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Research article

Definite integrals involving product of logarithmic functions and logarithm of square root functions expressed in terms of special functions

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Abstract: The derivation of integrals in the table of Gradshteyn and Ryzhik in terms of closed form solutions is always of interest. We evaluate several of these definite integrals of the form $\int_0^\infty \ln^k(\alpha y)\ln(R(y))dy$ in terms of a special function, where $R(y)$ is a general function and k and α are arbitrary complex numbers.

Keywords: logarithmic function; definite integral; Hankel contour; Cauchy integral; Gradshteyn and Ryzhik; Bierens de Haan
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1. Introduction

We will prove one of the formula in Gradshteyn and Ryzhik not previously done and derive other interesting integral formula similar to those in Gradshteyn and Ryzhik. We will evaluate the integrals $\int_0^\infty \ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right)\ln^k(\alpha y)\frac{dy}{y}$ and $\int_0^\infty \frac{\left(\frac{y^2}{z+\sqrt{z^2+y^2}}\right)^m}{y\sqrt{z^2+y^2}}\ln^k\left(\frac{\alpha y^2}{z+\sqrt{z^2+y^2}}\right)dy$ each in terms of a special function. The parameters k, α, a, m and b are arbitrary complex numbers subject to the restrictions given. These derivations are accomplished by the method used by us in [8].

2. Integrals involving the product of logarithmic functions

2.1. Definite integral of the contour integral

We use the method in [8] but because there are limits on w we have to define the contour C somewhat differently. We take the cut along the negative real axis a distance ϵ and then vertically to positive ∞. The path of the contour C is to the right of the cut from $(-\epsilon,\infty)$ to the negative real axis, along the axis and around the origin with zero radius, along the negative real axis to the cut and along the left of the cut to $(-\epsilon, \infty)$. Then we take the limit as $\epsilon \to 0$.

In Cauchy's integral formula we replace y by $\ln(\alpha y)$ and multiply both sides by $\frac{1}{y}\ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right)$ to get

$$\frac{\ln^k(\alpha y)}{yk!}\ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right) = \frac{1}{2\pi i}\int_C \frac{y^{w-1}\alpha^w}{w^{k+1}}\ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right)dw \tag{2.1}$$

the logarithmic function is defined in equation (4.1.2) in [1]. We then take the definite integral over $y \in [0, \infty)$ of both sides and factor the quadratic terms into their two roots to get

$$\begin{aligned}
\frac{1}{k!}\int_0^\infty \frac{\ln^k(\alpha y)}{y}\ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right)dy &= \frac{1}{2\pi i}\int_0^\infty \int_C \frac{y^{w-1}\alpha^w}{w^{k+1}}\ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right)dwdy \\
&= \frac{1}{2\pi i}\int_C\left(\int_0^\infty y^{w-1}\ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right)dy\right)\frac{\alpha^w dw}{w^{k+1}} \\
&= -\frac{1}{2i}\int_C \alpha^w\left(-b-\sqrt{b^2-a^2}\right)^w \csc(\pi w)\, w^{-k-2}dw \\
&+ \frac{1}{2i}\int_C \alpha^w\left(b-\sqrt{b^2-a^2}\right)^w \csc(\pi w)\, w^{-k-2}dw \\
&- \frac{1}{2i}\int_C \alpha^w\left(-b+\sqrt{b^2-a^2}\right)^w \csc(\pi w)\, w^{-k-2}dw \\
&+ \frac{1}{2i}\int_C \alpha^w\left(b+\sqrt{b^2-a^2}\right)^w \csc(\pi w)\, w^{-k-2}dw
\end{aligned} \tag{2.2}$$

from Eq (4.293.10) in [6] and the integral is valid for α, b and a complex, $-1 < Re(w) < 0$, $|\arg(\pm b \pm \sqrt{b^2-a^2})| < \pi$ and α is not real and negative. The condition on α prevents a singularity in the range of integration.

2.2. *Infinite sum of the contour integral*

Again, using the method in [8], replacing y with $\pi i(2p+1) + \ln(\alpha) + \ln(-b - \sqrt{b^2-a^2})$, we multiply both sides by $2\pi i$ to yield

$$2\pi i\frac{\left(\pi i(2p+1)+\ln(\alpha)+\ln\left(-b-\sqrt{b^2-a^2}\right)\right)^k}{k!} = 2\pi i\int_C \frac{e^{w\left(\pi i(2p+1)+\ln(\alpha)+\ln\left(-b-\sqrt{b^2-a^2}\right)\right)}}{w^{k+1}}dw \tag{2.3}$$

followed by taking the infinite sum of both sides of Eq (2.3) with respect to p over $[0, \infty)$ and replace k with $k+1$ to get

$$\begin{aligned}
\frac{(2\pi i)^{k+2}}{(k+1)!}\zeta\left(-k-1, \frac{1}{2} - \frac{i\left(\ln(\alpha)+\ln\left(-b-\sqrt{b^2-a^2}\right)\right)}{2\pi}\right) &= 2\pi i\sum_{p=0}^\infty \int_C \frac{e^{w\left(\pi i(2p+1)+\ln(\alpha)+\ln\left(-b-\sqrt{b^2-a^2}\right)\right)}}{w^{k+2}}dw \\
&= 2\pi i\int_C \sum_{p=0}^\infty \frac{e^{w\left(\pi i(2p+1)+\ln(\alpha)+\ln\left(-b-\sqrt{b^2-a^2}\right)\right)}}{w^{k+2}}dw \\
&= -\frac{1}{2i}\int_C \alpha^w\left(-b-\sqrt{b^2-a^2}\right)^w \csc(\pi w)\, w^{-k-2}dw
\end{aligned} \tag{2.4}$$

from (1.232.3) in [6] where $\operatorname{csch}(ix) = -i\csc(x)$ from (4.5.10) in [1] and $Im(w) > 0$ for the convergence of the sum and if the $Re(k) < 0$ then the argument of the sum over p cannot be zero for some value of p. We use (9.521.1) in [6] where $\zeta(s,u)$ is the Hurwitz Zeta function. Combining the log terms is not valid in general.

Similarly, by replacing y in turn by $\pi i(2p+1) + \ln(\alpha) + \ln\left(b - \sqrt{b^2-a^2}\right)$, $\pi i(2p+1) + \ln(\alpha) + \ln\left(-b + \sqrt{b^2-a^2}\right)$ and $\pi i(2p+1) + \ln(\alpha) + \ln\left(b + \sqrt{b^2-a^2}\right)$ and following the steps in (4) and (5) and simplifying the infinite sum on the left-hand side we get

$$\begin{aligned}\int_0^\infty \frac{\ln^k(\alpha y)}{y} \ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right) dy &= \frac{(2\pi i)^{k+2}}{(k+1)}\zeta\left(-k-1, \frac{1}{2} - \frac{i\left(\ln(\alpha) + \ln\left(-b-\sqrt{b^2-a^2}\right)\right)}{2\pi}\right) \\ &- \frac{(2\pi i)^{k+2}}{(k+1)}\zeta\left(-k-1, \frac{1}{2} - \frac{i\left(\ln(\alpha) + \ln\left(b-\sqrt{b^2-a^2}\right)\right)}{2\pi}\right) \\ &+ \frac{(2\pi i)^{k+2}}{(k+1)}\zeta\left(-k-1, \frac{1}{2} - \frac{i\left(\ln(\alpha) + \ln\left(-b+\sqrt{b^2-a^2}\right)\right)}{2\pi}\right) \\ &- \frac{(2\pi i)^{k+2}}{(k+1)}\zeta\left(-k-1, \frac{1}{2} - \frac{i\left(\ln(\alpha) + \ln\left(b+\sqrt{b^2-a^2}\right)\right)}{2\pi}\right)\end{aligned} \tag{2.5}$$

3. Special cases of the definite integral of the product of the logarithmic function

We will now use Eq (2.5) with a variety for values of the parameters to yield known integral forms. In the following sections we will make use of the Hurwitz Zeta function.

The Hurwitz zeta function $\zeta(s,a)$ is a generalization of the Riemann zeta function $\zeta(s)$ that is also known as the generalized zeta function. It is classically defined by the formula

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}$$

for $Re(s) > 1$, where any term with $k + a = 0$ is excluded and analytically continued by

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{at}(1-e^{-t})} dt$$

for $Re(s) > 1$, $Re(a) > 0$ and $s \neq 1$.

The Hurwitz Zeta function $\zeta(n,a)$ for n negative integers reduces to the Bernoulli polynomial given by $\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}$ for $n \geq 0$, from Eq (12.11.17) in [2] and (23.1) in [1], specifically we will use $\frac{B_2(a)}{2}$ and $\frac{B_3(a)}{3}$ polynomials from Table 5.2 in [9]. The table lists the Bernoulli polynomials up to $n = 6$ and gives the general formula for arbitrary integer n.

3.1. When $k = 0$ and $\alpha = 1$

$$
\begin{aligned}
\int_0^\infty \ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right)\frac{dy}{y} &= (2\pi i)^2\zeta\left(-1,\frac{1}{2}-\frac{i\ln(-b-\sqrt{b^2-a^2}}{2\pi}\right)\\
&- (2\pi i)^2\zeta\left(-1,\frac{1}{2}-\frac{i\ln(b-\sqrt{b^2-a^2}}{2\pi}\right)\\
&+ (2\pi i)^2\zeta\left(-1,\frac{1}{2}-\frac{i\ln(-b+\sqrt{b^2-a^2}}{2\pi}\right)\\
&- (2\pi i)^2\zeta\left(-1,\frac{1}{2}-\frac{i\ln(b+\sqrt{b^2-a^2}}{2\pi}\right)\\
&= \frac{1}{2}\left(-\left(\ln\left(-b-\sqrt{b^2-a^2}\right)\right)^2\right)\\
&+ \frac{1}{2}\left(\left(\ln\left(b-\sqrt{b^2-a^2}\right)\right)^2\right)\\
&+ \frac{1}{2}\left(-\left(\ln\left(-b+\sqrt{b^2-a^2}\right)\right)^2\right)\\
&+ \frac{1}{2}\left(\ln\left(\left(b+\sqrt{b^2-a^2}\right)\right)^2\right)\\
&= \frac{1}{2}\left(-\left(\pi i+\ln\left(b+\sqrt{b^2-a^2}\right)\right)^2\right)\\
&+ \frac{1}{2}\left(\left(\pi i+\ln\left(-b+\sqrt{b^2-a^2}\right)\right)^2\right)\\
&+ \frac{1}{2}\left(-\left(\ln\left(-b+\sqrt{b^2-a^2}\right)\right)^2\right)\\
&+ \frac{1}{2}\left(\ln\left(\left(b+\sqrt{b^2-a^2}\right)\right)^2\right)\\
&= \pi i\ln(-b+\sqrt{b^2-a^2}) - \pi i\ln(b+\sqrt{b^2-a^2})\\
&= \pi i\ln\left(\frac{-b+\sqrt{b^2-a^2}}{b+\sqrt{b^2-a^2}}\right)\\
&= -2\pi i\ln\left(i\frac{b}{a}+\sqrt{1-\frac{b^2}{a^2}}\right)\\
&= 2\pi\sin^{-1}\left(\frac{b}{a}\right)
\end{aligned}
\tag{3.1}
$$

from Eq (12.11.17) in [2] where $a \geq |b|$. Note we multiplied the ln-term in the third to last equation by $\frac{b-\sqrt{b^2-a^2}}{b-\sqrt{b^2-a^2}}$ and simplified.

3.2. When $k = 1$ and $\alpha = 1$

$$
\begin{aligned}
\int_0^\infty \frac{\ln(y)}{y} \ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right) dy &= \frac{(2\pi i)^3}{2}\zeta\left(-2, \frac{1}{2} - \frac{i\ln(-b-\sqrt{b^2-a^2})}{2\pi}\right) \\
&- \frac{(2\pi i)^3}{2}\zeta\left(-2, \frac{1}{2} - \frac{i\ln(b-\sqrt{b^2-a^2})}{2\pi}\right) \\
&+ \frac{(2\pi i)^3}{2}\zeta\left(-2, \frac{1}{2} - \frac{i\ln(-b+\sqrt{b^2-a^2})}{2\pi}\right) \\
&- \frac{(2\pi i)^3}{2}\zeta\left(-2, \frac{1}{2} - \frac{i\ln(b+\sqrt{b^2-a^2})}{2\pi}\right) \\
\\
&= \frac{1}{6}\left(-\pi^2\ln\left(-b-\sqrt{b^2-a^2}\right) - \left(\ln\left(-b-\sqrt{b^2-a^2}\right)\right)^3\right) \\
&+ \frac{1}{6}\left(\pi^2\ln\left(b-\sqrt{b^2-a^2}\right) + \left(\ln\left(b-\sqrt{b^2-a^2}\right)\right)^3\right) \\
&+ \frac{1}{6}\left(-\pi^2\ln\left(-b+\sqrt{b^2-a^2}\right) + \left(\ln\left(-b+\sqrt{b^2-a^2}\right)\right)^3\right) \\
&+ \frac{1}{6}\left(\pi^2\ln\left(b+\sqrt{b^2-a^2}\right) + \left(\ln\left(b+\sqrt{b^2-a^2}\right)\right)^3\right) \\
\\
&= -\frac{\pi^2}{6}\ln\left(-b-\sqrt{b^2-a^2}\right) + \frac{\pi^2}{6}\ln\left(b-\sqrt{b^2-a^2}\right) \\
&+ \frac{2\pi^2}{3}\ln\left(-b+\sqrt{b^2-a^2}\right) + \frac{\pi i}{2}\left(\ln\left(-b+\sqrt{b^2-a^2}\right)\right)^2 \\
&+ \frac{2\pi^2}{3}\ln\left(b+\sqrt{b^2-a^2}\right) - \frac{\pi i}{2}\left(\ln\left(b+\sqrt{b^2-a^2}\right)\right)^2 \\
\\
&= -\frac{\pi}{2}\left(\pi - i\ln\left(-a^2\right)\right)\ln\left(\frac{-b+\sqrt{b^2-a^2}}{b+\sqrt{b^2-a^2}}\right) \\
&= 2\pi\ln(a)\left(-i\ln\left(i\frac{b}{a} + \sqrt{1-\frac{b^2}{a^2}}\right)\right) \qquad (3.2) \\
&= 2\pi\ln(a)\sin^{-1}\left(\frac{b}{a}\right)
\end{aligned}
$$

from Eq (12.11.17) in [2] where $a \geq |b|$. Note in the fourth to last equation we employed the difference of two squares and the addition of logs to simplify. This result is equivalent to Eq (4.313.5) in [6].

3.3. Expressing integrals in terms of the log-gamma function when $\alpha = e^{i\alpha}$

We take the first partial derivative of Eq (2.5) with respect to k and set $k = -1$, rationalize the denominator on the left-hand side, combine the logarithmic and log-gamma functions on the right-hand side then simplify to get

$$\begin{aligned}\int_0^\infty \frac{\ln\left(\frac{a^2+2by+y^2}{a^2-2by+y^2}\right)}{y\ln\left(e^{i\alpha}y\right)}dy &= 2i\pi\ln\left[\frac{\Gamma\left(\frac{\alpha-i\ln\left(b-\sqrt{b^2-a^2}\right)+\pi}{2\pi}\right)\Gamma\left(\frac{\alpha-i\ln\left(b+\sqrt{b^2-a^2}\right)+\pi}{2\pi}\right)}{\Gamma\left(\frac{\alpha-i\ln\left(-b-\sqrt{b^2-a^2}\right)+\pi}{2\pi}\right)\Gamma\left(\frac{\alpha-i\ln\left(\sqrt{b^2-a^2}-b\right)+\pi}{2\pi}\right)}\right]\\ &-\frac{1}{2}i(\pi-2i\ln(2\pi))\left(\ln\left(-\sqrt{b^2-a^2}-b\right)-\ln\left(b-\sqrt{b^2-a^2}\right)\right)\\ &-\frac{1}{2}i(\pi-2i\ln(2\pi))\left(\ln\left(\sqrt{b^2-a^2}-b\right)-\ln\left(\sqrt{b^2-a^2}+b\right)\right)\end{aligned} \tag{3.3}$$

3.4. When $a = 1$, $\alpha = \pi$ and $b = 1/2$

Using Eq (3.3) and comparing the real and imaginary parts of the left-hand side we get

$$\int_0^\infty \frac{\ln\left(\frac{1+y+y^2}{1-y+y^2}\right)}{y(\pi^2+\ln^2(y))}dy = \ln\left[\frac{\Gamma^2\left(\frac{2}{3}\right)\Gamma^2\left(\frac{4}{3}\right)}{\Gamma^2\left(\frac{5}{6}\right)\Gamma^2\left(\frac{7}{6}\right)}\right] = \ln\left(\frac{4}{3}\right) \tag{3.4}$$

where the integral

$$\int_0^\infty \frac{\ln\left(\frac{1+y+y^2}{1-y+y^2}\right)\ln(y)}{y(\pi^2+\ln^2(y))}dy = 0. \tag{3.5}$$

3.5. When $a = 1$, $\alpha = \pi/2$ and $b = -1/2$

Using Eq (3.3) and comparing the real and imaginary parts of the left-hand side we get

$$\int_0^\infty \frac{\ln\left(\frac{1-y+y^2}{1+y+y^2}\right)}{y\left(\frac{\pi^2}{4}+\ln^2(y)\right)}dy = \ln\left[\frac{\Gamma^4\left(\frac{7}{12}\right)\Gamma^4\left(\frac{11}{12}\right)}{\Gamma^4\left(\frac{5}{12}\right)\Gamma^4\left(\frac{13}{12}\right)}\right] \tag{3.6}$$

where the integral

$$\int_0^\infty \frac{\ln\left(\frac{1+y+y^2}{1-y+y^2}\right)\ln(y)}{y\left(\frac{\pi^2}{4}+\ln^2(y)\right)}dy = 0. \tag{3.7}$$

4. Integrals involving the logarithm of the square root of a function

4.1. Definite integral of the contour integral

Using the method in [8] we replace y by $\ln\left(\frac{\alpha y^2}{z+\sqrt{z^2+y^2}}\right)$ and multiply both sides by $\frac{\left(\frac{y^2}{z+\sqrt{z^2+y^2}}\right)^m}{y\sqrt{z^2+y^2}}$ to get the Cauchy equation

$$\frac{\left(\frac{y^2}{z+\sqrt{z^2+y^2}}\right)^m \ln^k\left(\frac{\alpha y^2}{z+\sqrt{z^2+y^2}}\right)}{y\sqrt{z^2+y^2}(k!)} = \frac{1}{2\pi i}\int_C \frac{\left(\frac{\alpha y^2}{z+\sqrt{z^2+y^2}}\right)^{w+m}}{y\sqrt{z^2+y^2}w^{k+1}}dw \tag{4.1}$$

the logarithmic function is defined in section (4.1) in [1], where k, α, z and m are general complex numbers. We then take the definite integral over $y \in [0,\infty)$ of both sides to get

$$\begin{aligned}\int_0^\infty \frac{\left(\frac{y^2}{z+\sqrt{z^2+y^2}}\right)^m \ln^k\left(\frac{\alpha y^2}{z+\sqrt{z^2+y^2}}\right)}{y\sqrt{z^2+y^2}(k!)}dy &= \frac{1}{2\pi i}\int_0^\infty\left(\int_C \frac{\left(\frac{\alpha y^2}{z+\sqrt{z^2+y^2}}\right)^{w+m}}{y\sqrt{z^2+y^2}w^{k+1}}dw\right)dy \\ &= \frac{1}{2\pi i}\int_C\left(\int_0^\infty \frac{\left(\frac{\alpha y^2}{z+\sqrt{z^2+y^2}}\right)^{w+m}}{y\sqrt{z^2+y^2}}dy\right)\frac{dw}{w^{k+1}} \\ &= \frac{1}{2i}\int_C \alpha^w 2^{-1+m+w}z^{-1+m+w}\csc(\pi(m+w))dw\end{aligned} \tag{4.2}$$

from 2.2.11.11 in [7] where $Re(z) > 0$, $0 < Re(w+m) < 1$.

4.2. Infinite sum of the contour integral

We once again use the method in [8], replacing y with $\pi i(2p+1)+\ln(\alpha)+\ln(z)+\ln(2)$. Then we multiply both sides by $-2^m\pi i z^{m-1}e^{m\pi i(2p+1)}$ to yield

$$-\frac{2^m\pi i z^{m-1}}{e^{-m\pi i(2p+1)}}\frac{(\pi i(2p+1)+\ln(\alpha)+\ln(z)+\ln(2))^k}{k!} = -\frac{2^m\pi i z^{m-1}}{e^{-m\pi i(2p+1)}}\int_C \frac{e^{w(\pi i(2p+1)+\ln(\alpha)+\ln(z)+\ln(2))}}{w^{k+1}}dw \tag{4.3}$$

We now take the infinite sum of both sides of Eq (4.3) with respect to p over $[0,\infty)$ to get

$$\begin{aligned}-\frac{2^m\pi i}{z^{1-m}}\sum_{p=0}^{\infty}\frac{(\pi i(2p+1)+\ln(\alpha)+\ln(z)+\ln(2))^k}{e^{-m\pi i(2p+1)}k!} &= -\frac{2^m\pi i}{z^{1-m}}\sum_{p=0}^{\infty}\left(\int_C \frac{e^{w(\pi i(2p+1)+\ln(\alpha)+\ln(z)+\ln(2))}}{e^{-m\pi i(2p+1)}w^{k+1}}\right)dw \\ &= -\frac{2^m\pi i}{z^{1-m}}\int_C\left(\sum_{p=0}^{\infty}\frac{e^{w(\pi i(2p+1)+\ln(\alpha)+\ln(z)+\ln(2))}}{e^{-m\pi i(2p+1)}}\right)\frac{dw}{w^{k+1}} \\ &= \frac{1}{2i}\int_C \alpha^w 2^{-1+m+w}z^{-1+m+w}\csc(\pi(m+w))dw\end{aligned} \tag{4.4}$$

from (1.232.3) in [6] where $\text{csch}(ix) = -i\csc(x)$ from (4.5.10) in [1] and $Im(w) > 0$ for the sum to converge and if the $Re(k) < 0$ then the argument of the sum over p cannot be zero for some value of p. The log terms cannot be combined in general.

4.3. Equating the definite integral and infinite sum

Since the right hand side of Eqs (4.2) and (4.4) are equal we can equate the left hand sides of these equations to yield

$$\int_0^\infty \frac{\left(\frac{y^2}{z+\sqrt{z^2+y^2}}\right)^m \ln^k\left(\frac{\alpha y^2}{z+\sqrt{z^2+y^2}}\right)}{y\sqrt{z^2+y^2}(k!)}dy = -\frac{2^m \pi i}{z^{1-m}}\sum_{p=0}^{\infty}\frac{(\pi i(2p+1)+\ln(\alpha)+\ln(z)+\ln(2))^k}{e^{-m\pi i(2p+1)}k!} \tag{4.5}$$

We can simplify the infinite sum on the right hand side to get

$$\int_0^\infty \frac{\left(\frac{y^2}{z+\sqrt{z^2+y^2}}\right)^m \ln^k\left(\frac{\alpha y^2}{z+\sqrt{z^2+y^2}}\right)}{y\sqrt{z^2+y^2}}dy = -2^{k+m}e^{m\pi i}(\pi i)^{k+1}z^{m-1}\Phi\left(e^{2m\pi i}, -k, \frac{1}{2}-\frac{i(\ln(\alpha)+\ln(2z))}{2\pi}\right) \tag{4.6}$$

from (9.550) in [6] where $\Phi(r, s, u)$ is the Lerch function. Note the left-hand side of Eq (4.6) converges for all finite k. The integral in Eq (4.6) can be used as an alternative method to evaluating the Lerch function. The Lerch function has a series representation given by

$$\Phi(z, s, v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{4.7}$$

where $|z| < 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{4.8}$$

where $Re(v) > 0$, or $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

5. Special cases of the definite integral of the logarithm of the square root of a function

Using Eq (4.6) with various values of the parameters we will derive interesting integrals in terms of famous formula.

5.1. When $k = -1$, $\alpha = -1$, $m = 1/2$ and $z = 1/2$

$$\begin{aligned}\int_0^\infty \frac{\sqrt{\sqrt{1+4y^2}-1}}{y\sqrt{\frac{1}{2}+2y^2}\ln\left(\frac{1}{2}(1-\sqrt{1+4y^2})\right)}dy &= -2^{-1/2}e^{\pi i/2}(1/2)^{-1/2}\Phi\left(-1, 1, \frac{1}{2}-\frac{i(\ln(-1)}{2\pi}\right) \\ &= -i\ln(2)\end{aligned} \tag{5.1}$$

from (0.232.1) in [6].

5.2. When $k = -2$, $\alpha = -1$, $m = 1/2$ and $z = 1/2$

$$\int_0^\infty \frac{\sqrt{\sqrt{1+4y^2}-1}}{y\sqrt{\frac{1}{2}+2y^2}\ln^2\left(\frac{1}{2}(1-\sqrt{1+4y^2})\right)}dy = -2^{-3/2}e^{\pi i/2}(\pi i)^{-1}(1/2)^{-1/2}\Phi\left(-1,2,\frac{1}{2}-\frac{i(\ln(-1))}{2\pi}\right) = -\frac{\pi}{24} \tag{5.2}$$

from (0.234.1) in [6].

5.3. When $k = -2$, $\alpha = -1$, $m = 1/3$ and $z = 1/2$

$$\int_0^\infty \frac{2^{2/3}\left(\sqrt{1+4y^2}-1\right)^{1/3}}{y\sqrt{1+4y^2}\ln^2\left(\frac{1}{2}(1-\sqrt{1+4y^2})\right)}dy = \frac{ie^{\pi i/3}}{2\pi}\Phi\left(e^{2\pi i/3},2,1\right) = \frac{ie^{-\pi i/3}Li_2(e^{2\pi i/3})}{2\pi} \tag{5.3}$$

from (6) in [5], where $Li_2(z) = z\Phi(z,2,1)$ is the Dilogarithm, (1.11.14) in [3] also called Spence's function in honour of William Spence, the Scottish mathematician (1777-1815).

6. Discussion

In this article we derived the product of logarithmic functions in terms of the Hurwitz Zeta function. We also derived the definite integral of the logarithm of the square root of a function in terms of the Lerch function. Then we used these integral formula to derive known results for a famous integral and its generalized form. We were able to produce a closed form solution for Eq (25) Table 134 in Bierens de Haan [4] and Eq (4.313.5) in [6] not previously derived using Eq (2.5). The results presented were numerically verified for both real and imaginary values of the parameters in the integrals using Mathematica by Wolfram. In this work we used Mathematica software to numerically evaluate both the definite integral and associated Special function for complex values of the parameters k, α, a, m and b. We considered various ranges of these parameters for real, integer, negative and positive values. We compared the evaluation of the definite integral to the evaluated Special function and ensured agreement.

7. Conclusion

In this paper, we have derived a method for expressing definite integrals in terms of Special functions using contour integration. The contour we used was specific to solving integral representations in terms of the Hurwitz zeta function. We expect that other contours and integrals can be derived using this method.

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Conflict of interest

The authors declare no conflict of interest.

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