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Research article

Fixed point theorem for orthogonal contraction of Hardy-Rogers-type mapping on O-complete metric spaces

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Abstract: In this paper, we introduce the notion of an orthogonal (F, ψ)-contraction of Hardy-Rogers-type mapping and prove some fixed point theorem for such contraction mappings in orthogonally metric spaces. Our result extend and improve the main result of the paper by Sawangsup et al. [Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces, J. Fixed Point Theory Appl. (2020) 22:10].

Keywords: orthogonal metric space; fixed point; orthogonal contraction of Hardy-Rogers-type mapping
Mathematics Subject Classification: 47H10

1. Introduction

In 1922, Banach [1] introduced and established a fixed point result which is well known as "Banach fixed point theorem", it is one of the most famous and important results in metric fixed point theory. Since the simplicity and usefulness of the Banach's fixed point theorem, many authors have extended, improved and generalized Banach's fixed point theorem from different perspectives. For more details, we cite the readers to [2,3] and the references therein.

In [5], Gordji et al. introduced the concept of an orthogonal set and extended Banach's fixed point theorem to the setting of orthogonal sets as follows :

Theorem 1.1. *Let $(X, \perp, d)$ be an orthogonal complete metric space. Assume that $T : X \to X$ is a $\perp$-continuous mapping such that the following conditions hold :*

(i) there exists $0 < k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y), \quad \text{for all } x, y \in X \text{ with } x \perp y; \tag{1.1}$$

(ii) T is $\perp$-preserving.

Then, T has a unique fixed point in X. Moreover, for each $x \in X$, the Picard sequence $\{T^n x\}$ converges to the fixed point of T.

Theorem 1.1 is a real generalization of Banach's fixed point theorem, it can be applied to show the existence of a solution for a differential equation which can not be applied by the Banach's fixed point theorem, the results of Nieto and Rodríguez [6] and the results of Ran and Reurings [7].

We know that by using some control functions can generalize the Banach contractive condition (see [8,9,12,14,15]). Recently, Sawangsup et al. [10] modified the concept of F-contraction mappings to orthogonal sets and given some new fixed point theorems for F-contraction mappings in orthogonal complete metric spaces.

Definition 1.1. ([13]) Let $\mathcal{F}$ be the set of functions $F : (0, +\infty) \to \mathbb{R}$, such that

(i) F is strictly increasing;

(ii) For every sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to+\infty} F(t_n) = -\infty \iff \lim_{n\to+\infty} t_n = 0$;

(iii) There exists $k \in (0, 1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$.

Recently, Sawangsup et al. proved the following fixed point theorem in orthogonal complete orthogonal metric space.

Theorem 1.2. ([10]) *Let $(X, \perp, d)$ be an orthogonal complete orthogonal metric space with an orthogonal element x_0 and T be a self-mapping on X satisfying the following conditions:*

(i) T is $\perp$-preserving;

(ii) T is an $F_\perp$-contraction mapping;

(iii) T is $\perp$-continuous.

Then, T has a unique fixed point in X. Moreover, the Picard sequence $\{T^n x_0\}$ converges to the fixed point of T.

In [4], Cosentino and Vetro introduced the notion of F-contraction of Hardy-Rogers-type and given a new fixed point theorem which is generalized the result of Wardowski [13].

Definition 1.2. ([4]) Let (X, d) be a complete metric space. A mapping $T : X \to X$ is called an F-contraction of Hardy-Rogers-type if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(Tx, Ty)) \le F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + L d(y, Tx)), \tag{1.2}$$

holds for any $x, y \in X$ with $d(x, y) > 0$, where $\alpha, \beta, \gamma, \delta, L$ are non-negative numbers, $\gamma \ne 1$ and $\alpha + \beta + \gamma + 2\delta = 1$.

By the above definition, Cosentino and Vetro given the following new fixed point theorem in complete metric space.

Theorem 1.3. ([4]) *Let (X, d) be a complete metric space and let T be a self-mapping on X. Assume* *that T is an F-contraction of Hardy-Rogers-type. Then T has a fixed point. Moreover, if $\alpha + \delta + L \leq 1$, then the fixed point of T is unique.*

Inspired by the work of [4, 10, 11], we initiate the concept of an orthogonal generalized contraction of Hardy-Rogers-type mapping and establish some fixed point results for such contraction mapping in orthogonal metric spaces. Our main result is a generalization of Theorem 3.3 in [10].

2. Preliminaries

Firstly, we introduce the following control functions.

Let Ψ be the set of functions $\psi : [0, +\infty) \to (-\infty, 0)$ such that ψ is upper semi-continuous from the right. And let $\mathcal{F}$ be the set of functions as in Definition 1.1.

Now, we give the following concepts, which are used in this paper.

Definition 2.1. Let $X \neq \emptyset$ and $\perp \subset X \times X$ be a binary relation. If $\perp$ satisfies the following condition :

$$\exists x_0[(\forall y \in X, y \perp x_0) \text{ or } (\forall y \in X, x_0 \perp y)],$$

then it is called an orthogonal set (briefly O-set) and x_0 is called an orthogonal element. We denote this O-set by $(X, \perp)$.

Definition 2.2. The triplet $(X, \perp, d)$ is called an orthogonal metric space if $(X, \perp)$ is an O-set and (X, d) is a metric space.

Definition 2.3. Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall n \in \mathbb{N}, x_n \perp x_{n+1}) \text{ or } (\forall n \in \mathbb{N}, x_{n+1} \perp x_n).$$

Definition 2.4. Let $(X, \perp, d)$ be an orthogonal metric space. Then, a mapping $f : X \to X$ is said to be orthogonally continuous (or $\perp$-continuous) in $x \in X$ if, for each O-sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$, we have $f(x_n) \to f(x)$ as $n \to \infty$. Also, f is said to be $\perp$-continuous on X if f is $\perp$-continuous in each $x \in X$.

Remark 2.1. By [5], we know that every continuous mapping is $\perp$-continuous, but the converse is not true.

Definition 2.5. Let $(X, \perp, d)$ be an orthogonal metric space. Then, X is said to be orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent.

Remark 2.2. By [5], we know that every complete metric space is O-complete and the converse is not true.

Definition 2.6. Let $(X, \perp)$ be an O-set. A mapping $f : X \to X$ is said to be $\perp$-preserving if $f(x) \perp f(y)$ whenever $x \perp y$. Also, $f : X \to X$ is said to be weakly $\perp$-preserving if $f(x) \perp f(y)$ or $f(y) \perp f(x)$ whenever $x \perp y$.

Let (X, d) be a metric space. For a given mapping $T : X \to X$, set

$$M_T(x,y) = \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + Ld(y,Tx), \quad x,y \in X, \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta, L$ are non-negative numbers, $\gamma \neq 1$ and $\alpha + \beta + \gamma + 2\delta = 1$.

Now we give the concept of orthogonal (F,ψ)-contraction of Hardy-Rogers-type as follows:

Definition 2.7. Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $T : X \to X$ is said to be orthogonal (F,ψ)-contraction of Hardy-Rogers-type, if there exists $(F,\psi) \in \mathcal{F} \times \Psi$ such that

$$F(d(Tx,Ty)) \leq F(M_T(x,y)) + \psi(d(x,y)), \tag{2.2}$$

for all $(x,y) \in X \times X$ with $M_T(x,y) > 0$.

Remark 2.3. If $\psi \equiv -\tau$, $\alpha = 1$ and $\beta = \gamma = \delta = L = 0$, then Definition 2.7 reduced to the Definition 3.1 in [10].

3. Main results

In this section, we will prove some fixed point theorems for an orthogonal (F,ψ)-contraction of Hardy-Rogers-type mapping in an orthogonal metric space.

Theorem 3.1. *Let $(X, \perp, d)$ be an O-complete orthogonal metric space with an orthogonal element x_0 and $T : X \to X$ satisfying the following conditions:*

(i) T is $\perp$-preserving;

(ii) T is an orthogonal (F,ψ)-contraction of Hardy-Rogers-type mapping;

(iii) T is $\perp$-continuous.

Then T has a fixed point. Moreover, if $\beta + L \leq \gamma + \delta < 1$, then the fixed point of T is unique.

Proof. By the definition of the orthogonality, we have that $x_0 \perp T(x_0)$ or $T(x_0) \perp x_0$. Let $x_n := Tx_{n-1} = \cdots = T^n x_0$, for all $n \in \mathbb{N}$. If $x_{n_*} = x_{n_*+1}$ for some $n^* \in \mathbb{N} \cup \{0\}$, then x_{n_*} is a fixed point of T and so the proof is completed. Thus, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. So, we get $d(Tx_n, Tx_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is $\perp$-preserving, we have

$$x_n \perp x_{n+1} \quad \text{or} \quad x_{n+1} \perp x_n, \quad n \in \mathbb{N} \cup \{0\},$$

which implies that $\{x_n\}$ is O-sequence. Since T is an orthogonal (F,ψ)-contraction mapping, we have

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(d(Tx_{n-1}, Tx_n)) \leq F(M_T(x_{n-1}, x_n)) + \psi(d(x_{n-1}, x_n)) \\ &= F(\alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(x_n, Tx_n) + \delta d(x_{n-1}, Tx_n) \\ &\quad + Ld(x_n, Tx_{n-1})) + \psi(d(x_{n-1}, x_n)). \end{aligned} \tag{3.1}$$

For convenience, set $d_n = d(x_n, x_{n+1})$. Then, by (3.1) one has

$$\begin{aligned} F(d_n) &\leq F(\alpha d_{n-1} + \beta d_{n-1} + \gamma d_n + \delta d(x_{n-1}, x_{n+1})) + \psi(d_{n-1}) \\ &\leq F(\alpha d_{n-1} + \beta d_{n-1} + \gamma d_n + \delta(d_n + d_{n-1})) + \psi(d_{n-1}) \\ &\leq F((\alpha + \beta + \delta) d_{n-1} + (\gamma + \delta) d_n) + \psi(d_{n-1}), \end{aligned} \tag{3.2}$$

which implies that

$$F(d_n) < F((\alpha + \beta + \delta)d_{n-1} + (\gamma + \delta)d_n)$$

by $\psi \in \Psi$. It from the monotony of F, we obtain

$$d_n < (\alpha + \beta + \delta)d_{n-1} + (\gamma + \delta)d_n, \quad n \in \mathbb{N}. \tag{3.3}$$

By the conditions $\gamma + \delta < 1$ and $\alpha + \beta + \gamma + 2\delta = 1$, we get $1 - \gamma - \delta > 0$ and

$$d_n < \frac{\alpha + \beta + \delta}{1 - \gamma - \delta} d_{n-1} = d_{n-1}, \quad n \in N$$

by (3.3). So, the sequence $\{d_n\}$ is strictly decreasing, and $\lim_{n\to\infty} d_n = d$. Assume that $d > 0$. Since ψ is upper semi-continuous from the right, there exists some $q \in \mathbb{N}$ such that

$$\psi(d_j) < \psi(d) - \frac{\psi(d)}{2} = \frac{\psi(d)}{2} < 0, \quad j \geq q. \tag{3.4}$$

By (3.2), (3.4) and decreasing of $\{d_n\}$, for $n > q$, we have

$$\begin{aligned} F(d_n) &\leq F((\alpha + \beta + \gamma + 2\delta)d_{n-1}) + \tfrac{\psi(d)}{2} \\ &= F(d_{n-1}) + \tfrac{\psi(d)}{2} \leq F(d_{n-2}) + 2 \times \tfrac{\psi(d)}{2} \\ &\leq \cdots \leq F(d_q) + (n - q)\tfrac{\psi(d)}{2}. \end{aligned} \tag{3.5}$$

Letting $n \to \infty$ in the inequality (3.5), we get

$$\lim_{n\to\infty} F(d_n) = -\infty,$$

which implies from Definition 1.1 (ii) that

$$d = \lim_{n\to\infty} d_n = 0. \tag{3.6}$$

In the following, we will prove that $\{d_n\}$ is a Cauchy sequence. By (3.6) and Definition (iii), there exists some $k \in (0, 1)$ such that

$$\lim_{n\to+\infty} d_n^k F(d_n) = 0. \tag{3.7}$$

From (3.5), we get

$$d_n^k F(d_n) - d_n^k F(d_q) \leq \frac{n - q}{2} \psi(d) d_n^k \leq 0, \quad n > q.$$

Let $n \to +\infty$, and using (3.6) and (3.7), we obtain that

$$\lim_{n\to+\infty} n d_n^k = 0, \tag{3.8}$$

which implies that there exists $n_1 \in \mathbb{N}$ such that

$$d_n \leq \frac{1}{n^{1/k}}, \quad n \geq n_1. \tag{3.9}$$

Using (3.9) and the triangle inequality, for $n \geq n_1$ and $m \in \mathbb{N}^*$, we have

$$d(x_n, x_{n+m}) \leq \sum_{i=n}^{n+m-1} d(x_i, x_{i+1}) = \sum_{i=n}^{n+m-1} d_i \leq \sum_{i=n}^{n+m-1} \frac{1}{i^{1/k}}.$$

Since $0 < k < 1$, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1/k}}$ is converge, which yields $\{x_n\}$ is a Cauchy O–sequence in X. Since X is O-complete, there exists $x_* \in X$ such that $x_n \to x_*$ as $n \to \infty$. Thus, we have

$$Tx_* = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} x_{n+1} = x_*$$

by T is $\perp$-continuous. So, x_* is a fixed point of T.

Moreover, we will show that x_* is a unique fixed point of T. Suppose that x^* is an another fixed point of T. If $x_n \to x^*$ as $n \to \infty$, one has $x_* = x^*$. If x_n does not converge to x^* as $n \to \infty$, there is a subsequence $\{x_{n_k}\}$ such that $Tx_{n_k} \neq x^*$ for all $k \in \mathbb{N}$. By the choice of x_0 in the first of the proof, we have

$$(x_0 \perp x^*) \quad \text{or} \quad (x^* \perp x_0).$$

Since T is $\perp$-preserving and $T^n x^* = x^*$ for all $n \in \mathbb{N}$, we get

$$(T^n x_0 \perp x^*) \quad \text{or} \quad (x^* \perp T^n x_0), \quad \forall n \in \mathbb{N}.$$

Since T is an orthogonal (F, ψ)-contraction of Hardy-Rogers-type mapping, we obtain

$$\begin{aligned} F(d(x_{n_k}, x^*)) &= F(d(T^{n_k} x_0, x^*)) = F(d(T^{n_k} x_0, T^{n_k} x^*)) \\ &\leq F(M_T(T^{n_k-1} x_0, T^{n_k-1} x^*)) + \psi(d(T^{n_k-1} x_0, T^{n_k-1} x^*)) \\ &= F(M_T(x_{n_{k-1}}, x^*)) + \psi(d(x_{n_{k-1}}, x^*)) \\ &= F(\alpha d(x_{n_{k-1}}, x^*) + \beta d(x_{n_{k-1}}, x_{n_k}) + \delta d(x_{n_{k-1}}, x^*) + L d(x_{n_k}, x^*)) + \psi(d(x_{n_{k-1}}, x^*)) \\ &\leq F((\alpha + \beta + \delta) d(x_{n_{k-1}}, x^*) + (\beta + L) d(x_{n_k}, x^*)) + \psi(d(x_{n_{k-1}}, x^*)) \\ &< F((\alpha + \beta + \delta) d(x_{n_{k-1}}, x^*) + (\beta + L) d(x_{n_k}, x^*)). \end{aligned} \tag{3.10}$$

By condition $\beta + L \leq \delta + \gamma$, one has $1 - \beta - L > 0$, and by $\alpha + \beta + \gamma + 2\delta = 1$ we have $\alpha + 2\beta + \delta + L \leq 1$. Thus, by (3.10) and the monotony of F, we obtain

$$d(x_{n_k}, x^*) < \frac{\alpha + \beta + \delta}{1 - \beta - L} d(x_{n_{k-1}}, x^*) \leq d(x_{n_{k-1}}, x^*). \tag{3.11}$$

Let $\mu_{n_k} := d(x_{n_k}, x^*)$, (3.11) implies that the sequence $\{\mu_{n_k}\}$ is strictly decreasing. Thus, $\lim_{k\to\infty} \mu_{n_k} = \mu \geq 0$. Similar to the proof of (3.4), if $\mu > 0$, then there exists some $p \in \mathbb{N}$ such that

$$\psi(\mu_{n_j}) < \frac{\psi(\mu)}{2} < 0, \quad j \geq p, \tag{3.12}$$

By (3.10), (3.12) and the decreasing of μ_{n_k}, for $k > p$, we get

$$\begin{aligned} F(\mu_{n_k}) &\le F((\alpha + 2\beta + \delta + L)\mu_{n_{k-1}}) + \tfrac{1}{2}\psi(\mu) \\ &\le F(\mu_{n_{k-1}}) + \tfrac{1}{2}\psi(\mu) \le F(\mu_{n_{k-2}}) + 2 \times \tfrac{1}{2}\psi(\mu) \\ &\le \cdots \le F(\mu_{n_p}) + \tfrac{k-p}{2}\psi(\mu), \end{aligned}$$

which yields that $F(\mu_{n_k}) \to -\infty$ as $k \to \infty$ and so it follows from the condition of Definition 1.1 (ii) that $d(x_{n_k}, x^*) = \mu_{n_k} \to 0$ as $k \to \infty$. This implies that $x_n \to x^*$ as $n \to \infty$, which is a contradiction. Thus, T has a unique fixed point.

Remark 3.1. If $\psi \equiv -\tau$, $\alpha = 1$ and $\beta = \gamma = \delta = L = 0$, then Theorem 3.1 reduces to the Theorem 3.3 in [10].

We now illustrate Theorem 3.1 by the following example.

Example 3.1. Let $X = [0, \infty)$ be endowed with the metric $d(x, y) = |x - y|$ for all $x, y \in X$.

Define the sequence $\{S_n\}$ by $S_n = n2^n$ for all $n \in \mathbb{N} \cup \{0\}$. Define the orthogonality relation $\perp$ on X by

$$x \perp y \Leftrightarrow xy \in \{x, y\} \subset \{S_n\}.$$

Clearly, $(X, \perp, d)$ is an O-complete metric space.

Consider the mapping $T : X \to X$ given as

$$Tx = \begin{cases} S_0, & \text{if } S_0 \le x \le S_1, \\ \frac{S_{n-1}(S_{n+1}-x)+S_n(x-S_n)}{S_{n+1}-S_n}, & \text{if } S_n \le x \le S_{n+1} \text{ for each } n \ge 1. \end{cases}$$

Similar to Example in [10], it is easy to check that T is $\perp$-continuous and T is $\perp$-preserving. Now we will show that T is an orthogonal (F, ψ)-contraction of Hardy-Rogers-type mapping. Let $x, y \in X$ with $x \perp y$ and $d(Tx, Ty) > 0$. Without loss of generality, we may assume that $x < y$. This implies that $x \in \{S_0, S_1\}$ and $y = S_k$ for some $k \ge 2$. Thus, we get

$$d(x, y) \ge S_k - 1, \quad d(x, Ty) \ge S_{k-1} - 1, \quad d(Tx, Ty) = S_{k-1}.$$

Taking $\alpha = \frac{1}{3}, \beta = \gamma = L = 0$ and $\delta = \frac{1}{3}$. Then

$$M(x, y) = \frac{1}{3}d(x, y) + \frac{1}{3}d(x, Ty) \ge \frac{1}{3}(S_{k-1} + S_k - 2).$$

Hence we have

$$\frac{d(Tx, Ty)}{M(x, y)} e^{d(Tx,Ty)-M(x,y)} \le \frac{3S_{k-1}}{S_{k-1} + S_k - 2} e^{2S_{k-1}-S_k+2} < e^{-2},$$

which implies that

$$F(d(Tx, Ty)) \le F(M(x, y)) + \psi(d(x, y))$$

with $F(t) = t + \ln t$ and $\psi \equiv -2$. This means that T is an orthogonal (F, ψ)-contraction of Hardy-Rogers-type mapping. Hence, all the conditions of Theorem 3.1 are satisfied and so T has a unique fixed point $x = S_0$.

Remark 3.2. Theorem 3.1 is a powerful generalization of some fixed point theorems in orthogonal complete orthogonal metric space, the result of Example 3.1 can not be obtained by those classical fixed point theorems.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. S. Banach, *Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales*, Fund. Math., **3** (1922), 133–181.
2. V. Berinde, *Iterative approximation of Fixed Points*, Berlin, Heidelberg: Springer, 2007.
3. Y. J. Cho, *Survey on metric fixed point theory and applications*, In M. Ruzhansky, et al. Eds., Advances in Real and Complex Analysis with Applications, Trends in Math, Singapore: Springer, 2017.
4. V. Cosentino, P. Vetro, *Fixed point result for F-contractive mappings of Hardy-Rogers-type*, Filomat, **28** (2014), 715–722.
5. M. E. Gordji, M. Rameani, M. De La Sen, et al. *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory, **18** (2017), 569–578.
6. J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223–239.
7. A. C. M. Ran, M. C. B. Reuring, *A fixed point theorem in partially ordered sets and some appllications to matrix equations*, Proc. Am. Math. Soc., **132** (2004), 1435–1443.
8. S. Reich, *Some remarks concerning contraction mappings*, Can. Math. Bull., **14** (1971), 121–124.
9. L. Salazar, S. Aguirre Reich, *A remark on weakly contractive mappings*, J. Nonlinear Convex. Anal., **16** (2015), 767–773.
10. K. Sawangsup, W. Sintunavarat, Y. J. Cho, *Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces*, J. Fixed Point Theory Appl., **22** (2020), 10.
11. T. Senapati, L. K. Dey, B. Damjanovic, et al. *New fixed point results in orthogonal metric spaces with an application*, Kragujevac J. Math., **42** (2018), 505–516.
12. T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Am. Math. Soc., **136** (2008), 1861–1869.

13. D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric space*, Fixed Point Theory Appl., **2012** (2012), 94.

14. F. Yan, Y. Su, Q. Feng, *A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations*, Fixed Point Theory Appl., **2012** (2012), 152.

15. Q. Zhang, Y. Song, *Fixed point theory for generalized φ-weak contractions*, Appl. Math. Lett., **22** (2009), 75–78.

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