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Research article

Some generalized fractional integral Simpson's type inequalities with applications

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Abstract: In the article, we establish a Simpson-type generalized identity containing multi-parameters and derive some new estimates for the generalized Simpson's quadrature rule via the Raina fractional integrals. As applications, we provide several inequalities for the f-divergence measures and probability density functions.

Keywords: Simpson's inequality; cumulative distribution function; fractional integral; f-divergence measure
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1. Introduction

Over few years, the fractional calculus has attracted the attention of many researchers due to its has wide applications in pure and applied mathematics [1–7]. Like ordinary calculus, the fractional integral and derivative have not unique representation, with the passage of time, different authors have different representations. It is well-known that inequality is an indispensable research object in mathematics, it can give explicit error bounds for some known and some new quadrature formulae, for example, the Simpson's inequality [8], Jensen's inequality [9, 10], Hermite-Hadamard's inequality [11–15] and integral inequalities [16–21]. The following inequality is well known as Simpson's inequality which provides an error bound for the Simpson's rule.

Theorem 1.1. (See [7]) Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$ be a four times differentiable function on (a, b) such that $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}|(x) < \infty$. Then the inequality

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{\|f^{(4)}\|_\infty (b-a)^4}{2880}$$

holds.

In [22], Dragomir et al. improved Theorem 1.1 to the following Theorem 1.2.

Theorem 1.1. (See [22]) Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that its derivative is continuous on (a, b) and $\|f'\|_1 = \int_a^b |f'(x)|dx < \infty$. Then the inequality

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{3}\|f'\|_1$$

holds.

Recently, the Simpson type inequalities have been the subject of intensive research since many important inequalities can be obtained from the Simpson inequality. The main purpose of the article is to provide several generalized fractional integral versions of the Simpson's inequality and give their applications in f-divergence measures and probability density functions.

2. Preliminaries and assumptions

Definition 2.1. (See [23]) Let $p \in \mathbb{R}$ with $p \neq 0$ and $I \subseteq (0, \infty)$ be an interval. Then the real-valued function $f : I \to \mathbb{R}$ is said to be p-convex (concave) if the inequality

$$f(\sqrt[p]{tx^p + (1-t)y^p}) \leq (\geq) tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. Moreover, if $s \in (0, 1]$, then the function f is said to be (s, p)-convex (concave) if the inequality

$$f(\sqrt[p]{tx^p + (1-t)y^p}) \leq (\geq) t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Let $p > 0$, $s \in (0, 1]$ and $f : (0, \infty) \to (0, \infty)$ be defined by $f(x) = x^{sp}$. Then we clearly see that f is a (s, p)-convex function.

Definition 2.2. (See [24]) Let $\alpha > 0$, $[a, b] \subseteq \mathbb{R}$ be a finite interval and $f : [a, b] \to \mathbb{R}$ be a real-valued function such that $f \in L[a, b]$. Then the right-hand side and the left-hand side Riemann-Liouville fractional integrals $\mathcal{J}^\alpha_{a+}f$ and $\mathcal{J}^\alpha_{b-}f$ of order α are defined by

$$\left(\mathcal{J}^\alpha_{a+}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt \quad (x > a) \tag{2.1}$$

and

$$\left(\mathcal{J}_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt \quad (x < b), \tag{2.2}$$

respectively.

Definition 2.3. The gamma function Γ, beta function $\mathbb{B}$ and the hypergeometric function ${}_2F_1$ are defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^x dt \quad (x > 0),$$

$$\mathbb{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (x,y > 0)$$

and

$${}_2F_1(a,b;c,z) = \frac{1}{\mathbb{B}(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt \quad (|z| < 1),$$

respectively.

Definition 2.4 (See [25]) The hypergeometric function ${}_2F_1$ can be given by

$${}_2F_1(a,[b,c;y],x) = \frac{1}{\mathbb{B}(b,c-b)} \int_0^y t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt$$

for $y < 1$ and $Re(c) > Re(b) > 0$.

Definition 2.5 The incomplete beta function $\mathbb{B}(z;x,y)$ is defined by

$$\mathbb{B}(z;x,y) = \int_0^z t^{x-1}(1-t)^{y-1}dt \quad (Re(x) > Re(y) > 0, 0 \le z < 1).$$

Raina [26] introduced a class of functions as follows

$$\mathfrak{F}_{\rho,\lambda}^{\sigma}(x) = \mathfrak{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\cdots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k \quad (\rho,\lambda \in \mathbb{R}^+, x \in \mathbb{R}), \tag{2.3}$$

where the coefficients $\sigma(k) \in \mathbb{R}^+$, $k \in \mathbb{N}_0$ form a bounded sequence. By using (2.3), in [26, 27], the authors defined the left-side and right-sided fractional integral operators

$$\left(\mathfrak{J}_{\rho,\lambda,a+;w}^{\sigma}\phi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathfrak{F}_{\rho,\lambda}^{\sigma}[w(x-t)^{\rho}]\phi(t)dt \quad (x > a) \tag{2.4}$$

and

$$\left(\mathfrak{J}_{\rho,\lambda,b-;w}^{\sigma}\phi\right)(x) = \int_x^b (t-x)^{\lambda-1} \mathfrak{F}_{\rho,\lambda}^{\sigma}[w(t-x)^{\rho}]\phi(t)dt \quad (x < b), \tag{2.5}$$

respectively, where $w \in \mathbb{R}$ and ϕ is a function such that the integrals on right hand sides exit. It is easy to verify that $\mathfrak{J}_{\rho,\lambda,a+;w}^{\sigma}\phi(x)$ and $\mathfrak{J}_{\rho,\lambda,b-;w}^{\sigma}\phi(x)$ are bounded integral operators on $L(a,b)$ if $\mathfrak{M} = \mathfrak{F}_{\rho,\lambda+1}^{\sigma}[w(b-a)^{\rho}] < \infty$. In fact, if $\phi \in L(a,b)$, then we have

$$\left\|\mathfrak{J}_{\rho,\lambda,a+;w}^{\sigma}\phi\right\|_1 \le \mathfrak{M}(b-a)^{\lambda}\|\phi\|_1, \quad \left\|\mathfrak{J}_{\rho,\lambda,b-;w}^{\sigma}\phi\right\|_1 \le \mathfrak{M}(b-a)^{\lambda}\|\phi\|_1.$$

Let $\lambda \to \alpha$, adn $\sigma(0) \to 1$ and $w \to 0$ in (2.4) and (2.5), respectively, then we get (2.1) and (2.2). Before starting our main results, we introduce some notations as follows.

$$G_1(a,b;\xi,\varrho,\varepsilon) = {}_2F_1\left(\frac{\xi p-\xi}{p},\beta+\xi\rho k+\varepsilon+1;\xi\beta+\xi\rho k+\varrho+2,\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right),$$

$$G_2(a,b;\xi,\varrho,\varepsilon) = {}_2F_1\left(\frac{\xi p-\xi}{p},\left[\xi\beta+\xi\rho k+\varepsilon+1,\xi\beta+\xi\rho k+\varrho+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right),$$

$$H_{1,\lambda}(a,b;\pi,\varpi) = {}_2F_1\left(\frac{p-1}{p},\pi,\varpi,\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right),$$

$$H_{2,\lambda}(a,b;\pi,\varpi) = {}_2F_1\left(\frac{p-1}{p},\left[\pi,\varpi;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right),$$

$$\sigma_1 = \sigma(k)\left\{(\lambda|f'(a)|+(1-\lambda)|f'(b)|)\,\mathbb{B}(\beta+\rho k+1,s+1)G_1(a,b;1,s,0)\right.$$
$$\left.+G_1(a,b;1,s,s)|f'(b)|\mathbb{B}(\beta+\rho k+s+1,1)\right\},$$

$$\sigma_2 = \left\{[G_2(a,b;x,0,0)]^{\frac{1}{x}}\left[(2^{s+1}(1-\lambda)+\lambda)|f'(b)|^y+\lambda(2^{s+1}-1)|f'(a)|^y\right]^{\frac{1}{y}}\right.$$
$$\left.+[G_1(a,b;x,0,0)-G_2(a,b;x,0,0)]^{\frac{1}{x}}\left[(2^{s+1}-\lambda)|f'(b)|^y+\lambda|f'(a)|^y\right]^{\frac{1}{y}}\right\}$$
$$\times\sigma(k)\frac{[\mathbb{B}(x\beta+x\rho k+1,1)]^{\frac{1}{x}}}{[2^{s+1}(s+1)]^{\frac{1}{y}}},$$

$$\sigma_3 = \sigma(k)[\mathbb{B}(\beta+\rho k+1,1)]^{\frac{x-1}{x}}\left\{[G_2(a,b;1,0,0)]^{\frac{x-1}{x}}\left[\{\lambda|f'(a)|^x+(1-\lambda)|f'(b)|^x\}\right.\right.$$
$$\left.\times\mathbb{B}(\beta+\rho k+1,s+1)G_2(a,b;1,s,0)+G_2(a,b;1,s,s)|f'(b)|^x\mathbb{B}(\beta+\rho k+s+1,1)\right]^{\frac{1}{x}}$$
$$+[G_1(a,b;1,0,0)-G_2(a,b;1,0,0)]^{\frac{x-1}{x}}\left[\{\lambda|f'(a)|^x+(1-\lambda)|f'(b)|^x\}\mathbb{B}(\beta+\rho k+1,s+1)\right.$$
$$\times\{G_1(a,b;1,s,0)-G_2(a,b;1,s,0)\}+|f'(b)|^x\{G_1(a,b;1,s,s)-G_2(a,b;1,s,s)\}$$
$$\left.\left.\times\mathbb{B}(\beta+\rho k+s+1,1)\right]^{\frac{1}{x}}\right\}.$$

3. Main results

To establish our results for generalized Simpson's type inequality using (s,p)-convex function, we need the following lemma.

Lemma 3.1. Let $I \subseteq \mathbb{R}^+$ be an interval, I° be the interval of I, $a,b \in I$ with $a<b$, $\rho,\beta>0$, $p\in\mathbb{R}$ with $p\neq 0$, and $g(\xi)=\sqrt[p]{\xi}$ for $\xi>0$. Then the identity

$$\psi(t,a,b;f) = p\,[\lambda(b^p-a^p)]^\beta\left[\left\{2^{-\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{4}\right)^\rho\right]-u\right\}\right.$$
$$\left.\times f\left(\sqrt[p]{\frac{\lambda a^p+(2-\lambda)b^p}{2}}\right)+uf\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)\right]-p\left(\mathfrak{I}^{\sigma}_{\rho,\beta,\left[\frac{\lambda a^p+(2-\lambda)b^p}{2}\right]^-;\frac{w}{2^p}}f\circ g\right)$$

$$\times(\lambda a^p+(1-\lambda)b^p)+p\,[\lambda(b^p-a^p)]^\beta\left[\left\{\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho\right]-v\right\}f(b)\right.$$

$$-\left\{2^{-\beta}\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{4}\right)^\rho\right]-v\right\}f\left(\sqrt[p]{\frac{\lambda a^p+(2-\lambda)b^p}{2}}\right)\Bigg]-p\left\{\left(\mathfrak{J}^\sigma_{\rho,\beta,b^p-;\frac{w}{2^\rho}}f\circ g\right)\right.$$

$$\left.\times(\lambda a^p+(1-\lambda)b^p)-\left(\mathfrak{J}^\sigma_{\rho,\beta,\left[\frac{\lambda a^p+(2-\lambda)b^p}{2}\right]^-;\frac{w}{2^\rho}}f\circ g\right)(\lambda a^p+(1-\lambda)b^p)\right\}$$

$$=[\lambda(b^p-a^p)]^{1+\beta}\int_0^{\frac{1}{2}}\left\{t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]-u\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$

$$\times f'\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)dt+[\lambda(b^p-a^p)]^{1+\beta}$$

$$\times\int_{\frac{1}{2}}^1\left\{t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]-v\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$

$$\times f'\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)dt. \tag{3.1}$$

holds for $u,w\in\mathbb{R}$ and $\lambda\in[0,1]$.

Proof. Integrating by parts leads to

$$I_1=\int_0^{\frac{1}{2}}\left\{t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]-u\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$

$$\times f'\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)dt$$

$$=\left|\frac{pt^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]-pu}{\lambda(b^p-a^p)}f\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)\right|_0^{\frac{1}{2}}$$

$$-\int_0^{\frac{1}{2}}\frac{pt^{\beta-1}\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]}{\lambda(b^p-a^p)}f\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)dt$$

$$=\frac{p}{\lambda(b^p-a^p)}\left[\left\{2^{-\beta}\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{4}\right)^\rho\right]-u\right\}f\left(\sqrt[p]{\frac{\lambda a^p+(2-\lambda)b^p}{2}}\right)\right.$$

$$\left.+uf\left(\sqrt[p]{(\lambda a^p+(1-\lambda)b^p)}\right)\right]-\int_0^{\frac{1}{2}}\frac{pt^{\beta-1}\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]}{\lambda(b^p-a^p)}$$

$$\times f\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)dt.$$

Let $x=(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p$. Then $dx=[\lambda(b^p-a^p)]dt$, $0\le t\le 1/2$ is equivalent to $(\lambda a^p+(1-\lambda)b^p)\le x\le\frac{\lambda a^p+(2-\lambda)b^p}{2}$. Therefore, we get

$$I_1=\frac{p}{\lambda(b^p-a^p)}\left[\left\{2^{-\beta}\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{4}\right)^\rho\right]-u\right\}f\left(\sqrt[p]{\frac{\lambda a^p+(2-\lambda)b^p}{2}}\right)\right.$$

$$\left.+uf\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)\right]-\frac{p}{[\lambda(b^p-a^p)]^{1+\beta}}\int_{\lambda a^p+(1-\lambda)b^p}^{\frac{\lambda a^p+(2-\lambda)b^p}{2}}[x-(\lambda a^p+(1-\lambda)b^p)]^{\beta-1}$$

$$\times \mathfrak{F}^{\sigma}_{\rho,\beta}\left[w\left(\frac{x-(\lambda a^p+(1-\lambda)b^p)}{2}\right)^{\rho}\right](f\circ g)(x)dx$$

and

$$\frac{[\lambda(b^p-a^p)]^{1+\beta}}{p}I_1 = [\lambda(b^p-a^p)]^{\beta}\left[\left\{2^{-\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{4}\right)^{\rho}\right]-u\right\}\right.$$
$$\left.\times f\left(\sqrt[p]{\frac{\lambda a^p+(2-\lambda)b^p}{2}}\right)+uf\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)\right]$$
$$-\left(\mathfrak{J}^{\sigma}_{\rho,\beta,\left[\frac{\lambda a^p+(2-\lambda)b^p}{2}\right]^-;\frac{w}{2^{\rho}}}f\circ g\right)(\lambda a^p+(1-\lambda)b^p). \tag{3.2}$$

Again integrating by parts gives

$$I_2 = \int_{\frac{1}{2}}^{1}\left\{t^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}t^{\rho}\right]-v\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$
$$\times f'\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)dt$$
$$=\left|\frac{pt^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}t^{\rho}\right]-pv}{\lambda(b^p-a^p)}f\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)\right|_{\frac{1}{2}}^{1}$$
$$-\int_{\frac{1}{2}}^{1}\frac{pt^{\beta-1}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}t^{\rho}\right]}{\lambda(b^p-a^p)}f\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)dt$$
$$=\frac{p}{\lambda(b^p-a^p)}\left[\left\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}\right]-v\right\}f(b)\right.$$
$$\left.-\left\{2^{-\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{4}\right)^{\rho}\right]-v\right\}f\left(\sqrt[p]{\frac{\lambda a^p+(2-\lambda)b^p}{2}}\right)\right]$$
$$-\int_{\frac{1}{2}}^{1}\frac{pt^{\beta-1}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}t^{\rho}\right]}{\lambda(b^p-a^p)}f\left(\sqrt[p]{(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p}\right)dt.$$

Let $y=(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p$. Then one has

$$I_2 = \frac{p}{\lambda(b^p-a^p)}\left[\left\{\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}\right]-v\right\}f(b)\right.$$
$$\left.-\left\{2^{-\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{4}\right)^{\rho}\right]-v\right\}f\left(\sqrt[p]{\frac{\lambda a^p+(2-\lambda)b^p}{2}}\right)\right]$$
$$-\frac{p}{[\lambda(b^p-a^p)]^{1+\beta}}\int_{\frac{\lambda a^p+(2-\lambda)b^p}{2}}^{b^p}[y-(\lambda a^p+(1-\lambda)b^p)]^{\beta-1}$$
$$\times\mathfrak{F}^{\sigma}_{\rho,\beta}\left[w\left(\frac{y-(\lambda a^p+(1-\lambda)b^p)}{2}\right)^{\rho}\right](f\circ g)(y)dy$$

$$= \frac{p}{\lambda(b^p - a^p)} \left[\left\{ \mathfrak{F}^{\sigma}_{\rho,\beta+1} \left[w \left(\frac{\lambda(b^p - a^p)}{2} \right)^{\rho} \right] - v \right\} f(b) \right.$$

$$- \left\{ 2^{-\beta} \mathfrak{F}^{\sigma}_{\rho,\beta+1} \left[w \left(\frac{\lambda(b^p - a^p)}{4} \right)^{\rho} \right] - v \right\} f \left(\sqrt[p]{\frac{\lambda a^p + (2-\lambda) b^p}{2}} \right)$$

$$- \int_{\lambda a^p + (1-\lambda) b^p}^{\frac{\lambda a^p + (2-\lambda) b^p}{2}} [y - (\lambda a^p + (1-\lambda) b^p)]^{\beta - 1} \Bigg]$$

$$\times \mathfrak{F}^{\sigma}_{\rho,\beta} \left[w \left(\frac{y - (\lambda a^p + (1-\lambda) b^p)}{2} \right)^{\rho} \right] (f \circ g)(y) dy$$

and

$$\frac{[\lambda(b^p - a^p)]^{1+\beta}}{p} I_2 = [\lambda(b^p - a^p)]^{\beta} \left[\left\{ \mathfrak{F}^{\sigma}_{\rho,\beta+1} \left[w \left(\frac{\lambda(b^p - a^p)}{2} \right)^{\rho} \right] - v \right\} f(b) \right.$$

$$\left. - \left\{ 2^{-\beta} \mathfrak{F}^{\sigma}_{\rho,\beta+1} \left[w \left(\frac{\lambda(b^p - a^p)}{4} \right)^{\rho} \right] - v \right\} f \left(\sqrt[p]{\frac{\lambda a^p + (2-\lambda) b^p}{2}} \right) \right]$$

$$- \left\{ \left(\mathfrak{J}^{\sigma}_{\rho,\beta,b^{p-};\frac{w}{2^\rho}} f \circ g \right) (\lambda a^p + (1-\lambda) b^p) - \left(\mathfrak{J}^{\sigma}_{\rho,\beta,\left[\frac{\lambda a^p + (2-\lambda) b^p}{2}\right]^-;\frac{w}{2^\rho}} f \circ g \right) (\lambda a^p + (1-\lambda) b^p) \right\}. \tag{3.3}$$

Therefore, the desired inequality (3.1) can be obtained by adding (3.2) and (3.3). $\square$

Remark 3.1. From Lemma 3.1 we clearly see that
(1) Lemma 3.1 reduces to Lemma 1 of [8] if $\beta, \lambda, \sigma(0) \to 1$, $w \to 0$, $u \to \frac{1}{6}$ and $v \to \frac{5}{6}$;
(2) Lemma 3.1 leads to Lemma 2 of [8] if $p, \beta, \lambda, \sigma(0) \to 1$, $w \to 0$ and $a \to am$;
(3) Lemma 3.1 becomes Lemma 2.1 of [28] if $p, \beta, \lambda, \sigma(0) \to 1$, $w \to 0$; $u \to \frac{1}{6}$, $v \to \frac{5}{6}$ and $a \to am$.
(4) Lemma 3.1 degenerates into Lemma 3 of [8] if $\lambda, \sigma(0) \to 1$, $w \to 0$ and $p > 0$.

Theorem 3.1. Let $I \subseteq \mathbb{R}^+$ be an interval and I° be the interior of I, $a, b \in I^\circ$ with $a < b$, $\rho, \beta > 0$, $u, w \in \mathbb{R}$, $(s, \lambda) \in (0, 1] \times [0, 1]$, $g(\xi) = \sqrt[p]{\xi}$ for $\xi > 0$, and $f : I \to \mathbb{R}$ be a differentiable function on I° such that $|f'|$ is (s, p)-convex. Then one has

$$|\psi(t, a, b; f)| \leq \frac{[\lambda(b^p - a^p)]^{1+\beta}}{\left(\sqrt[p]{(\lambda a^p + (1-\lambda) b^p)}\right)^{p-1}} \left\{ \mathfrak{F}^{\sigma_1}_{\rho,\beta+1} \left[|w| \left(\frac{\lambda(b^p - a^p)}{2} \right)^{\rho} \right] \right.$$

$$+ \frac{(\lambda |f'(a)| + (1-\lambda)|f'(b)|)\{(|u| - |v|) H_{2,\lambda}(a, b; 1, s+2) + |v| H_{1,\lambda}(a, b; 1, s+2)\}}{s+1}$$

$$\left. + \frac{|f'(b)|\{(|u| - |v|) H_{2,\lambda}(a, b; s+1, s+2) + |v| H_{1,\lambda}(a, b; s+1, s+2)\}}{s+1} \right\}. \tag{3.4}$$

Proof. It follows from (3.1) and the (s, p)-convexity of $|f'|$ that

$$|\psi(t, a, b; f)| \leq [\lambda(b^p - a^p)]^{1+\beta} [|I_1| + |I_2|], \tag{3.5}$$

$$|I_1| = \left| \int_0^{\frac{1}{2}} \left\{ t^\beta \mathfrak{F}^{\sigma}_{\rho,\beta+1} \left[w \left(\frac{\lambda(b^p - a^p)}{2} \right)^{\rho} t^\rho \right] - u \right\} [(1-t)(\lambda a^p + (1-\lambda) b^p) + t b^p]^{\frac{1-p}{p}} \right.$$

$$
\begin{aligned}
&\times f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)dt\Bigg| \\
&\leq \int_0^{\frac{1}{2}}\left\{t^\beta \mathfrak{F}^\sigma_{\rho,\beta+1}\left[|w|\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]+|u|\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}} \\
&\times\left|f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)\right|dt \\
&=\int_0^{\frac{1}{2}}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}+|u|\right\} \\
&\times[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}\left|f'\left(\left((1-t)(\sqrt[p]{\lambda a^p+(1-\lambda)b^p})^p+tb^p\right)^{\frac{1}{p}}\right)\right|dt \\
&\leq \int_0^{\frac{1}{2}}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}+|u|\right\} \\
&\times\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{1-p}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}} \\
&\times[(1-t)^s\{\lambda|f'(a)|+(1-\lambda)|f'(b)|\}+t^s|f'(b)|]\,dt \\
&=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{p-1}\Gamma(\rho k+\beta+1)}\{(\lambda|f'(a)| \\
&+(1-\lambda)|f'(b)|)\int_0^{\frac{1}{2}}t^{\beta+\rho k}(1-t)^s\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt \\
&+|f'(b)|\int_0^{\frac{1}{2}}t^{\beta+\rho k+s}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\right\} \\
&+|u|\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{1-p}\{(\lambda|f'(a)|+(1-\lambda)|f'(b)|) \\
&\times\int_0^{\frac{1}{2}}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}(1-t)^s dt \\
&+|f'(b)|\int_0^{\frac{1}{2}}t^s\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\right\} \\
&=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{p-1}\Gamma(\rho k+\beta+1)} \\
&\times\{(\lambda|f'(a)|+(1-\lambda)|f'(b)|)\mathbb{B}(\beta+\rho k+1,s+1) \\
&\times{}_2F_1\left(\frac{p-1}{p},\left[\beta+\rho k+1,s+\beta+\rho k+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+{}_2F_1\left(\frac{p-1}{p},\left[\beta+\rho k+s+1,\beta+\rho k+s+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\\
&\times|f'(b)|\mathbb{B}\left(\beta+\rho k+s+1,1\right)\}+|u|\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{1-p}\\
&\times\{(\lambda|f'(a)|+(1-\lambda)|f'(b)|)\mathbb{B}(1,s+1)\\
&\times{}_2F_1\left(\frac{p-1}{p},\left[1,s+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\\
&+|f'(b)|\mathbb{B}(s+1,1){}_2F_1\left(\frac{p-1}{p},\left[s+1,s+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\Bigg\}
\end{aligned}\tag{3.6}
$$

and

$$
\begin{aligned}
|I_2|=&\left|\int_{\frac{1}{2}}^{1}\left\{t^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}t^{\rho}\right]-v\right\}\left[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p\right]^{\frac{1-p}{p}}\right.\\
&\left.\times f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)dt\right|\\
\leq&\int_{\frac{1}{2}}^{1}\left\{t^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[|w|\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}t^{\rho}\right]+|v|\right\}\left[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p\right]^{\frac{1-p}{p}}\\
&\times\left|f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)\right|dt\\
=&\int_{\frac{1}{2}}^{1}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}+|v|\right\}\\
&\times\left[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p\right]^{\frac{1-p}{p}}\\
&\times\left|f'\left(((1-t)(\sqrt[p]{\lambda a^p+(1-\lambda)b^p})^p+tb^p)^{\frac{1}{p}}\right)\right|dt\\
\leq&\int_{\frac{1}{2}}^{1}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}+|v|\right\}\\
&\times\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{1-p}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}\\
&\times\left[(1-t)^s\{\lambda|f'(a)|+(1-\lambda)|f'(b)|\}+t^s|f'(b)|\right]dt\\
=&\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{p-1}\Gamma(\rho k+\beta+1)}\{(\lambda|f'(a)|\\
&+(1-\lambda)|f'(b)|)\int_{\frac{1}{2}}^{1}t^{\beta+\rho k}(1-t)^s\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\\
&+|f'(b)|\int_{\frac{1}{2}}^{1}t^{\beta+\rho k+s}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\Bigg\}
\end{aligned}
$$

$$+|v|\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{1-p}\{(\lambda|f'(a)|+(1-\lambda)|f'(b)|)$$

$$\times\int_{\frac{1}{2}}^{1}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}(1-t)^s dt$$

$$+|f'(b)|\int_{\frac{1}{2}}^{1}t^s\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\right\}$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{p-1}}\{\mathbb{B}(\beta+\rho k+1,s+1)$$

$$\times\left[{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+1;s+\beta+\rho k+2,\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right.$$

$$\left.-{}_2F_1\left(\frac{p-1}{p},\left[\beta+\rho k+1,s+\beta+\rho k+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right]$$

$$\times(\lambda|f'(a)|+(1-\lambda)|f'(b)|)+|f'(b)|\mathbb{B}(\beta+\rho k+s+1,1)$$

$$\times\left[{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+s+1,\beta+\rho k+s+2,\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right.$$

$$\left.\left.-{}_2F_1\left(\frac{p-1}{p},\left[\beta+\rho k+s+1,\beta+\rho k+s+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right]\right\}$$

$$+\frac{|v|}{\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{p-1}}\{(\lambda|f'(a)|+(1-\lambda)|f'(b)|)$$

$$\times\mathbb{B}(1,s+1)\left[{}_2F_1\left(\frac{p-1}{p},1,s+2,\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right.$$

$$\left.-{}_2F_1\left(\frac{p-1}{p},\left[1,s+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right]+|f'(b)|$$

$$\times\mathbb{B}(s+1,1)\left[{}_2F_1\left(\frac{p-1}{p},s+1,s+2,\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right.$$

$$\left.\left.-{}_2F_1\left(\frac{p-1}{p},\left[s+1,s+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right]\right\}. \tag{3.7}$$

Therefore, inequality (3.4) follows easily from (3.5)–(3.7). □

Corollary 3.1. Let $I\subseteq\mathbb{R}^+$ be an interval and I° be the interior of I, $a,b\in I^\circ$ with $a<b$, $\beta>0$, $u,w\in\mathbb{R}^+$, $s\in(0,1]$, $p<0$, $g(\xi)=\sqrt[p]{\xi}$ for $\xi>0$, and $f:I\to\mathbb{R}$ be a differentiable function on I° such that $|f'|$ is (s,p)-convex. Then one has

$$\left|uf(a)+(1-v)f(b)+(v-u)f\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)-\frac{1}{b^p-a^p}\int_{a^p}^{b^p}(f\circ g)(x)dx\right|$$
$$\leq \frac{b^p-a^p}{p(s+1)a^{p-1}}\left[(u-v)\left\{|f'(a)|_2F_1\left(\frac{p-1}{p},\left[1,s+2;\frac{1}{2}\right],\frac{a^p-b^p}{a^p}\right)\right.\right.$$
$$\left.+|f'(b)|_2F_1\left(\frac{p-1}{p},\left[s+1,s+2;\frac{1}{2}\right],\frac{a^p-b^p}{a^p}\right)\right\}+v\left\{|f'(a)|\right.$$
$$\left.\left.\times{}_2F_1\left(\frac{p-1}{p},1,s+2,\frac{a^p-b^p}{a^p}\right)+|f'(b)|_2F_1\left(\frac{p-1}{p},s+1,s+2,\frac{a^p-b^p}{a^p}\right)\right\}\right]. \tag{3.8}$$

Proof. Let $\beta,\lambda,\sigma(0)\to 1$ and $w=0$. Then Corollary 3.1 follows directly from Theorem 3.1. □

Theorem 3.2. Let $I\subseteq\mathbb{R}^+$ be an interval and I° be the interior of I, $a,b\in I^\circ$ with $a<b$, $\rho,\beta>0$, $(s,\lambda)\in(0,1]\times[0,1]$, $p\in\mathbb{R}$ with $p\neq 0$, $u,w,\in\mathbb{R}$, $y>1$, $x=y/(y-1)$, $g(\xi)=\sqrt[p]{\xi}$ for $\xi>0$, and $f:I\to\mathbb{R}$ be a differentiable function on I° such that $|f'|^y$ is (s,p)-convex. Then the inequality

$$|\psi(t,a,b;f)|\leq\frac{[\lambda(b^p-a^p)]^{1+\beta}}{\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{x(p-1)}}\left\{\mathfrak{F}^{\sigma_2}_{\rho,\beta+1}\left[|w|\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho\right]\right.$$
$$+\frac{\left[\frac{\lambda a^p+(1-\lambda)b^p}{\lambda(a^p-b^p)}\right]^{\frac{1}{x}}}{[2^{s+1}(s+1)]^{\frac{1}{y}}}\left\{|u|\left[\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{2(\lambda a^p+(1-\lambda)b^p)};1,x\frac{p-1}{p}+1\right)\right]^{\frac{1}{x}}\right.$$
$$\times\left\{(2^{s+1}(1-\lambda)+\lambda)|f'(b)|^y+\lambda(2^{s+1}-1)|f'(a)|^y\right\}^{\frac{1}{y}}$$
$$+|v|\left[\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p};1,x\frac{p-1}{p}+1\right)-\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{2(\lambda a^p+(1-\lambda)b^p)};1,x\frac{p-1}{p}+1\right)\right]^{\frac{1}{x}}$$
$$\left.\left.\times\left\{(2^{s+1}-\lambda)|f'(b)|^y+\lambda|f'(a)|^y\right\}^{\frac{1}{y}}\right\}\right\} \tag{3.9}$$

holds.

Proof. It follows from the (s,p)-convexity of $|f'|^y$ and Hölder inequality that

$$|I_1|=\left|\int_0^{\frac{1}{2}}\left\{t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]-u\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}\right.$$
$$\left.\times f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)dt\right|$$
$$\leq\int_0^{\frac{1}{2}}\left\{t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[|w|\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]+|u|\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$
$$\times\left|f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)\right|dt$$

$$= \int_0^{\frac{1}{2}} \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)} t^{\beta+\rho k} + |u| \right\}$$

$$\times [(1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p]^{\frac{1-p}{p}}$$

$$\times \left| f'\left(\left((1-t)(\sqrt[p]{\lambda a^p + (1-\lambda)b^p})^p + tb^p \right)^{\frac{1}{p}} \right) \right| dt$$

$$= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}$$

$$\times \left\{ \int_0^{\frac{1}{2}} t^{x(\beta+\rho k)} \left[(1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p\right]^{\frac{x(1-p)}{p}} dt \right\}^{\frac{1}{x}}$$

$$\times \left\{ \int_0^{\frac{1}{2}} \left| f'\left(\left((1-t)(\sqrt[p]{\lambda a^p + (1-\lambda)b^p})^p + tb^p \right)^{\frac{1}{p}} \right) \right|^y dt \right\}^{\frac{1}{y}}$$

$$+ |u| \left\{ \int_0^{\frac{1}{2}} \left[(1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p\right]^{\frac{x(1-p)}{p}} dt \right\}^{\frac{1}{x}}$$

$$\times \left\{ \int_0^{\frac{1}{2}} \left| f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p + (1-\lambda)b^p}\right)^p + tb^p \right)^{\frac{1}{p}} \right) \right|^y dt \right\}^{\frac{1}{y}}$$

$$= \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[p]{\lambda a^p + (1-\lambda)b^p}\right)^{x(p-1)}} \right.$$

$$\times \left[\int_0^{\frac{1}{2}} t^{x(\beta+\rho k)} \left[1 - t\frac{\lambda(a^p-b^p)}{\lambda a^p + (1-\lambda)b^p}\right]^{\frac{x(1-p)}{p}} dt \right]^{\frac{1}{x}}$$

$$+ \frac{|u|}{\left(\sqrt[p]{\lambda a^p + (1-\lambda)b^p}\right)^{x(p-1)}}$$

$$\left. \times \left[\int_0^{\frac{1}{2}} \left[1 - t\frac{\lambda(a^p-b^p)}{\lambda a^p + (1-\lambda)b^p}\right]^{\frac{x(1-p)}{p}} dt \right]^{\frac{1}{x}} \right\}$$

$$\times \left\{ \int_0^{\frac{1}{2}} [(1-t)^s\{\lambda|f'(a)|^y + (1-\lambda)|f'(b)|^y\} + t^s|f'(b)|^y] dt \right\}^{\frac{1}{y}}$$

$$= \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[p]{\lambda a^p + (1-\lambda)b^p}\right)^{x(p-1)}} \sqrt[x]{\mathbb{B}(x\beta + x\rho k + 1, 1)} \right.$$

$$\times\left[{}_2F_1\left(x\frac{p-1}{p},\left[x\beta+x\rho k+1,x\beta+x\rho k+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right]^{\frac{1}{x}}$$

$$+\frac{|u|}{\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{x(p-1)}}$$

$$\times\left[\frac{\lambda a^p+(1-\lambda)b^p}{\lambda(a^p-b^p)}\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{2(\lambda a^p+(1-\lambda)b^p)};1,x\frac{p-1}{p}+1\right)\right]^{\frac{1}{x}}\Bigg\}$$

$$\times\left\{\frac{2^{s+1}(1-\lambda)+\lambda}{2^{s+1}(s+1)}|f'(b)|^y+\frac{\lambda(2^{s+1}-1)}{2^{s+1}(s+1)}|f'(a)|^y\right\}^{\frac{1}{y}} \tag{3.10}$$

and

$$|I_2|=\left|\int_{\frac{1}{2}}^{1}\left\{t^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}t^{\rho}\right]-v\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}\right.$$

$$\left.\times f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)dt\right|$$

$$\leq\int_{\frac{1}{2}}^{1}\left\{t^{\beta}\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[|w|\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho}t^{\rho}\right]+|v|\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$

$$\times\left|f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)\right|dt$$

$$=\int_{\frac{1}{2}}^{1}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}+|v|\right\}$$

$$\times[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$

$$\times\left|f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^p+tb^p\right)^{\frac{1}{p}}\right)\right|dt$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}$$

$$\times\left\{\int_{\frac{1}{2}}^{1}t^{x(\beta+\rho k)}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{x(1-p)}{p}}dt\right\}^{\frac{1}{x}}$$

$$\times\left\{\int_{\frac{1}{2}}^{1}\left|f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^p+tb^p\right)^{\frac{1}{p}}\right)\right|^y dt\right\}^{\frac{1}{y}}$$

$$+|v|\left\{\int_{\frac{1}{2}}^{1}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{x(1-p)}{p}}dt\right\}^{\frac{1}{x}}$$

$$\times\left\{\int_{\frac{1}{2}}^{1}\left|f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^p+tb^p\right)^{\frac{1}{p}}\right)\right|^y dt\right\}^{\frac{1}{y}}$$

$$
\begin{aligned}
&= \left\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{x(p-1)}}\right.\\
&\quad \times \left[\int_{\frac{1}{2}}^{1} t^{x(\beta+\rho k)}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{x(1-p)}{p}} dt\right]^{\frac{1}{x}}\\
&\quad \left.+\frac{|v|}{\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{x(p-1)}}\left[\int_{\frac{1}{2}}^{1}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{x(1-p)}{p}} dt\right]^{\frac{1}{x}}\right\}\\
&\quad \times\left\{\int_0^{\frac{1}{2}}\left[(1-t)^s\{\lambda|f'(a)|^y+(1-\lambda)|f'(b)|^y\}+t^s|f'(b)|^y\right]dt\right\}^{\frac{1}{y}}\\
&= \left\{\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{x(p-1)}}\sqrt[x]{\mathbb{B}(x\beta+x\rho k+1,1)}\right.\\
&\quad \times\left[{}_2F_1\left(x\frac{p-1}{p},\beta+x\rho k+1,x\beta+x\rho k+2,\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right.\\
&\quad \left.-{}_2F_1\left(x\frac{p-1}{p},\left[x\beta+x\rho k+1,x\beta+x\rho k+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right]^{\frac{1}{x}}\\
&\quad +\frac{|v|}{\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^{x(p-1)}}\left(\frac{\lambda a^p+(1-\lambda)b^p}{\lambda(a^p-b^p)}\left[\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p};1,x\frac{p-1}{p}+1\right)\right.\right.\\
&\quad \left.\left.\left.-\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{2(\lambda a^p+(1-\lambda)b^p)};1,x\frac{p-1}{p}+1\right)\right]\right)^{\frac{1}{x}}\right\}\\
&\quad \times\left\{\frac{2^{s+1}-\lambda}{2^{s+1}(s+1)}|f'(b)|^y+\frac{\lambda}{2^{s+1}(s+1)}|f'(a)|^y\right\}^{\frac{1}{y}}.
\end{aligned}
\tag{3.11}
$$

Therefore, the desired inequality (3.9) follows from (3.5) and (3.10) together with (3.11). □

Corollary 3.2. Let $I \subseteq \mathbb{R}^+$ be an interval and I° be the interior of I, $a, b \in I^\circ$ with $a < b$, $p < 0$, $u, v > 0$, $s \in (0, 1]$, $g(\xi) = \sqrt[p]{\xi}$ for $\xi > 0$, $y = \frac{x}{x-1} > 1$, and $f : I \to \mathbb{R}$ be a differentiable function on I° such that $|f'|^y$ is (s, p)-convex. Then the inequality

$$
\begin{aligned}
&\left|uf(a)+(1-v)f(b)+(v-u)f\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right)-\frac{1}{b^p-a^p}\int_{a^p}^{b^p}(f\circ g)(x)dx\right|\\
&\le \frac{(b^p-a^p)\left(\sqrt[x]{a^{p-x^2(p-1)}}\right)}{p\sqrt[x]{a^p-b^p}\sqrt[y]{2^{1+s}(s+1)}}\left[u\sqrt[x]{\mathbb{B}\left(\frac{a^p-b^p}{2a^p};1,\frac{xp-x+p}{p}\right)}\right.\\
&\quad \times\sqrt[y]{|f'(b)|^y+(2^{s+1}-1)|f'(a)|^y}
\end{aligned}
$$

$$+v\sqrt[x]{\mathbb{B}\left(\frac{a^p-b^p}{a^p};1,\frac{xp-x+p}{p}\right)-\mathbb{B}\left(\frac{a^p-b^p}{2a^p};1,\frac{xp-x+p}{p}\right)}$$
$$\times\sqrt[y]{(2^{s+1}-1)|f'(b)|^y+|f'(a)|^y}\Bigg] \tag{3.12}$$

holds.

Theorem 3.3. Let $I\subseteq\mathbb{R}^+$ be an interval and I° be the interior of I, $a,b\in I^\circ$ with $a<b$, $p,\rho,\beta>0$, $x\geq 1$, $u,w\in\mathbb{R}$, $(s,\lambda)\in(0,1]\times[0,1]$, $g(\xi)=\sqrt[p]{\xi}$, and $f:I\to\mathbb{R}$ be a differentiable function on I° such that $|f'|$ is (s,p)-convex. Then one has

$$|\psi(t,a,b;f)|\leq\frac{[\lambda(b^p-a^p)]^{1+\beta}}{\left(\sqrt[xp]{\lambda a^p+(1-\lambda)b^p}\right)^{(x-1)(p-1)}}\left\{\mathfrak{F}^{\sigma_3}_{\rho,\beta+1}\left[|w|\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho\right]\right.$$
$$+|u|\left\{\frac{\lambda a^p+(1-\lambda)b^p}{\lambda(a^p-b^p)}\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{2(\lambda a^p+(1-\lambda)b^p)},1,\frac{1}{p}\right)\right\}^{\frac{x-1}{x}}$$
$$\times\sqrt[x]{\frac{\{\lambda|f'(a)|^x+(1-\lambda)|f'(b)|^x\}H_{2,\lambda}(a,b;1,s+2)+|f'(b)|^xH_{2,\lambda}(a,b;s+1,s+2)}{s+1}}$$
$$+|v|\left\{\frac{\lambda a^p+(1-\lambda)b^p}{\lambda(a^p-b^p)}\left[\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p};1,\frac{1}{p}\right)\right.\right.$$
$$\left.\left.-\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{2(\lambda a^p+(1-\lambda)b^p)};1,\frac{1}{p}\right)\right]\right\}^{\frac{x-1}{x}}\{\{\lambda|f'(a)|^x+(1-\lambda)|f'(b)|^x\}$$
$$\times\frac{H_{1,\lambda}(a,b;1,s+2)-H_{2,\lambda}(a,b;1,s+2)}{s+1}$$
$$\left.\left.+\frac{|f'(b)|^x}{s+1}\left[H_{1,\lambda}(a,b;s+1,s+2)-H_{2,\lambda}(a,b;s+1,s+2)\right]\right\}^{\frac{1}{x}}\right\}. \tag{3.13}$$

Proof. It follows from the (s,p)-convexity of $|f'|$ and the power-mean inequality that

$$|I_1|=\left|\int_0^{\frac{1}{2}}\left\{t^\beta\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]-u\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}\right.$$
$$\left.\times f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)dt\right|$$
$$\leq\int_0^{\frac{1}{2}}\left\{t^\beta\mathfrak{F}^{\sigma}_{\rho,\beta+1}\left[|w|\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]+|u|\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$
$$\times\left|f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)\right|dt$$
$$=\int_0^{\frac{1}{2}}\left\{\sum_{k=0}^\infty\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}+|u|\right\}$$
$$\times[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$

$$\times\left|f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^p+tb^p\right)^{\frac{1}{p}}\right)\right|dt$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}$$

$$\times\left\{\int_0^{\frac{1}{2}}t^{\beta+\rho k}\left[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$

$$\times\left\{\int_0^{\frac{1}{2}}t^{\beta+\rho k}\left[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p\right]^{\frac{1-p}{p}}\right.$$

$$\left.\times\left|f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^p+tb^p\right)^{\frac{1}{p}}\right)\right|^x dt\right\}^{\frac{1}{x}}$$

$$+|u|\left\{\int_0^{\frac{1}{2}}\left[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$

$$\times\left\{\int_0^{\frac{1}{2}}\left[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p\right]^{\frac{1-p}{p}}\right.$$

$$\left.\times\left|f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^p+tb^p\right)^{\frac{1}{p}}\right)\right|^x dt\right\}^{\frac{1}{x}}$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[xp]{\lambda a^p+(1-\lambda)b^p}\right)^{(x-1)(p-1)}}$$

$$\times\left\{\int_0^{\frac{1}{2}}t^{\beta+\rho k}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$

$$\times\left\{\int_0^{\frac{1}{2}}t^{\beta+\rho k}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}\right.$$

$$\left.\times\left[(1-t)^s\{\lambda|f'(a)|^x+(1-\lambda)|f'(b)|^x\}+t^s|f'(b)|^x\right]dt\right\}^{\frac{1}{x}}$$

$$+\frac{|u|}{\left(\sqrt[xp]{\lambda a^p+(1-\lambda)b^p}\right)^{(x-1)(p-1)}}$$

$$\times\left\{\int_0^{\frac{1}{2}}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$

$$\times\left\{\int_0^{\frac{1}{2}}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}\right.$$

$$\times\left[(1-t)^{s}\{\lambda|f'(a)|^{x}+(1-\lambda)|f'(b)|^{x}\}+t^{s}|f'(b)|^{x}\right]dt\}^{\frac{1}{x}}$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}\left(\frac{\lambda(b^{p}-a^{p})}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[xp]{\lambda a^{p}+(1-\lambda)b^{p}}\right)^{(x-1)(p-1)}}$$

$$\times\left\{\int_{0}^{\frac{1}{2}}t^{\beta+\rho k}\left[1-t\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$

$$\times\{\{\lambda|f'(a)|^{x}+(1-\lambda)|f'(b)|^{x}\}$$

$$\times\int_{0}^{\frac{1}{2}}t^{\beta+\rho k}(1-t)^{s}\left[1-t\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right]^{\frac{1-p}{p}}dt$$

$$+|f'(b)|^{x}\int_{0}^{\frac{1}{2}}t^{\beta+\rho k+s}\left[1-t\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right]^{\frac{1-p}{p}}dt\Bigg\}^{\frac{1}{x}}$$

$$+\frac{|u|}{\left(\sqrt[xp]{\lambda a^{p}+(1-\lambda)b^{p}}\right)^{(x-1)(p-1)}}$$

$$\times\left\{\int_{0}^{\frac{1}{2}}\left[1-t\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}\{\{\lambda|f'(a)|^{x}+(1-\lambda)|f'(b)|^{x}\}$$

$$\times\int_{0}^{\frac{1}{2}}(1-t)^{s}\left[1-t\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right]^{\frac{1-p}{p}}dt$$

$$+|f'(b)|^{x}\int_{0}^{\frac{1}{2}}t^{s}\left[1-t\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right]^{\frac{1-p}{p}}dt\Bigg\}^{\frac{1}{x}}$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}\left(\frac{\lambda(b^{p}-a^{p})}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[xp]{\lambda a^{p}+(1-\lambda)b^{p}}\right)^{(x-1)(p-1)}}\{\mathbb{B}(\beta+\rho k+1,1)$$

$$\times{}_{2}F_{1}\left(\frac{p-1}{p},\left[\beta+\rho k+1,\beta+\rho k+2;\frac{1}{2}\right],\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)\Bigg\}^{1-\frac{1}{x}}$$

$$\times\{\{\lambda|f'(a)|^{x}+(1-\lambda)|f'(b)|^{x}\}\mathbb{B}(\beta+\rho k+1,s+1)$$

$$\times{}_{2}F_{1}\left(\frac{p-1}{p},\left[\beta+\rho k+1,s+\beta+\rho k+2;\frac{1}{2}\right],\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)+|f'(b)|^{x}$$

$$\times{}_{2}F_{1}\left(\frac{p-1}{p},\left[\beta+\rho k+s+1,\beta+\rho k+s+2;\frac{1}{2}\right],\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)$$

$$\times\mathbb{B}(\beta+\rho k+s+1,1)\}^{\frac{1}{x}}+\frac{|u|}{\left(\sqrt[xp]{\lambda a^{p}+(1-\lambda)b^{p}}\right)^{(x-1)(p-1)}}$$

$$\times\left\{\frac{\lambda a^p+(1-\lambda)b^p}{\lambda(a^p-b^p)}\mathbb{B}\left(\frac{\lambda(a^p-b^p)}{2(\lambda a^p+(1-\lambda)b^p)},1,\frac{1}{p}\right)\right\}^{1-\frac{1}{x}}$$
$$\times\{\{\lambda|f'(a)|^x+(1-\lambda)|f'(b)|^x\}\mathbb{B}(1,s+1)$$
$$\times{}_2F_1\left(\frac{p-1}{p},\left[1,s+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)+|f'(b)|^x\mathbb{B}(s+1,1)$$
$$\times{}_2F_1\left(\frac{p-1}{p},\left[s+1,s+2;\frac{1}{2}\right],\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right\}^{\frac{1}{x}} \tag{3.14}$$

and

$$|I_2|=\left|\int_{\frac{1}{2}}^1\left\{t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[w\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]-v\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}\right.$$
$$\left.\times f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)dt\right|$$
$$\leq\int_{\frac{1}{2}}^1\left\{t^\beta\mathfrak{F}^\sigma_{\rho,\beta+1}\left[|w|\left(\frac{\lambda(b^p-a^p)}{2}\right)^\rho t^\rho\right]+|v|\right\}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$
$$\times\left|f'\left(((1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p)^{\frac{1}{p}}\right)\right|dt$$
$$=\int_{\frac{1}{2}}^1\left\{\sum_{k=0}^\infty\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}t^{\beta+\rho k}+|v|\right\}$$
$$\times[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}$$
$$\times\left|f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^p+tb^p\right)^{\frac{1}{p}}\right)\right|dt$$
$$=\sum_{k=0}^\infty\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)}$$
$$\times\left\{\int_{\frac{1}{2}}^1 t^{\beta+\rho k}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$
$$\times\left\{\int_{\frac{1}{2}}^1 t^{\beta+\rho k}[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}\right.$$
$$\left.\times\left|f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^p+tb^p\right)^{\frac{1}{p}}\right)\right|^x dt\right\}^{\frac{1}{x}}$$
$$+|v|\left\{\int_{\frac{1}{2}}^1[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$
$$\times\left\{\int_{\frac{1}{2}}^1[(1-t)(\lambda a^p+(1-\lambda)b^p)+tb^p]^{\frac{1-p}{p}}\right.$$

$$\times\left|f'\left(\left((1-t)\left(\sqrt[p]{\lambda a^p+(1-\lambda)b^p}\right)^p+tb^p\right)^{\frac{1}{p}}\right)\right|^x dt\Bigg\}^{\frac{1}{x}}$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[xp]{\lambda a^p+(1-\lambda)b^p}\right)^{(x-1)(p-1)}}$$

$$\times\left\{\int_{\frac{1}{2}}^{1}t^{\beta+\rho k}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$

$$\times\left\{\int_{\frac{1}{2}}^{1}t^{\beta+\rho k}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}\right.$$

$$\times\left[(1-t)^s\{\lambda|f'(a)|^x+(1-\lambda)|f'(b)|^x\}+t^s|f'(b)|^x\right]dt\Big\}^{\frac{1}{x}}$$

$$+\frac{|v|}{\left(\sqrt[xp]{\lambda a^p+(1-\lambda)b^p}\right)^{(x-1)(p-1)}}$$

$$\times\left\{\int_{\frac{1}{2}}^{1}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$

$$\times\left\{\int_{\frac{1}{2}}^{1}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}\right.$$

$$\times\left[(1-t)^s\{\lambda|f'(a)|^x+(1-\lambda)|f'(b)|^x\}+t^s|f'(b)|^x\right]dt\Big\}^{\frac{1}{x}}$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k\left(\frac{\lambda(b^p-a^p)}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[xp]{\lambda a^p+(1-\lambda)b^p}\right)^{(x-1)(p-1)}}$$

$$\times\left\{\int_{\frac{1}{2}}^{1}t^{\beta+\rho k}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$

$$\times\{\{\lambda|f'(a)|^x+(1-\lambda)|f'(b)|^x\}$$

$$\times\int_{\frac{1}{2}}^{1}t^{\beta+\rho k}(1-t)^s\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt+|f'(b)|^x$$

$$\times\int_{\frac{1}{2}}^{1}t^{\beta+\rho k+s}\left[1-t\frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right]^{\frac{1-p}{p}}dt\Bigg\}^{\frac{1}{x}}$$

$$+\frac{|v|}{\left(\sqrt[xp]{\lambda a^p+(1-\lambda)b^p}\right)^{(x-1)(p-1)}}$$

$$\times\left\{\int_{\frac{1}{2}}^{1}\left[1-t\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right]^{\frac{1-p}{p}}dt\right\}^{1-\frac{1}{x}}$$

$$\times\left\{\{\lambda|f'(a)|^{x}+(1-\lambda)|f'(b)|^{x}\}\int_{\frac{1}{2}}^{1}(1-t)^{s}\left[1-t\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right]^{\frac{1-p}{p}}dt\right.$$

$$\left.+|f'(b)|^{x}\int_{\frac{1}{2}}^{1}t^{s}\left[1-t\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right]^{\frac{1-p}{p}}dt\right\}^{\frac{1}{x}}$$

$$=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}\left(\frac{\lambda(b^{p}-a^{p})}{2}\right)^{\rho k}}{\Gamma(\rho k+\beta+1)\left(\sqrt[xp]{\lambda a^{p}+(1-\lambda)b^{p}}\right)^{(x-1)(p-1)}}\{\mathbb{B}(\beta+\rho k+1,1)$$

$$\times\left[{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+1;\beta+\rho k+2,\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)\right.$$

$$\left.-{}_2F_1\left(\frac{p-1}{p},\left[\beta+\rho k+1,\beta+\rho k+2;\frac{1}{2}\right],\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)\right]\Bigg\}^{1-\frac{1}{x}}$$

$$\times\{\{\lambda|f'(a)|^{x}+(1-\lambda)|f'(b)|^{x}\}\mathbb{B}(\beta+\rho k+1,s+1)$$

$$\times\left[{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+1;s+\beta+\rho k+2,\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)\right.$$

$$\left.-{}_2F_1\left(\frac{p-1}{p},\left[\beta+\rho k+1,s+\beta+\rho k+2;\frac{1}{2}\right],\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)\right]+|f'(b)|^{x}$$

$$\times\left[{}_2F_1\left(\frac{p-1}{p},\beta+\rho k+s+1;\beta+\rho k+s+2,\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)\right.$$

$$\left.-{}_2F_1\left(\frac{p-1}{p},\left[\beta+\rho k+s+1,\beta+\rho k+s+2;\frac{1}{2}\right],\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)\right]$$

$$\times\mathbb{B}(\beta+\rho k+s+1,1)\}^{\frac{1}{x}}+\frac{|v|}{\left(\sqrt[xp]{\lambda a^{p}+(1-\lambda)b^{p}}\right)^{(x-1)(p-1)}}$$

$$\times\left[\left\{\frac{\lambda a^{p}+(1-\lambda)b^{p}}{\lambda(a^{p}-b^{p})}\left[\mathbb{B}\left(\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}};1,\frac{1}{p}\right)\right.\right.\right.$$

$$\left.\left.-\mathbb{B}\left(\frac{\lambda(a^{p}-b^{p})}{2(\lambda a^{p}+(1-\lambda)b^{p})};1,\frac{1}{p}\right)\right]\right\}^{1-\frac{1}{x}}\{\{\lambda|f'(a)|^{x}+(1-\lambda)|f'(b)|^{x}\}$$

$$\times\mathbb{B}(1,s+1)\left[{}_2F_1\left(\frac{p-1}{p},1;s+2,\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)\right.$$

$$\left.-{}_2F_1\left(\frac{p-1}{p},[1,s+2;\frac{1}{2}],\frac{\lambda(a^{p}-b^{p})}{\lambda a^{p}+(1-\lambda)b^{p}}\right)\right]+|f'(b)|^{x}\mathbb{B}(s+1,1)$$

$$\times\left[{}_2F_1\left(\frac{p-1}{p}, s+1; s+2, \frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right.$$

$$\left.\left.-{}_2F_1\left(\frac{p-1}{p}, \left[s+1, s+2; \frac{1}{2}\right], \frac{\lambda(a^p-b^p)}{\lambda a^p+(1-\lambda)b^p}\right)\right]\right\}^{\frac{1}{x}}\Bigg]. \tag{3.15}$$

Combining the inequalities (3.5), (3.14) and (3.15) gives the desired inequality (3.13). □

Corollary 3.3. Let $I \subseteq \mathbb{R}^+$ be an interval and I° be the interior of I, $a, b \in I^\circ$ with $a < b$, $p < 0$, $s \in (0, 1]$, $u, v > 0$, $g(\xi) = \sqrt[p]{\xi}$, $y = \frac{x}{x-1} > 1$ and $f : I \to \mathbb{R}$ be a differentiable function on I° such that $|f'|^y$ is (s, p)-convex. Then

$$\left|uf(a) + (1-v)f(b) + (v-u)f\left(\sqrt[p]{\frac{a^p+b^p}{2}}\right) - \frac{1}{b^p-a^p}\int_{a^p}^{b^p}(f\circ g)(x)dx\right|$$

$$\leq \frac{b^p-a^p}{p\left(\sqrt[x]{a}\right)^{(x-1)(p-1)}}\left\{u\left\{\frac{a^p}{a^p-b^p}\mathbb{B}\left(\frac{a^p-b^p}{2a^p}, 1, \frac{1}{p}\right)\right\}^{\frac{x-1}{x}}\right.$$

$$\times\sqrt[x]{\frac{|f'(a)|^x H_{2,1}(a, b; 1, s+2) + |f'(b)|^x H_{2,1}(a, b; s+1, s+2)}{s+1}}$$

$$+v\left\{\frac{a^p}{a^p-b^p}\left[\mathbb{B}\left(\frac{a^p-b^p}{a^p}; 1, \frac{1}{p}\right) - \mathbb{B}\left(\frac{a^p-b^p}{2a^p}; 1, \frac{1}{p}\right)\right]\right\}^{\frac{x-1}{x}}$$

$$\times\left\{\frac{|f'(a)|^x H_{1,1}(a, b; 1, s+2) H_{2,1}(a, b; 1, s+2)}{s+1}\right.$$

$$\left.\left.+\frac{|f'(b)|^x[H_{1,1}(a, b; s+1, s+2) - H_{2,1}(a, b; s+1, s+2)]}{s+1}\right\}^{\frac{1}{x}}\right\}. \tag{3.16}$$

Proof. Let $\beta, \lambda, \sigma(0) \to 1$ and $w = 0$. Then Corollary 3.3 follows easily from Theorem 3.3. □

4. Applications

In this section, we provide some applications on f-divergence measures and probability density functions by using the results obtained in Section 3.

4.1. f-divergence measures

Let ϕ be a set, μ be the σ finite measure, $\Omega = \{\chi | \chi : \phi \to \mathbb{R}, \chi(x) > 0, \int_\phi \chi(x)d\mu(x) = 1\}$ be the set of all probability densities on μ, and $f : (0, \infty) \to \mathbb{R}$ be a real-valued function. Then the Csisźar f-divergence $D_f(\chi, \psi)$ is defined by

$$D_f(\chi, \psi) = \int_\phi \chi(x) f\left[\frac{\psi(x)}{\chi(x)}\right] d\mu(x) \quad (\chi, \psi \in \Omega) \tag{4.1}$$

if f is convex, and the Hermite-Hadamard (HH) divergence $D^f_{HH}(\chi,\psi)$ is defined by

$$D^f_{HH}(\chi,\psi)=\int_\phi \chi(x)\frac{\int_1^{\frac{\psi(x)}{\chi(x)}} f(t)dt}{\frac{\psi(x)}{\chi(x)}-1}d\mu(x)\quad (\chi,\psi\in\Omega) \tag{4.2}$$

if f is convex with $f(1)=0$. Note that $D^f_{HH}(\chi,\psi)\geq 0$ and $D^f_{HH}(\chi,\psi)=0$ if and only if $\chi=\psi$.

Proposition 4.1. Let $I\subseteq\mathbb{R}^+$ be an interval and I° be the interior of I, $a,b\in I^\circ$ with $a<b$, $s\in(0,1]$ and $f:I\to\mathbb{R}$ be a differentiable function on I° such that $|f'|$ is s-convex and $f(1)=0$. Then

$$\begin{aligned}&\left|\frac{1}{6}\left[D_f(\chi,\psi)+4\int_\phi \chi(x)f\left(\frac{\psi(x)+\chi(x)}{2\chi(x)}\right)d\mu(x)\right]-D^f_{HH}(\chi,\psi)\right|\\
&\leq\frac{(b-a)|f'(a)|}{6(s+1)}\int_\phi \chi(x)\left[5{}_2F_1\left(0,1,s+2,\frac{|\psi(x)-\chi(x)|}{\chi(x)}\right)\right.\\
&\qquad\left.-4{}_2F_1\left(0,\left[1,s+2;\frac{1}{2}\right],\frac{|\psi(x)-\chi(x)|}{\chi(x)}\right)\right]d\mu(x)\\
&\quad+\frac{(b-a)|f'(b)|}{6(s+1)}\int_\phi \chi(x)\left[5{}_2F_1\left(0,s+1,s+2,\frac{|\psi(x)-\chi(x)|}{\chi(x)}\right)\right.\\
&\qquad\left.-4{}_2F_1\left(0,\left[s+1,s+2;\frac{1}{2}\right],\frac{|\psi(x)-\chi(x)|}{\chi(x)}\right)\right]d\mu(x).\end{aligned} \tag{4.3}$$

Proof. Let $\Phi_1=\{x\in\phi:\psi(x)>\chi(x)\}$, $\Phi_2=\{x\in\phi:\psi(x)<\chi(x)\}$ and $\Phi_3=\{x\in\phi:\psi(x)=\chi(x)\}$. Then we clearly see that inequality (4.3) holds if $x\in\Phi_3$.

For the case of $x\in\Phi_1$, taking $a,p\to 1$, $b\to\frac{\psi(x)}{\chi(x)}$, $u\to\frac{1}{6}$ and $v\to\frac{5}{6}$ in Corollary 3.1, multiplying both sides of the obtained result by $\chi(x)$ and integrating over Φ_1 lead to the conclusion that

$$\begin{aligned}&\left|\frac{1}{6}\left[4\int_{\Phi_1}\chi(x)f\left(\frac{\psi(x)+\chi(x)}{2\chi(x)}\right)d\mu(x)+\int_{\Phi_1}\chi(x)f\left(\frac{\psi(x)}{\chi(x)}\right)d\mu(x)\right]\right.\\
&\left.-\int_{\Phi_1}\chi(x)\frac{\int_1^{\frac{\psi(x)}{\chi(x)}} f(t)dt}{\frac{\psi(x)}{\chi(x)}-1}d\mu(x)\right|\leq\frac{(b-a)|f'(a)|}{6(s+1)}\int_{\Phi_1}\chi(x)\left[5{}_2F_1\left(0,1,s+2,\frac{\chi(x)-\psi(x)}{\chi(x)}\right)\right.\\
&\qquad\left.-4{}_2F_1\left(0,\left[1,s+2;\frac{1}{2}\right],\frac{\chi(x)-\psi(x)}{\chi(x)}\right)\right]d\mu(x)\\
&\quad+\frac{(b-a)|f'(b)|}{6(s+1)}\int_{\Phi_1}\chi(x)\left[5{}_2F_1\left(0,s+1,s+2,\frac{\chi(x)-\psi(x)}{\chi(x)}\right)\right.\\
&\qquad\left.-4{}_2F_1\left(0,\left[s+1,s+2;\frac{1}{2}\right],\frac{\chi(x)-\psi(x)}{\chi(x)}\right)\right]d\mu(x).\end{aligned} \tag{4.4}$$

Similarly, for the case of $x\in\Phi_2$, taking $b,p\to 1$, $a\to\frac{\psi(x)}{\chi(x)}$, $u\to\frac{1}{6}$ and $v\to\frac{5}{6}$ in Corollary 3.1, multiplying both sides to the obtained result by $\chi(x)$ and integrating over Φ_2, we get

$$\left|\frac{1}{6}\left[4\int_{\Phi_2}\chi(x)f\left(\frac{\psi(x)+\chi(x)}{2\chi(x)}\right)d\mu(x)+\int_{\Phi_2}\chi(x)f\left(\frac{\psi(x)}{\chi(x)}\right)d\mu(x)\right]\right.$$

$$\left.-\int_{\Phi_2}\chi(x)\frac{\int_1^{\frac{\psi(x)}{\chi(x)}}f(t)dt}{\frac{\psi(x)}{\chi(x)}-1}d\mu(x)\right|\leq\frac{(b-a)|f'(a)|}{6(s+1)}\int_{\Phi_2}\chi(x)\left[5{}_2F_1\left(0,1,s+2,\frac{\psi(x)-\chi(x)}{\chi(x)}\right)\right.$$

$$\left.-4{}_2F_1\left(0,\left[1,s+2;\frac{1}{2}\right],\frac{\psi(x)-\chi(x)}{\chi(x)}\right)\right]d\mu(x)$$

$$+\frac{(b-a)|f'(b)|}{6(s+1)}\int_{\Phi_2}\chi(x)\left[5{}_2F_1\left(0,s+1,s+2,\frac{\psi(x)-\chi(x)}{\chi(x)}\right)\right.$$

$$\left.-4{}_2F_1\left(0,\left[s+1,s+2;\frac{1}{2}\right],\frac{\psi(x)-\chi(x)}{\chi(x)}\right)\right]d\mu(x). \tag{4.5}$$

Therefore, the desired inequality (4.3) can be derived by adding inequalities (4.4) and (4.5) to together with the triangular inequality. □

4.2. *Probability density functions*

Let $a, b \in \mathbb{R}$ with $a < b$, $g : [a, b] \to [0, 1]$ be the probability density function of a continuous random variable X with the cumulative distribution function F given by

$$F(x) = P(X \leq x) = \int_a^x g(t)dt, \quad E(X) = \int_a^b tdF(t) = b - \int_a^b F(t)dt. \tag{4.6}$$

Then from Corollary 3.1 we clearly see that

$$\left|\frac{1}{6}\left[4P\left(X\leq\frac{a+b}{2}\right)+1\right]-\frac{1}{b-a}(b-E(X))\right|$$

$$\leq\frac{b-a}{s+1}\left\{|g(a)|\frac{5{}_2F_1\left(0,1,s+2,\frac{a-b}{a}\right)-4{}_2F_1\left(0,\left[1,s+2;\frac{1}{2}\right],\frac{a-b}{a}\right)}{6}\right.$$

$$\left.+|g(b)|\frac{5\cdot{}_2F_1\left(0,s+1,s+2,\frac{a-b}{a}\right)-4{}_2F_1\left(0,\left[s+1,s+2;\frac{1}{2}\right],\frac{a-b}{a}\right)}{6}\right\}$$

if $p \to 1$, $u \to \frac{1}{6}$ and $v \to \frac{5}{6}$.

5. Conclusions

We have established some new estimates for the generalized Simpson's quadrature rule via the Raina fractional integrals by use of a Simpson-type generalized identity with multi-parameters, and discovered several inequalities for the f-divergence measures and probability density functions. Our obtained results are the improvements and generalizations of some previous known results, our ideas and approach may lead to a lot of follow-up research.

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Conflict of interest

The authors declare no conflict of interest.

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