+1}\| + \|x_{n+1} - T y_{n,N}\| + \|T y_{n,N} - T x_n\| \\
\le\ & \|x_n - x_{n+1}\| + \|\lambda_n \xi(\alpha_n \psi(x_n) + (1-\alpha_n) S x_n) + (I - \lambda_n \mu F) T y_{n,N} - T y_{n,N}\| \\
& + \|y_{n,N} - x_n\| \\
=\ & \|x_n - x_{n+1}\| + \lambda_n \|\alpha_n(\xi\psi(x_n) - \mu F T y_{n,N}) + (1-\alpha_n)(\xi S x_n - \mu F T y_{n,N})\| \\
& + \|y_{n,N} - x_n\| \\
\le\ & \|x_n - x_{n+1}\| + \lambda_n[\|\xi\psi(x_n) - \mu F T y_{n,N}\| + \|\xi S x_n - \mu F T y_{n,N}\|] + \|y_{n,N} - x_n\| \\
\le\ & \|x_n - x_{n+1}\| + \lambda_n[\|\xi\psi(x_n) - \mu F T y_{n,N}\| + \|\xi S x_n - \mu F T y_{n,N}\|] \\
& + \sum_{k=2}^{N} \beta_{n,k}\|S_k u_n - y_{n,k-1}\| + \|y_{n,1} - x_n\|.
\end{aligned} \tag{3.42}$$

Since $\|x_n - x_{n+1}\| \to 0$, $\lambda_n \to 0$, $\|y_{n,1} - x_n\| \to 0$, $\beta_{n,k} \to \beta_k$ and $\|S_k u_n - y_{n,k-1}\| \to 0$ for all $k \in \{1, \dots, N\}$, we obtain

$$\lim_{n\to\infty} \|y_{n,N} - x_n\| = \lim_{n\to\infty} \|x_n - T x_n\| = 0. \tag{3.43}$$

Thus, by the demiclosedness principle, we have $q \in Fix(T)$.

In addition, we rewrite $u_{n_k} = J_{\lambda_1}^{B_1}[x_{n_k} + \gamma A^*(\mathcal{U} - I)A x_{n_k}]$ as

$$\frac{x_{n_k} - u_{n_k} + \gamma A^*(\mathcal{U} - I)A x_{n_k}}{\lambda_1} \in B_1 u_{n_k}. \tag{3.44}$$

Taking $k \to \infty$ in (3.44) and using (3.24), (3.28) and the fact that the graph of a maximal monotone operator is weakly strongly closed, we have $0 \in B_1 q$, i.e., $q \in SOLVIP(B_1)$. Furthermore, since x_n and u_n have the same asymptotical behavior, $A x_{n_k}$ weakly converges to Aq. It follows from (3.24), the nonexpansion of $\mathcal{U}$, and Lemma 2.8 that $(I - \mathcal{U})Aq = 0$. Thus, by Proposition 2.4, we have $0 \in f(Aq) + B_2(Aq)$, i.e., $Aq \in SOLVIP(B_2)$. As a result, $q \in \Gamma$. This shows that $q \in \Omega$. Therefore, we obtain the claim.

Step 6. We show that $\{x_n\}$ converges strongly to a unique solution x^* to Problem 1.2.

Indeed, according to $\|x_{n+1} - x_n\| \to 0$, we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying

$$\begin{aligned}\limsup_{n\to\infty}\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle &= \limsup_{n\to\infty}\langle(\xi\psi - \mu F)x^*, x_n - x^*\rangle \\ &= \lim_{j\to\infty}\langle(\xi\psi - \mu F)x^*, x_{n_j} - x^*\rangle. \end{aligned} \tag{3.45}$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \tilde{x}$; then, $\tilde{x} \in \Omega$, as we have just proved. Since x^* is a solution to Problem 1.2, we obtain

$$\limsup_{n\to\infty}\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle = \langle(\xi\psi - \mu F)x^*, \tilde{x} - x^*\rangle \le 0. \tag{3.46}$$

Repeating the same argument as that of (3.46), we have

$$\limsup_{n\to\infty}\langle(\xi S - \mu F)x^*, x_{n+1} - x^*\rangle \le 0. \tag{3.47}$$

From (3.2) and (3.9), it follows (noticing that $x_{n+1} = P_C z_n$ and $0 < \xi \le \tau$) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
= \ & \langle z_n - x^*, x_{n+1} - x^*\rangle + \langle P_C z_n - z_n, P_C z_n - x^*\rangle \\
\le \ & \langle z_n - x^*, x_{n+1} - x^*\rangle \\
= \ & \langle (I - \lambda_n\mu F)Ty_{n,N} - (I - \lambda_n\mu F)x^*, x_{n+1} - x^*\rangle \\
& +\alpha_n\lambda_n\xi\langle\psi(x_n) - \psi(x^*), x_{n+1} - x^*\rangle + \lambda_n(1 - \alpha_n)\xi\langle Sx_n - Sx^*, x_{n+} - x^*\rangle \\
& +\alpha_n\lambda_n\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle + \lambda_n(1 - \alpha_n)\langle(\xi S - \mu F)x^*, x_{n+1} - x^*\rangle \\
\le \ & [1 - \lambda_n\tau + \alpha_n\lambda_n\xi\rho + \lambda_n(1 - \alpha_n)\xi]\|x_n - x^*\|\|x_{n+1} - x^*\| \\
& +\alpha_n\lambda_n\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle + \lambda_n(1 - \alpha_n)\langle(\xi S - \mu F)x^*, x_{n+1} - x^*\rangle \\
\le \ & [1 - \alpha_n\lambda_n\xi(1 - \rho)]\|x_n - x^*\|\|x_{n+1} - x^*\| \\
& +\alpha_n\lambda_n\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle + \lambda_n(1 - \alpha_n)\langle(\xi S - \mu F)x^*, x_{n+1} - x^*\rangle \\
\le \ & [1 - \alpha_n\lambda_n\xi(1 - \rho)]\frac{1}{2}(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
& +\alpha_n\lambda_n\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle + \lambda_n(1 - \alpha_n)\langle(\xi S - \mu F)x^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.48}$$

It turns out that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\le \ & \frac{1 - \alpha_n\lambda_n\xi(1 - \rho)}{1 + \alpha_n\lambda_n\xi(1 - \rho)}\|x_n - x^*\|^2 + \frac{2}{1 + \alpha_n\lambda_n\xi(1 - \rho)}[\alpha_n\lambda_n\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle \\
& +\lambda_n(1 - \alpha_n)\langle(\xi S - \mu F)x^*, x_{n+1} - x^*\rangle] \\
\le \ & [1 - \alpha_n\lambda_n\xi(1 - \rho)]\|x_n - x^*\|^2 + \frac{2}{1 + \alpha_n\lambda_n\xi(1 - \rho)}[\alpha_n\lambda_n\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle \\
& +\lambda_n(1 - \alpha_n)\langle(\xi S - \mu F)x^*, x_{n+1} - x^*\rangle] \\
= \ & [1 - \alpha_n\lambda_n\xi(1 - \rho)]\|x_n - x^*\|^2 + \alpha_n\lambda_n\xi(1 - \rho)\Big\{\frac{2}{\xi(1 - \rho)[1 + \alpha_n\lambda_n\xi(1 - \rho)]}
\end{aligned}$$

$$
\times\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle
+\frac{2(1-\alpha_n)}{\alpha_n\xi(1-\rho)[1+\alpha_n\lambda_n\xi(1-\rho)]}\langle(\xi S - \mu F)x^*, x_{n+1} - x^*\rangle\Big\}. \tag{3.49}
$$

Put $s_n = \|x_n - x^*\|^2, \xi_n = \alpha_n\lambda_n\xi(1-\rho)$ and

$$
\begin{aligned}
\delta_n \;=\; & \frac{2}{\xi(1-\rho)[1+\alpha_n\lambda_n\xi(1-\rho)]}\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle \\
& + \frac{2(1-\alpha_n)}{\alpha_n\xi(1-\rho)[1+\alpha_n\lambda_n\xi(1-\rho)]}\langle(\xi S - \mu F)x^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

Then, (3.49) can be rewritten as

$$
s_{n+1} \leq (1-\gamma_n)s_n + \xi_n\delta_n.
$$

From conditions $(C1)$ and $(C2)$, we conclude from $0 < 1-\rho \leq 1$ that

$$
\{\xi_n\} \subset [0,1] \text{ and } \sum_{n=1}^{\infty}\xi_n = \infty.
$$

Note that

$$
\frac{2}{\xi(1-\rho)[1+\alpha_n\lambda_n\xi(1-\rho)]} \leq \frac{2}{\xi(1-\rho)}
$$

and

$$
\frac{2(1-\alpha_n)}{\alpha_n\xi(1-\rho)[1+\alpha_n\lambda_n\xi(1-\rho)]} \leq \frac{2}{a\xi(1-\rho)}.
$$

Consequently, utilizing Lemma 2.5, we find that

$$
\begin{aligned}
\limsup_{n\to\infty}\delta_n \;\leq\; & \limsup_{n\to\infty}\frac{2}{\xi(1-\rho)[1+\alpha_n\lambda_n\xi(1-\rho)]}\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle \\
& + \limsup_{n\to\infty}\frac{2(1-\alpha_n)}{\alpha_n\xi(1-\rho)[1+\alpha_n\lambda_n\xi(1-\rho)]}\langle(\xi\psi - \mu F)x^*, x_{n+1} - x^*\rangle \\
\leq\; & 0.
\end{aligned}
$$

Thus, this, together with Lemma 2.5, leads to $\lim_{n\to\infty}\|x_n - x^*\| = 0$. The proof is complete.

□

In Theorem 3.1, if $\lambda_1 = \lambda_2 = \lambda$ and $f = 0$, the we obtain the following corollary immediately.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A, and r be the spectral radius of the operator A^*A. Let $B_1 : C \to 2^{H_1}, B_2 : H_2 \to 2^{H_2}$ be two multivalued maximal monotone operators, and $F : C \to H_1$ be k-Lipschitzian and η-strongly monotone. Let $\psi : C \to H_1$ be a ρ-contraction with a coefficient $\rho \in [0,1)$ and $S_i, S, T : C \to C$ be nonexpansive mappings for all $i \in \{1,\ldots,N\}$. Let $\{\lambda_n\}, \{\alpha_n\}, \{\beta_{n,i}\}, i = 1,\ldots,N$ be sequences in $(0,1)$ such that $\beta_{n,i} \to \beta_i \in (0,1)$ as $n \to \infty$ for all $i \in \{1,\ldots,N\}$, $0 < \mu < \frac{2\eta}{k^2}$ and*

$0 < \xi \leq \tau$, *where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. *Then, the sequence* $\{x_n\}$ *is generated from an arbitrary initial point* $x_1 \in C$ *by the following:*

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n], \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, i = 2, \ldots, N, \\ x_{n+1} = P_C[\lambda_n \xi(\alpha_n \psi(x_n) + (1 - \alpha_n) S x_n) + (I - \lambda_n \mu F) T y_{n,N}],\ n \geq 1. \end{cases} \tag{3.50}$$

Suppose that the following conditions are satisfied:

(C1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(C2) $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(C3) $\sum_{n=2}^{\infty} |\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}|}{\lambda_n} = 0$;

(C4) $\sum_{n=2}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} = 0$;

(C5) $\sum_{n=2}^{\infty} |\beta_{n,i} - \beta_{n-1,i}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\lambda_n} = 0$ *for all* $i \in \{1, \ldots, N\}$;

(C6) $0 < \gamma < \frac{1}{r}$.

Then, $\{x_n\}$ *converges strongly to a unique solution* $x^* \in Fix(T) \cap (\bigcap_i Fix(S_i)) \cap SVIP$.

4. Numerical illustration and application

Here as a numerical illustration, we consider a split common fixed points of a family of nonexpansive mappings, which is a particular case of problem 1.2. To that end, we have the following, which is an equivalent formulation of Theorem 3.1.

Let C be a nonempty closed convex subset of a real Hilbert space H_1 and Q be a nonempty closed convex subset of a real Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A, and r be the spectral radius of the operator A^*A. Let $f : H_2 \to H_2$ be a ς-inverse strongly monotone operator, and $F : C \to H_1$ be k-Lipschitzian and η-strongly monotone. Let $\psi : C \to H_1$ be a ρ-contraction with a coefficient $\rho \in [0, 1)$ and $S_i, S, T : C \to C$ be nonexpansive mappings for all $i \in \{1, \ldots, N\}$. Let $\{\lambda_n\}, \{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \ldots, N$ be sequences in $(0, 1)$ such that $\beta_{n,i} \to \beta_i \in (0, 1)$ as $n \to \infty$ for all $i \in \{1, \ldots, N\}$, $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \xi \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Then, the sequence $\{x_n\}$ is generated from an arbitrary initial point $x_1 \in C$ by the following:

$$\begin{cases} u_n = x_n + \gamma A^*(I - \lambda_2 f)Ax_n, \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, i = 2, \ldots, N, \\ x_{n+1} = P_C[\lambda_n \xi(\alpha_n \psi(x_n) + (1 - \alpha_n) S x_n) + (I - \lambda_n \mu F) T y_{n,N}],\ n \geq 1. \end{cases} \tag{4.1}$$

Assume that the problem

$$\langle (\mu F - \xi\psi)x^*, x - x^* \rangle \geq 0, \forall x \in \Omega, \tag{4.2}$$

has a solution. Suppose that the following conditions are satisfied:

(C1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(C2) $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(C3) $\sum_{n=2}^{\infty} |\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| < \infty$ or $\lim_{n\to\infty} \frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}|}{\lambda_n} = 0$;

(C4) $\sum_{n=2}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ or $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} = 0$;

(C5) $\sum_{n=2}^{\infty} |\beta_{n,i} - \beta_{n-1,i}| < \infty$ or $\lim_{n\to\infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\lambda_n} = 0$ for all $i \in \{1, \ldots, N\}$;

(C6) $\lambda_1 > 0, 0 < \lambda_2 < 2\varsigma, 0 < \gamma < \frac{1}{r}$.

Then, $\{x_n\}$ converges strongly to a unique solution $x^* \in \Omega$ of Problem (4.2). Suppose $H = C = \mathbb{R}$, for each $x \in \mathbb{R}$ the mappings S_i and T_i are defined as follows

$$S_i x = \frac{i}{i+1} x$$

and

$$T_i(x) = \begin{cases} x, & x \in (-\infty, 0), \\ 2x, & x \in [0, \infty). \end{cases} \tag{4.3}$$

Observe that S_i for $i \geq 1$ are nonexpansive and T is $\frac{1}{3}$-demicontractive mapping [29]. Take $\beta_{n,i} = \frac{6}{n^2 i^2}, \alpha_n = \frac{3}{n^2}$ and $\lambda_n = \frac{1}{n^2+2}$. Also define $\psi(x) = \frac{2x}{3}$ and $Ax = 2x$ with $\|A\| = 2$. Therefore it can be seen that the sequences satisfy the conditions in the (C1) - (C6).

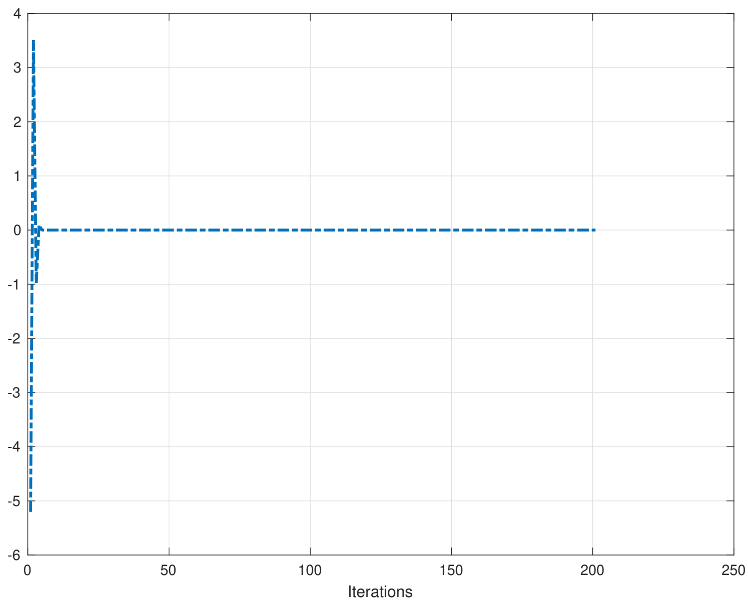

Figure 1. Plot of the iterative sequence after 200 iterations.

It can be observed from Figure1, that the sequence $\{x_n\}$ generated converges to 0, which is the only element of the solution set, i.e $\Omega = \{0\}$.

5. Conclusion

In this paper, we first propose triple hierarchical variational inequality problem (4.1) in Theorem 3.1 and then we prove some strong convergence of the sequence $\{x_n\}$ generated by (4.1) to a common solution of variational inequality problem, split monotone variational inclusion problem and fixed point problems. We divide the proof into 6 steps and our theorem is extends and improves the corresponding results of Jitsupa et al. [1] and Kazmi and Rizvi [22].

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Conflict of interest

The authors declare that they have no competing interests.

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