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*Research article*

# New results for nonlinear fractional jerk equations with resonant boundary value conditions

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**Abstract:** A novel fractional-order jerk equation with resonant boundary value conditions is proposed. Using coincidence degree theory, we obtain the existence of solutions of nonlinear fractional jerk equation with two-point boundary conditions. This paper enriches some existing literatures. Finally, an example is given to demonstrate the effectiveness of our main result.

**Keywords:** fractional jerk equation; boundary value conditions; coincidence degree theory; resonance
**Mathematics Subject Classification:** 26A33, 34B15

## 1. Introduction

It is well known that a three-dimensional dynamical system:

$$\begin{cases} \dot{x}(t) = y, \\ \dot{y}(t) = z, \\ \dot{z}(t) = f(x, y, z), \end{cases}$$

can be transformed into a third-order differential equation of the form:

$$\dddot{x}(t) = f(x, \dot{x}, \ddot{x}).$$

The above equation involves a third derivative of the variable $x$, which in a mechanical system is a rate of change of acceleration, it is called a jerk equation. Jerk equations are widely used in applied science and engineering, such as it can describe a three-order dynamic vibration model. In recent years, the discussion of jerk equations have attracted the attention of scholars and some results have been obtained. For recent publication on jerk equations, we refer the reader to see [1–7].

In 2017, by using classical contraction mapping theorem, Elsonbaty and El-Sayed [1] obtained the existence and uniqueness of the solution of the system:

$$\begin{cases} \dot{x_1}(t) = x_2, \\ \dot{x_2}(t) = x_3, \\ \dot{x_3}(t) = bx_1 - cx_2 - x_3 - x_1^3, \end{cases}$$

where the dot denotes differentiation with respect to time, $a$ and $c$ are positive parameters, and $c \in \mathbb{R}$.

In [2], a second approximate solution of nonlinear jerk equation:

$$\begin{cases} \dddot{x}(t) + f(x, \dot{x}, \ddot{x}) = 0, \\ x(0) = 0,\ \dot{x}(0) = B,\ \ddot{x}(0) = 0, \end{cases}$$

was obtained by using modified harmonic balance method.

Very recently, Liu and Chang [3] developed two iterative algorithms to determine the periodic solutions of the following nonlinear jerk equation:

$$\begin{cases} \dddot{x}(t) = J(x, \dot{x}, \ddot{x}), \\ x(0) = x_0,\ \dot{x}(0) = \dot{x}_0,\ \ddot{x}(0) = \ddot{x}_0, \end{cases}$$

where the initial values $x_0, \dot{x}_0$ and $\ddot{x}_0$ for the periodic solution of the above are unknown and the period is denoted by $T > 0$.

What is interesting to us is the appearance of the article [4]. Prakash and Singh presented a new fractional order jerk equations which do not have equilibrium point:

$$\begin{cases} D_{0+}^{\alpha} x(t) = y(t), \\ D_{0+}^{\beta} y(t) = z(t), \\ D_{0+}^{\gamma} z(t) = f(x, y, z), \end{cases}$$

where $f = -y(t)+3y^2(t)-x^2(t)-x(t)z(t)+\beta-W(x(t))y(t)$, $\alpha, \beta, \gamma$ are fractional order and $0 < \alpha, \beta, \gamma \leq 1$. In addition, the authors successfully designed a fractional-order backstepping controller to stabilise the chaos in the system. This proposed system is novel in the sense that the jerk systems involving fractional order. It enlightens us to think the problem: how to observe the existence result of fractional jerk differential equation.

Differential equations with fractional order arise from a variety of applications in various fields of science and engineering, such as electromagnetic, mechanics, biology, chemistry etc. (see [8]). Over last two decades, there has been significant progress in the area of fractional differential equations. Though many works have been done dealing with the existence of solutions to fractional differential equation (see [9–21]), no results on boundary value problem of fractional jerk equation are reported in the literature. To the best of the authors' knowledge, this is the first paper dealing with resonant boundary value problem of fractional jerk equations.

In this paper, we are interested in the solutions to the following fractional jerk equations:

$$\begin{cases} D_{0+}^{\alpha}u(t) = y(t), \\ D_{0+}^{\beta}y(t) = z(t), \\ D_{0+}^{\gamma}z(t) = f(t, u(t), u'(t), u''(t)), \end{cases} \tag{1.1}$$

with boundary value conditions given as

$$u(0) = u(\xi), D_{0+}^{\alpha}u(0) = (D_{0+}^{\beta}(D_{0^+}^{\alpha}u))(0) = 0, \tag{1.2}$$

where $t \in (0, 1)$, $0 < \alpha, \beta, \gamma \leq 1$, $0 < \xi \leq 1$, $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\gamma}$ denote the Caputo fractional derivatives and $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

It is easy to check that the following fractional jerk equation:

$$\begin{cases} (D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0^+}^{\alpha}u)))(t) = f(t, u(t), u'(t), u''(t)), \\ u(0) = u(\xi), D_{0+}^{\alpha}u(0) = (D_{0+}^{\beta}(D_{0^+}^{\alpha}u))(0) = 0, \end{cases} \tag{1.3}$$

is equivalent to (1.1). Due to conditions (1.2), Eq (1.3) happens to be at resonance in the sense that the associated linear homogeneous equation:

$$\begin{cases} (D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0^+}^{\alpha}u)))(t) = 0, \\ u(0) = u(\xi), D_{0+}^{\alpha}u(0) = (D_{0+}^{\beta}(D_{0^+}^{\alpha}u))(0) = 0, \end{cases}$$

has $u(t) = c, c \in \mathbb{R}$ as nontrivial solutions.

In addition, we can see if $\alpha = \beta = \gamma = 1$, the Eq (1.3) can be rewritten as

$$\begin{cases} \dddot{u}(t) = f(t, u, \dot{u}, \ddot{u}), \\ u(0) = u(\xi), \ \dot{u}(0) = \ddot{u}(0) = 0, \end{cases}$$

which is a standard nonlinear jerk equation. Furthermore, if $\xi \neq 1$, then the above equation is a nonlocal initial value problems of nonlinear jerk equation; if $\xi = 1$, then it is a nonlinear jerk equations with periodic boundary conditions. If $u(t), t \in [0, 1]$ satisfies $u(0) = u(1)$, then boundary value condition $u(0) = u(1)$ is called periodic boundary conditions. According to different values of $\xi, \alpha, \beta, \gamma$, we will get diversiform types of boundary value conditions. Thus, this paper enriches and extends the existing literatures, such as [4–7].

The remaining part of this article is organized as follows. In Section 2, we will state some necessary notations, definitions and lemmas. In Section 3, by applying the coincidence degree theory due to Mawhin [22], we obtain the existence of solutions of (1.3). In the last section, an example is given to illustrate our results.

## 2. Preliminaries

We present the necessary definitions and lemmas from fractional calculus theory that will be used to prove our main theorems.

**Definition 2.1** ([8]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

**Definition 2.2** ([8]). The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n-1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

**Lemma 2.3** ([8]). Let $n-1 < \alpha \leq n$, $u \in C(0,1) \cap L^1(0,1)$, then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$.

**Lemma 2.4.** *If* $\beta > 0,\ \alpha + \beta > 0$*, then the equation*

$$I_{0+}^{\alpha} I_{0+}^{\beta} f(x) = I_{0+}^{\alpha+\beta} f(x)$$

*is satisfied for continuous function* $f$*.*

Firstly, we briefly recall some definitions on the coincidence degree theory. For more details, see [22].

Let $Y, Z$ be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that

$$\operatorname{Ker} L = \operatorname{Im} P,\ \ \operatorname{Im} L = \operatorname{Ker} Q,\ \ Y = \operatorname{Ker} L \oplus \operatorname{Ker} P,\ \ Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse of this map by $K_P$. If $\Omega$ is an open bounded subset of $Y$, the map $N$ will be called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_{P,Q}N = K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

**Theorem 2.5** ([22]). Let $L$ be a Fredholm operator of index zero and $N$ be $L$-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied:

(1) $Lx \neq \lambda Nx$ for each $(x, \lambda) \in [(\operatorname{dom} L \backslash \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;
(2) $Nx \notin \operatorname{Im} L$ for each $x \in \operatorname{Ker} L \cap \partial\Omega$;
(3) $\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $Q : Z \to Z$ is a continuous projection as above with $\operatorname{Im} L = \operatorname{Ker} Q$ and $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is any isomorphism.

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

## 3. Main results

In this section, we will discuss the existence of solutions to Eq (1.3).

To obtain our main results, we impose the following condition:

$(\mathrm{H}_0)$ $\alpha+\beta+\gamma>2$.

We define $E=C[0,1]$ with the norm $\|u\|_\infty=\max\limits_{0\le t\le 1}|u(t)|$ and $X=\{u(t): u(t),u'(t),u''(t)\in E\}$ with the norm $\|u\|_X=\|u\|_\infty+\|u'\|_\infty+\|u''(t)\|_\infty$. It is clear that $(E,\|\cdot\|_\infty)$ and $(X,\|\cdot\|_X)$ are Banach spaces.

Define

$$L:\mathrm{dom}L\to E, u\mapsto (D^\gamma_{0+}(D^\beta_{0+}(D^\alpha_{0^+}u)))(t), \tag{3.1}$$
$$N: X\to E, u\mapsto f(t,u(t),u'(t),u''(t)), \tag{3.2}$$

where

$$\mathrm{dom}L=\{u\in X:\ u(0)=u(\xi), D^\alpha_{0+}u(0)=(D^\beta_{0+}(D^\alpha_{0^+}u))(0)=0\}.$$

Then the three-point boundary value problem (1.3) can be written as

$$Lu=Nu.$$

**Lemma 3.1.** *$L$ is defined as (3.1), then*

$$\mathrm{Ker}L=\{u\in X:\ u=c,\ c\in\mathbb{R}\}, \tag{3.3}$$
$$\mathrm{Im}L=\left\{x\in E:\ I^{\alpha+\beta+\gamma}_{0+}x(\xi)=0\right\}. \tag{3.4}$$

*Proof.* It is obvious that the operator $L$ is linear. By $Lu=0$ and Lemmas 2.3, we have $(D^\beta_{0+}(D^\alpha_{0^+}u))(t)=c_0, c_0\in\mathbb{R}$. In view of $(D^\beta_{0+}(D^\alpha_{0^+}u))(0)=0$, we get $c_0=0$. Then, $(D^\beta_{0+}(D^\alpha_{0^+}u))(t)=0$. Thus, by Lemmas 2.3 again, we obtain $D^\alpha_{0^+}u(t)=c_1, c_1\in\mathbb{R}$. In view of $D^\alpha_{0^+}u(0)=0$, we get $c_1=0$. So, $D^\alpha_{0^+}u(t)=0$. Then, according to $u(0)=u(\xi)$, we have $u(t)=c, c\in\mathbb{R}$. Hence, we obtain (3.3).

Next, we prove (3.4) hold. Let $x\in\mathrm{Im}L$, so there exists $u\in\mathrm{dom}L$ such that $x(t)=(D^\gamma_{0+}(D^\beta_{0+}(D^\alpha_{0^+}u)))(t)$. By Lemma 2.3 and the definition of $\mathrm{dom}L$, we have

$$u(t)=I^{\alpha+\beta+\gamma}_{0+}x(t)+I^{\beta+\alpha}_{0+}c_0+I^\alpha_{0+}c_1+c_2.$$

In view of $D^\beta_{0+}u(0)=(D^\beta_{0+}(D^\alpha_{0^+}u))(0)=0$, we get $c_0=c_1=0$. Hence,

$$u(t)=I^{\alpha+\beta+\gamma}_{0+}x(t)+c_2.$$

According to $u(0)=u(\xi)$, we have $I^{\alpha+\beta+\gamma}_{0+}x(\xi)=0$. On the other hand, suppose $x$ satisfies the above equations. Let $u(t)=I^{\alpha+\beta+\gamma}_{0+}x(t)$, we can prove $u(t)\in\mathrm{dom}L$ and $Lu(t)=x$. Then, (3.4) holds. $\square$

For simplicity of notation, we write $p=\frac{1+(\alpha+\beta+\gamma)^2}{\Gamma(1+\alpha+\beta+\gamma)}$.

**Lemma 3.2.** *The mapping $L:\mathrm{dom}L\subset Y\to Z$ is a Fredholm operator of index zero.*

*Proof.* The linear continuous projector operator $P$ can be defined as

$$Pu = u(0).$$

Obviously, $P^2 = P$. It is clear that

$$\mathrm{Ker}P = \{u : u(0) = 0\}.$$

It follows from $u = u - Pu + Pu$ that $Y = \mathrm{Ker}P + \mathrm{Ker}L$. For $u \in \mathrm{Ker}L \cap \mathrm{Ker}P$, then $u = c, c \in \mathbb{R}$. Furthermore, by the definition of $\mathrm{Ker}P$, we have $c = 0$. Thus,

$$Y = \mathrm{Ker}L \oplus \mathrm{Ker}P.$$

The linear operator $Q$ can be defined as

$$Qx(t) = \frac{\Gamma(1+\alpha+\beta+\gamma)}{\xi^{\alpha+\beta+\gamma}} I_{0+}^{\alpha+\beta+\gamma} x(\xi).$$

Obviously, $Qx \cong \mathbb{R}$. For $x(t) \in E$, we have

$$Q(Qx(t)) = Qx(t) \cdot \frac{\Gamma(1+\alpha+\beta+\gamma)}{\xi^{\alpha+\beta+\gamma}} (I_{0+}^{\alpha+\beta+\gamma} 1)_{t=\xi} = Qx(t).$$

So, the operator $Q$ is idempotent. It follows from $x = x - Qx + Qx$ that $E = \mathrm{Im}L + \mathrm{Im}Q$. Moreover, by $\mathrm{Ker}Q = \mathrm{Im}L$ and $Q^2 = Q$, we get $\mathrm{Im}L \cap \mathrm{Im}Q = \{0\}$. Hence,

$$E = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Now, $\mathrm{Ind}\, L = \dim \mathrm{Ker}\, L - \mathrm{codim}\, \mathrm{Im}\, L = 0$, and so $L$ is a Fredholm mapping of index zero. □

For every $u \in X$, we have

$$\|Pu\|_X = |u(0)|. \tag{3.5}$$

Furthermore, the operator $K_P : \mathrm{Im}\, L \to \mathrm{dom}\, L \cap \mathrm{Ker}P$ can be defined by

$$K_P x(t) = I_{0+}^{\alpha+\beta+\gamma} x(t).$$

For $x(t) \in \mathrm{Im}L$, we have

$$LK_P x(t) = LI_{0+}^{\alpha+\beta+\gamma} x(t) = (D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0^+}^{\alpha} I_{0+}^{\alpha+\beta+\gamma} x)))(t) = x(t). \tag{3.6}$$

On the other hand, for $u \in \mathrm{dom}L \cap \mathrm{Ker}P$, according to Lemma 2.3 and the definitions of $\mathrm{dom}L$ and $\mathrm{Ker}P$, we have

$$I_{0+}^{\alpha+\beta+\gamma} Lu(t) = I_{0+}^{\alpha+\beta+\gamma}(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0^+}^{\alpha} u)))(t) = u(t). \tag{3.7}$$

Combining (3.6) and (3.7), we have $K_P = (L_{\mathrm{dom}L\cap\mathrm{Ker}P})^{-1}$.

For $x \in \mathrm{Im}L$ and $\alpha + \beta + \gamma > 2$, we have

$$\|K_P x\|_X = \|I_{0+}^{\alpha+\beta+\gamma} x\|_X = \|I_{0+}^{\alpha+\beta+\gamma} x\|_\infty + \|(I_{0+}^{\alpha+\beta+\gamma} x)'\|_\infty + \|(I_{0+}^{\alpha+\beta+\gamma} x)''\|_\infty$$

$$
\begin{aligned}
&= \left[\frac{1}{\Gamma(1+\alpha+\beta+\gamma)} + \frac{1}{\Gamma(\alpha+\beta+\gamma)} + \frac{1}{\Gamma(\alpha+\beta+\gamma-1)}\right]\|x\|_\infty \\
&= \left[\frac{1+(\alpha+\beta+\gamma)^2}{\Gamma(1+\alpha+\beta+\gamma)}\right]\|x\|_\infty \\
&= p\|x\|_\infty. \qquad (3.8)
\end{aligned}
$$

Again for $u \in \Omega_1$, $u \in \mathrm{dom}(L) \setminus \mathrm{Ker}(L)$, then $(I-P)u \in \mathrm{dom}L \cap \mathrm{Ker}P$ and $LPu = 0$, thus from (3.8), we have

$$\|(I-P)u\|_X = \|K_P L(I-P)u\|_X = \|K_P Lu\|_X = p\|Nu\|_\infty.$$

With the similar proof showed in [13], we have the following lemma.

**Lemma 3.3.** $K_P(I-Q)N : Y \to Y$ is completely continuous.

**Theorem 3.4.** Assume the following conditions hold:
$(H_1)$ There exist nonnegative functions $\psi(t), \varphi_0(t), \varphi_1(t), \varphi_2(t) \in E$, such that for all $t \in [0,1]$, $(u_1, u_2, u_3) \in \mathbb{R}^3$, one has

$$|f(t, u_1, u_2, u_3)| \le \psi(t) + \varphi_0(t)|u_1| + \varphi_1(t)|u_2| + \varphi_2(t)|u_3|.$$

$(H_2)$ There exists $A > 0$ such that for $(t, u, u', u'')$, if $|u| > A$ for all $t \in [0,1]$, one has

$$QN(u) \neq 0.$$

$(H_3)$ There exists $B > 0$ such that if $|c| > B, c \in \mathbb{R}$, one has either

$$cQN(c) > 0$$

or

$$cQN(c) < 0.$$

Then, BVP (1.3) has at least a solution in $X$ provided that

$$\varphi < \frac{\Gamma(\alpha+\beta+\gamma+1)}{2+(\alpha+\beta+\gamma+1)^2} \qquad (3.9)$$

where $\varphi = \max\{\|\varphi_0(t)\|_\infty, \|\varphi_1(t)\|_\infty, \|\varphi_2(t)\|_\infty\}$.

*Proof.* Let

$$\Omega_1 = \{u \in \mathrm{dom}L \setminus \mathrm{Ker}L : \ Lu = \lambda Nu, \lambda \in (0,1)\}.$$

For $Lu = \lambda Nu \in \mathrm{Im}L = \mathrm{Ker}Q$, by (3.2) and the definition of $\mathrm{Ker}Q$, we have

$$I_{0+}^{\alpha+\beta+\gamma} f(t, u, u', u'')(\xi) = 0.$$

According to $(H_2)$, there exists $t_0 \in (0,1)$ such that $|u(t_0)| \le A$. By $Lu = \lambda Nu,\ u \in \mathrm{dom}L \setminus \mathrm{Ker}L$, that is $(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0^+}^{\alpha} u)))(t) = \lambda Nu$, we have

$$u(t) = \frac{\lambda}{\Gamma(\alpha+\beta+\gamma)} \int_0^t (t-s)^{\alpha+\beta+\gamma-1} f(s, u, u', u'')ds + c_0.$$

Substituting $t = t_0$ into the above equation, we get

$$u(t_0) = \frac{\lambda}{\Gamma(\alpha+\beta+\gamma)} \int_0^{t_0} (t_0 - s)^{\alpha+\beta+\gamma-1} f(s, u, u', u'')ds + c_0.$$

So, we have

$$\begin{aligned} u(t) - u(t_0) = &\frac{\lambda}{\Gamma(\alpha+\beta+\gamma)} \int_0^{t} (t - s)^{\alpha+\beta+\gamma-1} f(s, u, u', u'')ds \\ &- \frac{\lambda}{\Gamma(\alpha+\beta+\gamma)} \int_0^{t_0} (t_0 - s)^{\alpha+\beta+\gamma-1} f(s, u, u', u'')ds. \end{aligned}$$

Substituting $t = 0$ into the above equation, together with $|u(t_0)| \leq A$ and $(\mathrm{H}_1)$, we have

$$u(0) = u(t_0) - \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_0} (t_0 - s)^{\alpha+\beta+\gamma-1} f(s, u, u', u'')ds.$$

So

$$\begin{aligned} |u(0)| &\leq |u(t_0)| + \frac{\lambda}{\Gamma(\alpha+\beta+\gamma)} \int_0^{t_0} (t_0 - s)^{\alpha+\beta+\gamma-1} \Big|f(s, u, u', u'')\Big|ds \\ &\leq A + \frac{1}{\Gamma(\alpha+\beta+\gamma)} \int_0^{t_0} (t_0 - s)^{\alpha+\beta+\gamma-1} \Big|f(s, u, u', u'')\Big|ds \\ &\leq A + \frac{1}{\Gamma(\alpha+\beta+\gamma)} \int_0^{t_0} (t_0 - s)^{\alpha+\beta+\gamma-1} (|\psi| + |\varphi_0||u| + |\varphi_1||u'| + |\varphi_2||u''|)ds \\ &\leq A + \frac{1}{\Gamma(\alpha+\beta+\gamma)} \cdot (||\psi||_\infty + ||u||_X \cdot \varphi) \cdot \int_0^1 (1 - s)^{\alpha+\beta+\gamma-1}\, ds \\ &= A + \frac{1}{\Gamma(\alpha+\beta+\gamma+1)} (||\psi||_\infty + ||u||_X \cdot \varphi). \end{aligned} \tag{3.10}$$

By (3.10) and $(\mathrm{H}_1)$, we have

$$\begin{aligned} ||u||_X &= ||Pu + (I - P)u||_X \leq ||Pu||_Y + ||(I - P)u||_X \leq \Big|u(0)\Big| + p||Nu||_\infty \\ &\leq A + \frac{1}{\Gamma(\alpha+\beta+\gamma+1)} (||\psi||_\infty + ||u||_X \cdot \varphi) + p\Big|f(s, u, u', u'')\Big|_\infty \\ &\leq A + \frac{1}{\Gamma(\alpha+\beta+\gamma+1)} (||\psi||_\infty + ||u||_X \cdot \varphi) + p(||\psi||_\infty + ||u||_X \cdot \varphi) \\ &\leq A + \frac{1 + p\Gamma(\alpha+\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)} (||\psi||_\infty + ||u||_X \cdot \varphi) \\ &= A + \frac{1 + p\Gamma(\alpha+\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)} ||\psi||_\infty + ||u||_X \cdot \frac{1 + p\Gamma(\alpha+\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)} \varphi \\ &= A + \frac{2 + (\alpha+\beta+\gamma)^2}{\Gamma(\alpha+\beta+\gamma+1)} ||\psi||_\infty + ||u||_X \cdot \frac{2 + (\alpha+\beta+\gamma)^2}{\Gamma(\alpha+\beta+\gamma+1)} \varphi \end{aligned}$$

According to (3.9), we can derive

$$||u||_X \leq \frac{A + \frac{2+(\alpha+\beta+\gamma)^2}{\Gamma(\alpha+\beta+\gamma+1)} ||\psi||_\infty}{1 - \frac{2+(\alpha+\beta+\gamma)^2}{\Gamma(\alpha+\beta+\gamma+1)} \varphi} := M \tag{3.11}$$

Thus, we have $\Omega_1$ is bounded.

Let

$$\Omega_2 = \{u \in \mathrm{Ker}L : Nu \in \mathrm{Im}L\}.$$

For $u \in \mathrm{Ker}L$, then $u(t) = c, c \in \mathbb{R}$. In view of $Nu \in \mathrm{Im}L = \mathrm{Ker}Q$, we have $Q(Nu) = 0$, that is

$$I_{0+}^{\alpha+\beta+\gamma} f(t, u, u', u'')(\xi) = 0.$$

By ($H_2$), there exist constants $t_0 \in [0, 1]$ such that

$$|u(t_0)| = |c| \leq A. \tag{3.12}$$

Therefore, $\Omega_2$ is bounded.

Let

$$\Omega_3 = \{u \in \mathrm{Ker}L : \lambda u + (1 - \lambda)QNu = 0, \lambda \in [0, 1]\}.$$

For $u \in \mathrm{Ker}L$, then $u(t) = c_0$. By the definition of the set $\Omega_3$, we have

$$\lambda c_0 + (1 - \lambda)QN(c_0) = 0. \tag{3.13}$$

Next, the proof is divided into three cases.

**Case** 1. If $\lambda = 1$, then $c_0 = 0$.

**Case** 2. If $\lambda = 0$, similar to the proof of the boundness of $\Omega_2$, we have $|c_0| \leq A$.

**Case** 3. If $\lambda \in (0, 1)$, we also have $|c_0| \leq B$. Otherwise, if $|c_0| > B$, in view of the first part of ($H_3$), we obtain

$$\lambda c_0^2 + (1 - \lambda)c_0 \cdot QN(c_0) > 0,$$

which contradicts (3.13). Thus, $\Omega_3$ is bounded.

If the second part of ($H_3$) holds, we can prove the set

$$\Omega_3' = \{u \in \mathrm{Ker}L : -\lambda u + (1 - \lambda)QNu = 0, \lambda \in [0, 1]\}$$

is bounded.

Let $\Omega = \{x \in X : \|x\|_X < \max\{M, A, B\} + 1\}$. It follows from Lemma 3.2 and Lemma 3.3 that $L$ is a Fredholm operator of index zero and $N$ is $L-$compact on $\bar{\Omega}$. By (3.11) and (3.12), we get that the following two conditions of Theorem 2.5 are satisfied

(1). $Lu \neq \lambda Nu$, for every $u \in [(\mathrm{dom}L \setminus KerL) \cap \partial\Omega] \times (0, 1)$;

(2). $Nu \notin \mathrm{Im}L$ for every $u \in \mathrm{Ker}L \cap \partial\Omega$.

Let $H(u, \lambda) = \pm\lambda Iu + (1 - \lambda)JQNu$, where $I$ is the identical operator. By the proof of the boundness of $\Omega_3$, we have that $H(u, \lambda) \neq 0$ for $u \in \mathrm{Ker}L \cap \partial\Omega$. Therefore, via the homotopy property of degree, we obtain

$$\begin{aligned}
\deg(JQN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot, 0), \Omega \cap \mathrm{Ker}L, 0) \\
&= \deg(H(\cdot, 1), \Omega \cap \mathrm{Ker}L, 0) \\
&= \deg(I, \Omega \cap \mathrm{Ker}L, 0) \\
&= 1 \neq 0.
\end{aligned}$$

So, the condition (3) of Theorem 2.5 is satisfied. Applying Theorem 2.5, we conclude that $Lu = Nu$ has at least one solution in $\mathrm{dom}\, L \cap \overline{\Omega}$. Then fractional jerk equation (1.1) has at least one solution and the proof is complete. □

## 4. Example

*Example* 4.1. Let us consider the following fractional jerk equation:

$$\begin{cases}(D_{0+}^{0.6}(D_{0+}^{0.7}(D_{0+}^{0.8}u)))(t) = f(t, u(t), u'(t), u''(t)),\\ u(0) = u(\frac{1}{3}), D_{0+}^{0.8}u(0) = (D_{0+}^{0.7}(D_{0^+}^{0.8}u))(0) = 0,\end{cases} \tag{4.1}$$

where

$$f(t, u(t), u'(t), u''(t)) = \frac{t}{20} + \frac{u(t)}{10} + \frac{\cos u'(t)}{20} + \frac{\arctan^2 u''(t)}{5\pi^2}.$$

Corresponding to BVP (1.3), we have that $\alpha = 0.8, \beta = 0.7, \gamma = 0.6, \alpha + \beta + \gamma = 2.1 > 2$. By calculation, we have

$$\frac{\Gamma(\alpha+\beta+\gamma+1)}{2+(\alpha+\beta+\gamma+1)^2} = \frac{\Gamma(3.1)}{11.61} \approx 0.19.$$

By a simple proof, we have

$$\begin{aligned}|f(t, u(t), u'(t), u''(t))| &= \left|\frac{t}{20} + \frac{u(t)}{10} + \frac{\cos u'(t)}{20} + \frac{\arctan^2 u''(t)}{5\pi^2}\right|\\ &\leq \frac{3}{20} + \frac{|u(t)|}{10}.\end{aligned}$$

Choose $\varphi_0(t) = \frac{3}{20}$, $\varphi_1(t) = \frac{1}{10}$, $\varphi_2 = \varphi_3 = 0$, then ($H_1$) is satisfied. Furthermore,

$$\varphi = \max\{\|\varphi_0(t)\|_\infty, \|\varphi_1(t)\|_\infty, \|\varphi_2(t)\|_\infty\} = 0.15 < 0.19.$$

Hence, (3.9) holds. In addition, by a simple calculation, by choosing $A = B = \frac{\sqrt{2}}{2}$, then condition ($H_2$) and ($H_3$) are satisfied. Consequently, By Theorem 3.4, BVP (4.1) has at least one solution.

## 5. Conclusions

In this article, we considered a class of nonlinear fractional jerk equations with boundary value conditions at resonance. By using the theory of Mawhin's continuation theorem, we obtained the existence of solutions of boundary value problems (1.1). An example is presented to illustrate the main results. To the authors' knowledge, the existence of solutions for fractional jerk BVPs at resonance has not been reported. So, it is novel in term of fractional jerk equations and our result enriches and extends some related results in the literature. There is still more work to be done in the future on this interesting problem. For example, seeking the existence of positive solutions for fractional jerk equations with resonant boundary conditions.

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## Conflict of interest

The authors declare no conflicts of interest in this paper.

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