Mathematics

## Research article

# Existence of $W_{0}^{1,1}(\Omega)$ solutions to nonlinear elliptic equation with singular natural growth term 

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Abstract: In this paper, we investigate the existence of $W_{0}^{1,1}(\Omega)$ solutions to the following elliptic equation with principal part having noncoercivity and singular quadratic term

$$
\left\{\begin{aligned}
-\operatorname{div}\left(\frac{\nabla u}{\left.(1+|l|)^{y}\right)}\right)+\frac{|\nabla u|^{2}}{u^{\theta}} & =f, & x \in \Omega, \\
u & =0, & x \in \partial \Omega,
\end{aligned}\right.
$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^{N}(N \geq 3), \gamma>0, \frac{N}{N-1} \leq \theta<2, f \in L^{m}(\Omega)(m \geq 1)$ is a nonnegative function.

Keywords: noncoercivity; existence; nonlinear elliptic equation
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## 1. Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^{N}(N \geq 3)$ with smooth boundary $\partial \Omega$. In this paper, we consider the existence of $W_{0}^{1,1}(\Omega)$ solutions to the following elliptic problem

$$
\left\{\begin{align*}
-\operatorname{div}(M(x, u) \nabla u)+\frac{|\nabla u|^{2}}{u^{\theta}} & =f, & & x \in \Omega,  \tag{1.1}\\
u & =0, & & x \in \partial \Omega,
\end{align*}\right.
$$

where $\frac{N}{N-1} \leq \theta<2, M: \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{N^{2}}$ is a symmetric Carathéodory matrix function, which satisfies
the following assumptions: for some real constants $\gamma>0, \alpha>0, \beta>0$,

$$
\begin{equation*}
|M(x, s)| \leq \beta, \quad M(x, s) \xi \cdot \xi \geq \frac{\alpha}{(a(x)+\mid s)^{\gamma}}|\xi|^{2}, \tag{1.2}
\end{equation*}
$$

for almost every $x \in \Omega,(s, \xi) \in \mathbb{R} \times \mathbb{R}^{N}$, where $a(x)$ is a measurable function, such that

$$
\begin{equation*}
0<\zeta \leq a(x) \leq \rho, \tag{1.3}
\end{equation*}
$$

for some positive constants $\zeta, \rho$.
We note that there are two difficulties in dealing with (1.1), the first one is the fact that, due to hypothesis (1.2), the differential operator $A(u)=-\operatorname{div}(M(x, u) \nabla u)$ is well defined in $H_{0}^{1}(\Omega)$, but it not coercive on $H_{0}^{1}(\Omega)$ when $u$ is large enough. Therefore, the classical Leray-Lions theorem cannot be applied even if $f$ is sufficiently regular. The second difficulty is dealing with lower order term which singular natural growth with respect to the gradient. In order to overcome these difficulties, we approximate problem (1.1) by means of truncations in $M(x, s)$ to get a coercive differential operator on $H_{0}^{1}(\Omega)$.

The existence of $W_{0}^{1,1}(\Omega)$ solution to elliptic problem has been studied by many authors. Boccardo and Croce [4] proved the existence of $W_{0}^{1,1}(\Omega)$ solutions to problem

$$
\left\{\begin{aligned}
-\operatorname{div}\left(\frac{a(x) \nabla u}{\left.(1+\mid u)^{\gamma}\right)}\right. & =f, & & x \in \Omega, \\
u & =0, & & x \in \partial \Omega,
\end{aligned}\right.
$$

where $a: \Omega \rightarrow \mathbb{R}$ is a measurable function which satisfies (1.3), $f \in L^{m}(\Omega)$ with

$$
m=\frac{N}{N+1-\gamma(N-1)}, \quad \frac{1}{N-1}<\gamma<1 .
$$

In the literature [6], the authors considered the existence and regularity of solutions to the following elliptic equation with noncoercivity

$$
\left\{\begin{array}{rlrl}
-\operatorname{div}(a(x, u) \nabla u) & =f, & x \in \Omega,  \tag{1.4}\\
u & =0, & & x \in \partial \Omega,
\end{array}\right.
$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^{N}(N \geq 3), f \in L^{m}(\Omega)$ and $a(x, s): \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function which satisfies

$$
\frac{\alpha}{(1+\mid s)^{\gamma}} \leq a(x, s) \leq \beta,
$$

where $0 \leq \gamma<1$.The existence results of solutions to problem (1.4)are as following:

- There exists a weak solution $u \in H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$ to (1.4) if $m>\frac{N}{2}$.
- There exists a weak solution $u \in H_{0}^{1}(\Omega) \cap L^{r}(\Omega)$ to (1.4) with $r=\frac{N m(1-\gamma)}{N-2 m}$ if

$$
\frac{2 N}{N+2-\gamma(N-2)} \leq m<\frac{N}{2} .
$$

- There exists a distributional solution $u \in W_{0}^{1, q}(\Omega)$ to (1.4) with $q=\frac{N m(1-\gamma)}{N-m(1+\gamma)}<2$ if

$$
\frac{N}{N+1-\gamma(N-1)}<m<\frac{2 N}{N+2-\gamma(N-2)} .
$$

In [15], the under the assumption (1.2)-(1.3), Souilah proved the existence results of solutions to problem

$$
\left\{\begin{array}{cl}
-\operatorname{div}(M(x, u) \nabla u)+\frac{|\nabla u|^{2}}{u^{\theta}}=f+\lambda u^{r}, & x \in \Omega,  \tag{1.5}\\
u=0, & x \in \partial \Omega,
\end{array}\right.
$$

where $0<\theta<1,0<r<2-\theta, \lambda>0, f \in L^{m}(\Omega)(m \geq 1)$. There exists at least a solution to problem (1.5):

- If $\frac{2 N}{2 N-\theta(N-2)} \leq m<\frac{N}{2}$, then $u \in H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$.
- If $1<m<\frac{2 N}{2 N-\theta(N-2)}$, then $u \in W_{0}^{1, q}(\Omega)$ with $q=\frac{N m(2-\theta)}{N-m \theta}$.
- If $m \geq \frac{N}{2}$, then $u \in H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$.

Moreover, the existence of solutions $u \in H_{0}^{1}(\Omega)$ to problem (1.5) with $\lambda=0$ have been obtained in [9]. Some other related results see [1,3,5,7,10-12, 14, 16].

Based on the above research results, the aim of this article is to study the existence of $W_{0}^{1,1}(\Omega)$ solution to problem (1.1).

In order to state the main results of this paper, the following definition need to be introduced. We use the following notion of distributional solution to problem (1.1).

Definition 1.1. We say that $u \in W_{0}^{1,1}(\Omega)$ is a distributional solution to problem (1.1) if $u>0$ in $\Omega$, $\frac{|\nabla u|^{2}}{u^{\theta}} \in L^{1}(\Omega)$ and

$$
\int_{\Omega} M(x, u) \nabla u \cdot \nabla \varphi+\int_{\Omega} \frac{|\nabla u|^{2}}{u^{\theta}} \varphi=\int_{\Omega} f \varphi,
$$

for every $\varphi \in C_{0}^{\infty}(\Omega)$.
Our main results are following:
Theorem 1.2. Assume that (1.2)-(1.3) hold, $f \in L^{m}(\Omega)$ is a nonnegative function with

$$
\begin{equation*}
m=\frac{N}{2 N-\theta(N-1)}, \quad \frac{N}{N-1}<\theta<2 . \tag{1.6}
\end{equation*}
$$

Then there exists a distributional solution $u \in W_{0}^{1,1}(\Omega)$ to problem (1.1).
Remark 1.3. Notice that the result of previous theorem do not depend on $\gamma$.
Remark 1.4. Observe that, $m>1$ if and only if $\theta>\frac{N}{N-1}$.
For $f \in L^{1}(\Omega)$, we have the following theorem.
Theorem 1.5. Assume (1.2)-(1.3) hold, $f \in L^{1}(\Omega)$ is a nonnegative function and $\theta=\frac{N}{N-1}$. Then there exists a distributional solution $u \in W_{0}^{1,1}(\Omega)$ to problem (1.1).

The paper is organized as follows. In section 2, we collect some definitions and useful tools. The proof of Theorem 1.2 and 1.5 be given in section 3 .

## 2. Preliminaries

In order to prove our main results, we need to introduce a basic definition and some lemmas.
Definition 2.1. For all $k \geq 0$, the truncation function defined by

$$
T_{k}(s)=\max \{-k, \min \{k, s\}\}, \quad G_{k}(s)=s-T_{k}(s) .
$$

Let $0<\varepsilon<1$, we approximate problem (1.1) by the following non-singular problem

$$
\left\{\begin{align*}
-\operatorname{div}\left(M\left(x, T_{\frac{1}{\varepsilon}}\left(u_{\varepsilon}\right)\right) \nabla u_{\varepsilon}\right)+\frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(\mid u_{\varepsilon}+\varepsilon \varepsilon\right)^{0+1}} & =f_{\varepsilon}, & x \in \Omega,  \tag{2.1}\\
u_{\varepsilon} & =0, & x \in \partial \Omega,
\end{align*}\right.
$$

where $f_{\varepsilon}=T_{\frac{1}{\varepsilon}}(f)$. Problem (2.1) admits at least a solution $u_{\varepsilon} \in H_{0}^{1}(\Omega) \cap L^{\infty}(\Omega)$ by Theorem 2 of [8]. Due to the fact that $f_{\varepsilon} \geq 0$ and quadratic lower order term has the same sign of the solution, it is easy to prove that $u_{\varepsilon} \geq 0$ by taking $u_{\varepsilon}^{-}$as a test function in (2.1).

Lemma 2.2. Let $u_{\varepsilon}$ be the solutions to problem (2.1). Then

$$
\begin{equation*}
\int_{\Omega} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \leq \int_{\Omega} f . \tag{2.2}
\end{equation*}
$$

Proof. For fixed $h>0$, taking $\frac{T_{h}\left(u_{s}\right)}{h}$ as a test function in (2.1). Dropping the first term, we obtain

$$
\int_{\Omega} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \frac{T_{h}\left(u_{\varepsilon}\right)}{h} \leq \int_{\Omega} f_{\varepsilon} \frac{T_{h}\left(u_{\varepsilon}\right)}{h} .
$$

Using the fact that $f_{\varepsilon} \leq f$ and $\frac{T_{h}\left(u_{\varepsilon}\right)}{h} \leq 1$, then

$$
\int_{\Omega} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \frac{T_{h}\left(u_{\varepsilon}\right)}{h} \leq \int_{\Omega} f
$$

Letting $h \rightarrow 0$, we deduce (2.2) by the Fatou Lemma.
Lemma 2.3. Let $\delta>0$ and $0<\varepsilon<1$. Then there exists $C>0$, such that

$$
\frac{\alpha \delta(t+\varepsilon)^{\theta-2}}{(\rho+t)^{\gamma}}+\frac{t}{t+\varepsilon} \geq C
$$

for every $t \geq 0$.
Proof. Clearly, if $t \geq \varepsilon$, we have $\frac{t}{t+\varepsilon} \geq \frac{1}{2}$, while if $t<\varepsilon$, we have

$$
\frac{\alpha \delta(t+\varepsilon)^{\theta-2}}{(\rho+t)^{\gamma}} \geq \frac{\alpha \delta}{(\rho+t)^{\gamma}(2 \varepsilon)^{2-\theta}} \geq \frac{\alpha \delta}{2^{2-\theta}(\rho+1)^{\gamma}},
$$

since $\varepsilon<1$. Therefore, Lemma 2.3 is proved.

## 3. Proof of main results

In this section, $C$ denotes a generic constant whose value might change from line to line. We prove the existence results of Theorems 1.2 and 1.5 by considering the following approximate problem

$$
\left\{\begin{align*}
-\operatorname{div}\left(M\left(x, T_{\frac{1}{\varepsilon}}\left(u_{\varepsilon}\right)\right) \nabla u_{\varepsilon}\right)+\frac{\left.u_{\varepsilon} \nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon \varepsilon^{\rho+1}\right.}=f_{\varepsilon}, & x \in \Omega,  \tag{3.1}\\
u_{\varepsilon}=0, & x \in \partial \Omega
\end{align*}\right.
$$

Proof of Theorem 1.2. Step 1: Let $\delta=\theta-\frac{N}{N-1}$, then $\delta>0$ by (1.6). Choosing $\left(u_{\varepsilon}+\varepsilon\right)^{\delta}-\left(u_{\varepsilon}+\varepsilon\right)^{\delta-1}$ as a test function in the approximate problem (3.1), we find

$$
\begin{aligned}
& \int_{\Omega} M\left(x, T_{\frac{1}{\varepsilon}}\left(u_{\varepsilon}\right)\right) \nabla u_{\varepsilon} \cdot \nabla u_{\varepsilon}\left[\delta\left(u_{\varepsilon}+\varepsilon\right)^{\delta-1}+(1-\delta)\left(u_{\varepsilon}+\varepsilon\right)^{\delta-2}\right]+\int_{\Omega} \frac{u_{\varepsilon}\left(u_{\varepsilon}+\varepsilon\right)^{\delta}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \\
= & \int_{\Omega} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}}\left(u_{\varepsilon}+\varepsilon\right)^{\delta-1}+\int_{\Omega} f_{\varepsilon}\left[\left(u_{\varepsilon}+\varepsilon\right)^{\delta}-\left(u_{\varepsilon}+\varepsilon\right)^{\delta-1}\right] .
\end{aligned}
$$

Combining (1.2)-(1.3) and dropping the positive term, we obtain

$$
\begin{aligned}
& \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\left(u_{\varepsilon}+\varepsilon\right)^{\delta-\theta}\left[\frac{\alpha(1-\delta)\left(u_{\varepsilon}+\varepsilon\right)^{\theta-2}}{\left(\rho+u_{\varepsilon}\right)^{\gamma}}+\frac{u_{\varepsilon}}{u_{\varepsilon}+\varepsilon}\right] \\
\leq & \int_{\Omega} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}}\left(u_{\varepsilon}+\varepsilon\right)^{\delta-1}+\int_{\Omega} f_{\varepsilon}\left(u_{\varepsilon}+\varepsilon\right)^{\delta} .
\end{aligned}
$$

Since $1-\delta>0$, according to Lemma 2.3, we have

$$
C \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\left(u_{\varepsilon}+\varepsilon\right)^{\delta-\theta} \leq \int_{\Omega} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}}\left(u_{\varepsilon}+\varepsilon\right)^{\delta-1}+\int_{\Omega} f_{\varepsilon}\left(u_{\varepsilon}+\varepsilon\right)^{\delta} .
$$

Using the fact that $u_{\varepsilon} \geq 0, f_{\varepsilon} \leq f$ and (2.2), we obtain

$$
\begin{align*}
C \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\left(u_{\varepsilon}+\varepsilon\right)^{\delta-\theta} & \leq \varepsilon^{\delta-1} \int_{\Omega} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}}+\int_{\Omega} f\left(u_{\varepsilon}+\varepsilon\right)^{\delta} \\
& \leq \varepsilon^{\delta-1} \int_{\Omega} f+\int_{\Omega} f\left(u_{\varepsilon}+\varepsilon\right)^{\delta} . \tag{3.2}
\end{align*}
$$

Observe that the left hand side of (3.2) can be rewritten as

$$
\begin{equation*}
C \int_{\Omega}\left|\nabla\left[\left(u_{\varepsilon}+\varepsilon\right)^{\frac{\delta-\theta+2}{2}}-\varepsilon^{\frac{\delta-\theta+2}{2}}\right]\right|^{2} . \tag{3.3}
\end{equation*}
$$

Then, (3.2) and (3.3) imply

$$
\begin{equation*}
C \int_{\Omega}\left|\nabla\left[\left(u_{\varepsilon}+\varepsilon\right)^{\frac{\delta-\theta+2}{2}}-\varepsilon^{\frac{\delta-\theta+2}{2}}\right]\right|^{2} \leq \varepsilon^{\delta-1} \int_{\Omega} f+\int_{\Omega} f\left(u_{\varepsilon}+\varepsilon\right)^{\delta} . \tag{3.4}
\end{equation*}
$$

By the Sobolev inequality, satisfy

$$
\begin{equation*}
\left[\int_{\Omega}\left|\left(u_{\varepsilon}+\varepsilon\right)^{\frac{\delta-\theta+2}{2}}-\varepsilon^{\frac{\delta-\theta+2}{2}}\right|^{z^{\frac{z}{2}}}\right]^{\frac{2}{2 \pi}} \leq C \int_{\Omega}\left|\nabla\left[\left(u_{\varepsilon}+\varepsilon\right)^{\frac{\delta-\theta+2}{2}}-\varepsilon^{\frac{\delta-\theta+2}{2}}\right]\right|^{2} . \tag{3.5}
\end{equation*}
$$

Using the Hölder inequality and (3.4)-(3.5), we get

$$
\left[\int_{\Omega}\left|\left(u_{\varepsilon}+\varepsilon\right)^{\frac{\delta-\theta+2}{2}}-\varepsilon^{\frac{\delta-\theta+2}{2}}\right|^{2^{*}}\right]^{\frac{2}{2^{*}}} \leq C\|f\|_{L^{m}(\Omega)}+C\|f\|_{L^{m}(\Omega)}\left[\int_{\Omega}\left(u_{\varepsilon}+\varepsilon\right)^{\delta m^{\prime}}\right]^{\frac{1}{m^{\prime}}}
$$

Since $\left|(t+\varepsilon)^{s}-\varepsilon^{s}\right|^{2^{*}} \geq C\left[(t+\varepsilon)^{2^{*} s}-1\right]$ for every $t \geq 0$ and for suitable constant $C$ independent on $\varepsilon$, then we find

$$
\begin{equation*}
\left(\int_{\Omega}\left[\left(u_{\varepsilon}+\varepsilon\right)^{\frac{2^{*}(\delta-\theta+2)}{2}}-1\right]\right)^{\frac{2}{2^{*}}} \leq C\|f\|_{L^{m}(\Omega)}+C\|f\|_{L^{m}(\Omega)}\left[\int_{\Omega}\left(u_{\varepsilon}+\varepsilon\right)^{\delta m^{\prime}}\right]^{\frac{1}{m^{\prime}}} . \tag{3.6}
\end{equation*}
$$

Thanks to the choice of $\delta$, we have

$$
\frac{2^{*}(\delta-\theta+2)}{2}=\delta m^{\prime}=\frac{N}{N-1} .
$$

Moreover $\frac{2}{2^{*}}>\frac{1}{m^{\prime}}$ since $m<\frac{N}{2}$. Then (3.6) implies that

$$
\begin{equation*}
\int_{\Omega} u_{\varepsilon}^{\frac{N}{N-1}} \leq C \tag{3.7}
\end{equation*}
$$

Observe that $\delta-\theta=-\frac{N}{N-1}$, then, (3.2), (3.7) follow

$$
\begin{equation*}
\int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(\varepsilon+u_{\varepsilon}\right)^{\frac{N}{N-1}}} \leq C \tag{3.8}
\end{equation*}
$$

Combining (3.7)-(3.8) with the Hölder inequality, we obtain

$$
\begin{aligned}
\int_{\Omega}\left|\nabla u_{\varepsilon}\right|=\int_{\Omega} \frac{\nabla u_{\varepsilon}}{\left(\varepsilon+u_{\varepsilon}\right)^{\frac{N}{2 N-2}}}\left(\varepsilon+u_{\varepsilon}\right)^{\frac{N}{2 N-2}} & \leq\left[\int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(\varepsilon+u_{\varepsilon}\right)^{\frac{N}{N-1}}}\right]^{\frac{1}{2}}\left[\int_{\Omega}\left(\varepsilon+u_{\varepsilon}\right)^{\frac{N}{N-1}}\right]^{\frac{1}{2}} \\
& \leq C .
\end{aligned}
$$

Then we get that $\left\{u_{\varepsilon}\right\}$ is bounded in $W_{0}^{1,1}(\Omega)$. Hence, there exists a subsequence $\left\{u_{\varepsilon}\right\}$, which converges to a measurable function $u$ a.e. in $L^{r}(\Omega)$ with $1 \leq r<\frac{N}{N-1}$.

Step 2: First, we are going to estimate $\int_{\left\{u_{\varepsilon} \gtrless k\right\}}\left|\nabla u_{\varepsilon}\right|$. Choosing $\left[\left(u_{\varepsilon}+\varepsilon\right)^{\delta}-(k+\varepsilon)^{\delta}\right]^{+}$as a test function in (3.1). By (1.2)-(1.3) and Lemma 2.3, we have

$$
\begin{aligned}
\int_{\left\{u_{\varepsilon} \geq k\right\}} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(\varepsilon+u_{\varepsilon}\right)^{\frac{N}{N-1}}} & \leq\left(\int_{\left\{u_{\varepsilon} \geq k\right\}}|f|^{m}\right)^{\frac{1}{m}}\left(\int_{\left\{u_{\varepsilon} \geq k\right\}}\left(\varepsilon+u_{\varepsilon}\right)^{\frac{N}{N-1}}\right)^{\frac{1}{m^{\prime}}} \\
& \leq C\left(\int_{\left\{u_{\varepsilon} \geq k\right\}}|f|^{m}\right)^{\frac{1}{m}}
\end{aligned}
$$

Using the Hölder inequality and (3.7), we find

$$
\begin{equation*}
\int_{\left\{u_{\varepsilon} \geq k\right\}}\left|\nabla u_{\varepsilon}\right|=\int_{\left\{u_{\varepsilon} \geq k\right\}} \frac{\nabla u_{\varepsilon}}{\left(\varepsilon+u_{\varepsilon}\right)^{\frac{N}{2 N-2}}}\left(\varepsilon+u_{\varepsilon}\right)^{\frac{N}{2 N-2}} \leq C\left(\int_{\left\{u_{\varepsilon} \geq k\right\}}|f|^{m}\right)^{\frac{1}{2 m}} . \tag{3.9}
\end{equation*}
$$

Choosing $T_{k}\left(u_{\varepsilon}\right)$ as a test function in (3.1). Dropping the nonnegative lower order term, by (1.2)(1.3) and the boundedness of $u_{\varepsilon}$ in $L^{\frac{N}{N-1}}(\Omega)$, we get

$$
\begin{equation*}
\int_{\Omega}\left|\nabla T_{k}\left(u_{\varepsilon}\right)\right|^{2} \leq \frac{k(\rho+k)^{\gamma}}{\alpha}\|f\|_{L^{\prime}(\Omega)} . \tag{3.10}
\end{equation*}
$$

This implies that $T_{k}\left(u_{\varepsilon}\right) \rightharpoonup T_{k}(u)$ weakly in $W_{0}^{1,2}(\Omega)$.
Let $E$ be a measurable subset of $\Omega$, and $i=1, \cdots, N$. By the Hölder inequality and (3.9)-(3.10), we obtain

$$
\begin{align*}
\int_{E}\left|\frac{\partial u_{\varepsilon}}{\partial x_{i}}\right| & \leq \int_{E}\left|\nabla u_{\varepsilon}\right| \leq \int_{E}\left|\nabla T_{k}\left(u_{\varepsilon}\right)\right|+\int_{\left\{u_{\varepsilon} \geq k\right\}}\left|\nabla u_{\varepsilon}\right| \\
& \leq \operatorname{meas}(E)^{\frac{1}{2}}\left(\int_{E}\left|\nabla T_{k}\left(u_{\varepsilon}\right)\right|^{2}\right)^{\frac{1}{2}}+C\left(\int_{\left\{u_{\varepsilon} \geq k\right\}}|f|^{m}\right)^{\frac{1}{2 m}} \tag{3.11}
\end{align*}
$$

The estimates (3.7) and (3.11) shows that the sequence $\left\{\frac{\partial u_{\varepsilon}}{\partial x_{i}}\right\}$ is equi-integrable. Thus, by the DunfordPettis theorem, there exists a subsequence $\left\{u_{\varepsilon}\right\}$ and $V_{i}$ in $L^{1}(\Omega)$, such that $\frac{\partial u_{n}}{\partial x_{i}} \rightharpoonup V_{i}$ in $L^{1}(\Omega)$. Since $\frac{\partial u_{\varepsilon}}{\partial x_{i}}$ is the distributional partial derivative of $u_{\varepsilon}$, then we have

$$
\int_{\Omega} \frac{\partial u_{\varepsilon}}{\partial x_{i}} \varphi=-\int_{\Omega} u_{\varepsilon} \frac{\partial \varphi}{\partial x_{i}}, \quad \forall \varphi \in C_{0}^{\infty}(\Omega),
$$

for every $\varepsilon>0$.
Since $\frac{\partial u_{\varepsilon}}{\partial x_{i}} \rightharpoonup V_{i}$ in $L^{1}(\Omega)$ and $u_{\varepsilon} \rightarrow u$ in $L^{1}(\Omega)$, we find

$$
\int_{\Omega} V_{i} \varphi=-\int_{\Omega} u \frac{\partial \varphi}{\partial x_{i}}, \quad \forall \varphi \in C_{0}^{\infty}(\Omega) .
$$

This implies that $V_{i}=\frac{\partial u}{\partial x_{i}}$ for every $i$.
Step 3: We prove that $\frac{\left.u_{\varepsilon} \nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon \varepsilon^{\phi+1}\right.}$ is equi-integrable. Let $E \subset \subset \Omega$, then

$$
\int_{E} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \leq \int_{E \cap\left\{u_{\varepsilon} \leq k\right\}} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}}+\int_{E \cap\left\{u_{\varepsilon} \geq k\right\}} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} .
$$

For every subset $E \subset \subset$,

$$
\int_{E \cap\left\{u_{\varepsilon} \leq k\right\}} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \leq \int_{E \cap\left\{u_{\varepsilon} \leq k\right\}} \frac{1}{u_{\varepsilon}^{\theta}}\left|\nabla T_{k}\left(u_{\varepsilon}\right)\right|^{2} \leq C \int_{E \cap\left\{u_{\varepsilon} \leq k\right\}}\left|\nabla T_{k}\left(u_{\varepsilon}\right)\right|^{2},
$$

since $u_{\varepsilon} \geq C>0$ in $E$ by Proposition 2 of [9]. Moreover, since $T_{k}\left(u_{\varepsilon}\right) \rightharpoonup T_{k}(u)$ weakly in $W_{0}^{1,2}(\Omega)$, then there exists $\varepsilon_{n}, \delta>0$, such that

$$
\begin{equation*}
\int_{E \cap\left\{u_{s} \leq k\right\}}\left|\nabla T_{k}\left(u_{\varepsilon}\right)\right| d x \leq \frac{\epsilon}{2}, \quad \forall \varepsilon \geq \varepsilon_{n}, \tag{3.12}
\end{equation*}
$$

for every $\epsilon>0$ if $\mu(E)<\delta$.

Choosing $T_{1}\left(u_{\varepsilon}-T_{k-1}\left(u_{\varepsilon}\right)\right)$ as a test function in the approximate problem (3.1), dropping the nonnegative term, we have

$$
\begin{equation*}
\int_{\left\{u_{z} \geq k\right\}} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \leq \int_{\left\{u_{\varepsilon} \geq k-1\right\}} f . \tag{3.13}
\end{equation*}
$$

Observe there exists a constant $C>0$, such that $\mu\left(u_{\varepsilon} \geq k-1\right) \leq \frac{C}{k-1}$. As $u_{\varepsilon}$ are uniformly bounded in $L^{\frac{N}{N-1}}(\Omega)$. This implies the right hand side of (3.13) converges to 0 as $k \rightarrow \infty$. Thus, we deduce there exists $k_{0}>1$, such that

$$
\begin{equation*}
\int_{\left\{u_{\varepsilon}<k\right\}} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \leq \frac{\epsilon}{2}, \quad \forall k>k_{0}, \tag{3.14}
\end{equation*}
$$

for every $\epsilon>0$. The (3.12), (3.14) imply that $\frac{u_{s}\left|\nabla u_{s}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon \varepsilon^{\theta+1}\right.}$ is equi-integrable and converges a.e. to $\frac{|\nabla u|^{2}}{u^{\theta}}$.
Let $u$ the weak limit of the sequence of approximated solutions $u_{\varepsilon}$. Thanks to (2.2), we have

$$
\int_{\Omega} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \leq \int_{\Omega} f .
$$

Using the Fatou lemma, that $u_{\varepsilon}$ convergence to $u$ a.e, $\nabla u_{\varepsilon}$ convergence to $\nabla u$ a.e and the strict positivity of $u_{\varepsilon}$ imply

$$
\int_{\Omega} \frac{|\nabla u|^{2}}{u^{\theta}} \leq \int_{\Omega} f \leq C
$$

This show that $\frac{|\nabla u|^{2}}{u^{\theta}} \in L^{1}(\Omega)$.
Since $u_{\varepsilon}$ is bounded and $\nabla u_{\varepsilon} \rightarrow \nabla u$ a.e, it follow $M\left(x, T_{\frac{1}{\varepsilon}}\left(u_{\varepsilon}\right) \nabla u_{\varepsilon} \rightarrow M(x, u) \nabla u\right.$ a.e. Hence, we can pass to the limit in (3.1). Thus prove that $u \in W_{0}^{1,1}(\Omega)$ is a distributional solution of (1.1) and yields the conclusion of the proof of Theorem 1.1.

Proof of Theorem 1.5. Step 1: For $0<\varepsilon<1$, according to Lemma 2.2, we have

$$
\begin{equation*}
\frac{1}{2^{\theta+1}} \int_{\left\{u_{\varepsilon} \geq 1\right\}} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{u_{\varepsilon}^{\theta}} \leq \int_{\left\{u_{\varepsilon} \geq 1\right\}} \frac{u_{\varepsilon}\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+\varepsilon\right)^{\theta+1}} \leq\|f\|_{L^{1}(\Omega)} \tag{3.15}
\end{equation*}
$$

By the Sobolev inequality, (3.15) lead to

$$
\begin{equation*}
\left[\int_{\Omega}\left|u_{\varepsilon}^{\frac{2-\theta}{2}}-1\right|^{2^{*}}\right]^{\frac{2}{2^{*}}} \leq C\|f\|_{L^{1}(\Omega)} \tag{3.16}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\left[\int_{\Omega} u_{\varepsilon}^{\frac{(-\theta-\theta)^{*}}{2}}\right]^{\frac{2}{2^{*}}} \leq C+C\|f\|_{L^{1}(\Omega)} \tag{3.17}
\end{equation*}
$$

Observe that $\theta=\frac{(2-\theta)^{2}}{2}=\frac{N}{N-1}$. Then (3.17) shows that

$$
\begin{equation*}
\int_{\Omega} u_{\varepsilon}^{\frac{N}{N-1}} \leq C \tag{3.18}
\end{equation*}
$$

Using the Hölder inequality and (3.15), (3.18), we obtain

$$
\begin{aligned}
\int_{\Omega}\left|\nabla G_{1}\left(u_{\varepsilon}\right)\right|=\int_{\left\{u_{\varepsilon} \geq 1\right\}} \frac{\left|\nabla G_{1}\left(u_{\varepsilon}\right)\right|}{u_{\varepsilon}^{\frac{\theta}{2}}} u_{\varepsilon}^{\frac{\theta}{2}} & \leq\left[\int_{\left\{u_{s} \geq 1\right\}} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{u_{\varepsilon}^{\theta}}\right]^{\frac{1}{2}}\left[\int_{\left\{u_{\varepsilon} \geq 1\right\}} u_{\varepsilon}^{\theta}\right]^{\frac{1}{2}} \\
& \leq C\|f\|_{L^{1}(\Omega)} .
\end{aligned}
$$

This fact show that $G_{1}\left(u_{\varepsilon}\right)$ is bounded in $W_{0}^{1,1}(\Omega)$.
Choosing $T_{1}\left(u_{\varepsilon}\right)$ as a test function in (3.1), it is easy to prove that $T_{1}\left(u_{\varepsilon}\right)$ is bounded in $H_{0}^{1}(\Omega)$ ), hence in $W_{0}^{1,1}(\Omega)$. Since $u_{\varepsilon}=G_{1}\left(u_{\varepsilon}\right)+T_{1}\left(u_{\varepsilon}\right)$, we deduce that $u_{\varepsilon}$ is bounded in $W_{0}^{1,1}(\Omega)$.

Moreover, due to (3.15) and the Hölder inequality, we have

$$
\begin{equation*}
\int_{\left\{u_{z} \geq k\right\}}\left|\nabla u_{\varepsilon}\right|=\int_{\left\{u_{\varepsilon} \geq k\right\}} \frac{\left|\nabla u_{\varepsilon}\right|}{u_{\varepsilon}^{\frac{\theta}{2}}} u_{\varepsilon}^{\frac{\theta}{2}} \leq C\|f\|_{L^{1}(\Omega)}^{\frac{1}{2}} \tag{3.19}
\end{equation*}
$$

That (3.10), (3.19) implies, for every measurable subset $E$, we have

$$
\begin{aligned}
\int_{E}\left|\frac{\partial u_{\varepsilon}}{\partial x_{i}}\right| & \leq \int_{E}\left|\nabla u_{\varepsilon}\right| \leq \int_{E}\left|\nabla T_{k}\left(u_{\varepsilon}\right)\right|+\int_{\left\{u_{\varepsilon} \geq k\right\}}\left|\nabla u_{\varepsilon}\right| \\
& \leq \operatorname{meas}(E)^{\frac{1}{2}}\left[\frac{k(\rho+k)^{\gamma}}{\alpha}\|f\|_{L^{1}(\Omega)}\right]^{\frac{1}{2}}+C\|f\|_{L^{1}(\Omega)}^{\frac{1}{2}} .
\end{aligned}
$$

Thus, we prove that $u_{\varepsilon} \rightharpoonup u$ in $W_{0}^{1,1}(\Omega)$. Then pass to the limit in problem (3.1), as in the proof of Theorem 1.2, it is sufficient to observe that $u \in W_{0}^{1,1}(\Omega)$ is a distributional solution of (1.1). This concludes the proof the Theorem 1.5.

## 4. Conclusions

In this paper, we main consider the existence of $W_{0}^{1,1}(\Omega)$ solutions to a elliptic equation with principal part having noncoercivity. The main results show that the singular quadratic term has an important impact on this existence.

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## Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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