



Research article

Some integrals involving Coulomb functions associated with the three-dimensional proper Lorentz group

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Abstract: For two continual bases in the representation space, we obtain the matrix elements of the linear operator transforming the first basis into the second. These elements are expressed in terms of Coulomb wave functions. Computing the matrix elements of subrepresentations to some subgroups or their separate elements and using the connection between above bases, we evaluate some integrals involving Coulomb wave functions.

Keywords: Coulomb functions; Bessel-Clifford functions; the first and the second Bessel-Clifford integral transforms; Bessel-Macdonald integral transform; Appell function F_1 ; Whittaker function; proper Lorentz group; bases transformations; matrix elements of representation

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1. Introduction and preliminaries

The Coulomb wave functions $F_\sigma(\rho, \lambda)$ and $H_\sigma^\pm(\rho, \lambda)$ are functions belonging to the kernel of the Coulomb differential operator (see, e.g., [18, Chapter 33])

$$\mathfrak{d} := \frac{d^2}{d\lambda^2} + 1 - \frac{2\rho}{\lambda} - \frac{\sigma(\sigma+1)}{\lambda^2},$$

where $\lambda \in \mathbb{R}^+$, $\rho \in \mathbb{R}$ (Sommerfeld parameter), and $\sigma \in \mathbb{N}_0$ (angular momentum quantum number). Here and throughout, we denote $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ by the sets of complex numbers, real numbers,

positive real numbers, integers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$. The functions $F_\sigma(\rho, \lambda)$ and $H_\sigma^\pm(\rho, \lambda)$ satisfy the following formulas (see [6]):

$$F_\sigma(\rho, \lambda) = 2^{-\sigma-1} N_\sigma(\rho) (\mp \mathbf{i})^{\sigma+1} M_{\pm \mathbf{i}\rho, \sigma+\frac{1}{2}}(\pm 2\mathbf{i}\lambda) \quad (1.1)$$

and

$$H_\sigma^\pm(\rho, \lambda) = (\mp \mathbf{i})^\sigma \exp\left(\frac{\pi\rho}{2} \pm \mathbf{i}n_\sigma(\rho)\right) W_{\mp \mathbf{i}\rho, \sigma+\frac{1}{2}}(\mp 2\mathbf{i}\lambda), \quad (1.2)$$

where $\mathbf{i} = \sqrt{-1}$, $M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ are Whittaker functions of the first kind and the second kind (see, e.g., [1, Sections 4.3 and 4.4]), respectively, and the normalizing constant (Gamow factor) $N_\sigma(\rho)$ and Coulomb phase shift $n_\sigma(\rho)$ are given as follows:

$$N_\sigma(\rho) = 2^\sigma \exp\left(-\frac{\pi\rho}{2}\right) [\Gamma(2\sigma+2)]^{-1} |\Gamma(\sigma+1+\mathbf{i}\rho)| \quad (1.3)$$

and

$$n_\sigma(\rho) = \arg \Gamma(\sigma+1+\mathbf{i}\rho),$$

where Γ is the familiar Gamma function (see, e.g., [27, Section 1.1]). The functions $F_\sigma(\rho, \lambda)$ and $H_\sigma^\pm(\rho, \lambda)$ may also be continued analytically to $\lambda \in \mathbb{C} \setminus \{0\}$, and $\rho, \sigma \in \mathbb{C}$ (see, e.g., [5, 8]; see also [18, Entry 33.13]).

Recall the generalized hypergeometric series ${}_pF_q(p, q \in \mathbb{N}_0)$ defined by (see, e.g., [22, p. 73]):

$$\begin{aligned} {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z), \end{aligned}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol which is defined (for $\lambda, \nu \in \mathbb{C}$), in terms of the familiar Gamma function Γ , by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$.

Since the two Whittaker functions are determined by the confluent hypergeometric function ${}_1F_1$ (see, e.g., [7, Entries 6.5(7), 6.9(1) and 6.9(2)]) which satisfies the following Kummer's transformation formula (e.g., [7, Entry 6.3(7)])

$${}_1F_1(a; b; z) = e^z {}_1F_1(b-a; b; -z),$$

the choice of ambiguous signs in (1.1) and (1.2) is immaterial, provided that *either* all upper signs are taken *or* all lower signs are taken (see, e.g., the comment after [18, Entry 33.2.5]).

Also recall (see, e.g., [18, Entry 33.2(iii)]) that $H_\sigma^+(\rho, \lambda)$ and $H_\sigma^-(\rho, \lambda)$ are complex conjugates, and their real and imaginary parts are given by

$$H_\sigma^\pm(\rho, \lambda) = G_\sigma(\rho, \lambda) \pm \mathbf{i} F_\sigma(\rho, \lambda). \quad (1.4)$$

Here $F_\sigma(\rho, \lambda)$ is the regular solution of the Coulomb wave equation $\mathfrak{d}[y] = 0$ while $G_\sigma(\rho, \lambda)$ and $H_\sigma^\pm(\rho, \lambda)$ are its irregular solutions.

Since the dimension of the space $\text{Ker } \mathfrak{d}$ is 2, the linearly independent functions $F_\sigma(\rho, \lambda)$ and $G_\sigma(\rho, \lambda)$ form a basis in $\text{Ker } \mathfrak{d}$. Another basis consists of $H_\sigma^+(\rho, \lambda)$ and $H_\sigma^-(\rho, \lambda)$. Curtis [4] considered two other bases in $\text{Ker } \mathfrak{d}$ consisting of functions which are still named Coulomb functions (see also [11, 14]).

Recall another differential operator (see [12]):

$$\mathfrak{b} := \lambda \frac{d^2}{d\lambda^2} + (\sigma + 1) \frac{d}{d\lambda} + 1.$$

The so-called Bessel-Clifford function of the first kind

$$C_\sigma(\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{k! \Gamma(\sigma + k + 1)}$$

is one of the functions belonging to $\text{Ker } \mathfrak{b}$. The function $C_\sigma(\lambda)$ in (1) is related to the Bessel function J_σ as follows:

$$C_\sigma(\lambda) = \lambda^{-\frac{\sigma}{2}} J_\sigma(2\sqrt{\lambda}). \quad (1.5)$$

Likewise the modified Bessel-Clifford function of the second kind $\mathcal{K}_\sigma(\lambda)$ can be expressed via the Macdonald function K_σ :

$$\mathcal{K}_\sigma(\lambda) = \lambda^{-\frac{\sigma}{2}} K_\sigma(2\sqrt{\lambda}). \quad (1.6)$$

We also need the following integral transformations:

The first Hankel-Clifford integral transform (see [16, Eq. (2.7)])

$$H_\sigma^{(1)}[f](\lambda) = \lambda^\sigma \int_0^\infty C_\sigma(\lambda \hat{\lambda}) f(\hat{\lambda}) d\hat{\lambda} \quad (\lambda \in \mathbb{R}^+);$$

The second Hankel-Clifford integral transform (see [13]; see also [16, Eq. (2.9)])

$$H_\sigma^{(2)}[f](\lambda) = \int_0^\infty \hat{\lambda}^\sigma C_\sigma(\lambda \hat{\lambda}) f(\hat{\lambda}) d\hat{\lambda} \quad (\lambda \in \mathbb{R}^+);$$

The Macdonald-Clifford transform

$$K_\sigma[f](\lambda) = \int_0^\infty \hat{\lambda}^\sigma \mathcal{K}_\sigma(\lambda \hat{\lambda}) f(\hat{\lambda}) d\hat{\lambda} \quad (\lambda \in \mathbb{R}^+).$$

In this paper, we aim to present new formulas which involve Coulomb functions and are related to a representation of the three-dimensional proper Lorentz group in one appropriate carrier functional space (called here the representation space). The paper is organized as follows: In Section 2 we describe the representation space and two bases in it. In Section 3 we express the connection between above bases in terms of Coulomb functions (basically, in terms of $F_\sigma(\rho, \lambda)$). In Sections 4–8 we consider some restrictions of our representation to some subgroups or their separate elements and, using the connections between bases, obtain some integral formulas. In particular, for $\text{diag}(-1, 1, -1)$ and the (circle) rotations through π and $\frac{\pi}{2}$ in the plane Ox_2x_3 we obtain the Bessel–Clifford and Mellin transforms or Coulomb functions, respectively, which are expressed in terms of Coulomb functions and Gaussian hypergeometric functions. Considering the product of the maximal nilpotent subgroup

and the subgroup of hyperbolic rotations in the plane Ox_1x_2 , we express one integral involving product of Coulomb functions in terms of Appell hypergeometric function.

The same problem for two different pairs of bases in the similar representation space for the four-dimensional analogue of our group has been considered in the papers [25] and [26], in which the related theorems have been formulated in terms of ${}_4F_3$ and Macdonald functions, respectively.

2. Representation space, its bases, and functionals F_1 and F_2

We recall that the three-dimensional Lorentz group is the subgroup of matrices (g_{ij}) in $GL(3, \mathbb{R})$ satisfying the equalities

$$g_{i1}^2 - g_{i2}^2 - g_{i3}^2 = (-1)^{E(\frac{i}{2})} \quad (i \in \{1, 2, 3\}),$$

where $E(\mu)$ denotes the integer part of $\mu \in \mathbb{R}$, and $g_{i1}g_{j1} - g_{i2}g_{j2} - g_{i3}g_{j3} = 0$ for different $i, j \in \{1, 2, 3\}$. In this paper, we consider the intersection G of this group with $SL(3, \mathbb{R})$, calling G the proper Lorentz group. Let $\sigma \in \mathbb{R}$ and T be the representation of G in the linear space $\mathfrak{D}$ consisting of σ -homogeneous and infinitely differentiable functions defined on the cone

$$\Lambda = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3^2 = 0 \right\}$$

acting according to rule $T(g)[f(x)] = f(g^{-1}x)$ (see [30]). The similar realization of the G -representation can be considered on hyperboloid $x_1^2 - x_2^2 - x_3^2 = r^2$ (see [32]).

We recall that the functions $x_{\pm}^{\mu}$ on $\mathbb{R}$, which generate the generalized functions $(x_{\pm}^{\mu}, f)$, are defined as follows (see [9]): For $\mu \in \mathbb{C}$, $x_{\pm}^{\mu}$ is equal to $|x|^{\mu}$ for $x \in \mathbb{R}^+$ and coincides with zero function otherwise. Here we deal with the bases (see [31])

$$B_1 = \left\{ f_{\lambda}^{\sigma}(x) = (x_1 + x_2)^{\sigma} \exp \frac{\lambda x_3}{x_1 + x_2} \mid \lambda \in \mathbb{R} \right\}$$

and

$$B_2 = \left\{ f_{\rho, \pm}^{\sigma}(x) = (x_2)_{\pm}^{\sigma - i\rho} (x_1 + x_3)^{i\rho} \mid \rho \in \mathbb{R} \right\}.$$

In the following, we use two bilinear functionals defined on pairs of representation spaces in the same way as in [24]. In order to introduce them, we define the following subsets on Λ : parabola $\gamma_1 : x_1 + x_2 = 1$ and hyperbola $\gamma_2 = \gamma_{2,+} \cup \gamma_{2,-}$, where $\gamma_{2,\pm} : x_2 = \pm 1$. Let H_i be a subgroup of G , which acts transitively on γ_i . We define F_1 and F_2 as

$$F_i : (\mathfrak{D}, \hat{\mathfrak{D}}) \longrightarrow \mathbb{C}, \quad (f, g) \longmapsto \int_{\gamma_i} f(x) g(x) d\gamma_i,$$

where $d\gamma_i$ is a H_i -invariant measure on γ_i . Let γ_1 and $\gamma_{2,\pm}$ be parameterized as follows:

$$\gamma_1 = \begin{cases} x_1 = \frac{1}{2} (1 + \alpha_1^2), \\ x_2 = \frac{1}{2} (1 - \alpha_1^2), \\ x_3 = \alpha_1, \end{cases} \quad \gamma_{2,\pm} = \begin{cases} x_1 = \cosh \alpha_2, \\ x_2 = \pm 1, \\ x_3 = \sinh \alpha_2, \end{cases}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. Since the subgroups H_1 and H_2 consist of matrices, respectively,

$$h_1(\theta_1) = \frac{1}{2} \begin{pmatrix} 2 + \theta_1^2 & \theta_1^2 & 2\theta_1 \\ -\theta_1^2 & 2 - \theta_1^2 & -2\theta_1 \\ 2\theta_1 & 2\theta_1 & 2 \end{pmatrix}$$

and

$$h_2(\theta_2) = \begin{pmatrix} \cosh \theta_2 & 0 & \sinh \theta_2 \\ 0 & 1 & 0 \\ \sinh \theta_2 & 0 & \cosh \theta_2 \end{pmatrix},$$

where $\theta_1, \theta_2 \in \mathbb{R}$, and

$$\begin{aligned} T(h_1(\theta_1))[f_\lambda(\alpha_1)] &= f_\lambda(\alpha_1 - \theta_1), \\ T(h_2(\theta_2))[f_{\rho,\pm}(\alpha_2)] &= f_{\rho,\pm}(\alpha_2 - \theta_2), \end{aligned}$$

we have $d\gamma_i = d\alpha_i$. Also these identities mean that f_λ is an eigenfunction of the operator $T(h_1(\theta_1))$ and $f_{\rho,\pm}$ is an eigenfunction of $T(h_2(\theta_2))$ with eigenvalues $\exp(-\theta_1\lambda)$ and $\exp(-\theta_2\rho)$, respectively. It was shown in [24] that F_1 and F_2 coincide on pairs $(\mathfrak{D}, \mathfrak{D}^\bullet)$ such that degree of homogeneity of $\mathfrak{D}^\bullet$ is equal to $-\sigma - 1$.

3. Matrix elements of $B_1 \rightleftharpoons B_2$ and $B_1^\bullet \rightleftharpoons B_2^\bullet$ transformations in terms of Coulomb functions

Express a function $f_\lambda \in B_1^\bullet$ as a linear combination of vectors belonging to $B_2^\bullet$:

$$f_\lambda^\bullet(x) = \int_{\mathbb{R}} [c_{\lambda,\rho,+}^\bullet f_{\rho,+}^\bullet(x) + c_{\lambda,\rho,-}^\bullet f_{\rho,-}^\bullet(x)] d\rho. \quad (3.1)$$

Since

$$f_{\rho,\pm}|_{\gamma_{2,\pm}} = f_{\rho,\pm}^\bullet|_{\gamma_{2,\pm}} = \exp(i\rho\alpha_2) \quad \text{and} \quad f_{\rho,\pm}|_{\gamma_{2,\mp}} = f_{\rho,\pm}^\bullet|_{\gamma_{2,\mp}} = 0, \quad (3.2)$$

we have

$$\begin{aligned} F_i(f_\lambda^\bullet, f_{\hat{\rho},\pm}) &= \int_{-\infty}^{\infty} c_{\lambda,\rho,\pm}^\bullet F_2(f_{\rho,\pm}^\bullet, f_{\hat{\rho},\pm}) d\rho \\ &= \int_{-\infty}^{\infty} c_{\lambda,\rho,\pm}^\bullet d\rho \int_{-\infty}^{\infty} \exp(i(\rho + \hat{\rho})\alpha_2) d\alpha_2 = 2\pi \int_{-\infty}^{\infty} c_{\lambda,\rho,\pm}^\bullet \delta(\rho + \hat{\rho}) d\rho = 2\pi c_{\lambda,-\hat{\rho},\pm}^\bullet, \end{aligned}$$

where $\delta(\rho + \hat{\rho})$ is the $\hat{\rho}$ -delayed Dirac delta function. Therefore,

$$c_{\lambda,\rho,\pm}^\bullet = \frac{1}{2\pi} F_i(f_\lambda^\bullet, f_{-\hat{\rho},\pm}). \quad (3.3)$$

Likewise, if

$$f_{\rho,\pm}^\bullet(x) = \int_{-\infty}^{\infty} c_{\rho,\pm,\lambda}^\bullet f_\lambda^\bullet(x) d\lambda,$$

then

$$c_{\rho,\pm,\lambda}^\bullet = \frac{1}{2\pi} F_i(f_{\rho,\pm}^\bullet, f_{-\lambda}) = c_{-\lambda,-\rho,\pm}^\bullet. \quad (3.4)$$

Considering that σ is the third argument (after ρ and λ) of $c_{\rho,\pm,\lambda}^\bullet$, we derive from (3.4) that

$$c_{\rho,\pm,\lambda}^\bullet(\sigma) = c_{-\lambda,-\rho,\pm}^\bullet(-\sigma - 1). \quad (3.5)$$

Theorem 3.1. The $c_{\lambda,\rho,+}^\bullet$ in (3.3) is expressed in terms of the Coulomb wave function in (1.1) as follows:

$$c_{\lambda,\rho,+}^\bullet = \frac{|\Gamma(\sigma + 1 + \mathbf{i}\rho)|}{\pi \lambda^{\sigma+1}} \exp\left(\frac{\pi\rho}{2}\right) F_\sigma(\rho, \lambda) \quad (\lambda \neq 0, \sigma > -1). \quad (3.6)$$

Proof. Recall a known formula (see, e.g., [10, Entry 3.383-1] and [19, Entry 2.3.6-1])

$$\int_0^a x^{\nu-1} (a-x)^{\mu-1} e^{-px} dx = B(\mu, \nu) a^{\mu+\nu-1} {}_1F_1(\nu; \mu + \nu; -ap) \quad (a \in \mathbb{R}_0, \min\{\Re(\mu), \Re(\nu)\} > 0), \quad (3.7)$$

where $B(\alpha, \beta)$ is the familiar Beta function given by (see, e.g., [27, Section 1.1])

$$B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt & (\min\{\Re(\alpha), \Re(\beta)\} > 0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{cases} \quad (3.8)$$

Then

$$\begin{aligned} c_{\lambda,\rho,+}^\bullet &= \frac{1}{2\pi} F_1(f_\lambda^\bullet, f_{-\rho,+}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1-\alpha_1^2}{2}\right)_+^{\sigma+\mathbf{i}\rho} \left(\frac{1+\alpha_1^2}{2} + \alpha_1\right)^{-\mathbf{i}\rho} \exp(\mathbf{i}\lambda\alpha_1) d\alpha_1 \\ &= \frac{2^{-\sigma-1}}{\pi} \int_{-1}^1 (1-\alpha_1)^{\sigma+\mathbf{i}\rho} (1+\alpha_1)^{\sigma-\mathbf{i}\rho} \exp(\mathbf{i}\lambda\alpha_1) d\alpha_1 \\ &= \frac{2^{-\sigma-1}}{\pi} \exp(-\mathbf{i}\lambda) \int_0^2 t^{\sigma-\mathbf{i}\rho} (2-t)^{\sigma+\mathbf{i}\rho} \exp(\mathbf{i}\lambda t) dt \\ &= \frac{2^\sigma}{\pi} \exp(-\mathbf{i}\lambda) B(\sigma+1+\mathbf{i}\rho, \sigma+1-\mathbf{i}\rho) {}_1F_1(\sigma+1-\mathbf{i}\rho; 2\sigma+2; 2\mathbf{i}\lambda), \end{aligned}$$

where (3.7) is applied for the fifth equality. For the ${}_1F_1$ of the last expression, using the following known relation (see, e.g., [1, Eq. (4.3.2)], [17, p. 290]):

$$M_{\mu,\nu}(z) = z^{\nu+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) {}_1F_1\left(\nu-\mu+\frac{1}{2}; 2\nu+1; z\right), \quad (3.9)$$

we find

$$c_{\lambda,\rho,+}^\bullet = \frac{(\mathbf{i}\lambda)^{-\sigma-1}}{2\pi} B(\sigma+1+\mathbf{i}\rho, \sigma+1-\mathbf{i}\rho) M_{\mathbf{i}\rho, \sigma+\frac{1}{2}}(2\mathbf{i}\lambda),$$

which, in view of (3.8), is equivalent to

$$c_{\lambda,\rho,+}^\bullet = \frac{(\mathbf{i}\lambda)^{-\sigma-1}}{2\pi} \frac{\Gamma(\sigma+1+\mathbf{i}\rho)\Gamma(\sigma+1-\mathbf{i}\rho)}{\Gamma(2\sigma+2)} M_{\mathbf{i}\rho, \sigma+\frac{1}{2}}(2\mathbf{i}\lambda). \quad (3.10)$$

Since $\Gamma(z)$ is analytic in the half plane $\Re(z) > 0$ whose domain is symmetric with respect to the real axis $x = \Re(z) \in \mathbb{R}^+$ and $\Gamma(x) \in \mathbb{R}$ on the real axis $x = \Re(z) \in \mathbb{R}^+$, by reflection principle (see, e.g., [3, p. 57]), we have $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ for each z in the half plane $\Re(z) > 0$. Therefore (3.10) is rewritten as follows:

$$c_{\lambda,\rho,+}^\bullet = \frac{(\mathbf{i}\lambda)^{-\sigma-1}}{2\pi} \frac{|\Gamma(\sigma+1+\mathbf{i}\rho)|^2}{\Gamma(2\sigma+2)} M_{\mathbf{i}\rho, \sigma+\frac{1}{2}}(2\mathbf{i}\lambda). \quad (3.11)$$

From (1.1) and (1.3),

$$M_{\mathbf{i}\rho, \sigma + \frac{1}{2}}(2\mathbf{i}\lambda) = \frac{2(-\mathbf{i})^{-\sigma-1} \exp\left(\frac{\pi\rho}{2}\right) \Gamma(2\sigma+2)}{|\Gamma(\sigma+1+\mathbf{i}\rho)|} F_{\sigma}(\rho, \lambda). \quad (3.12)$$

Finally, using (3.12) in (3.11), we obtain the desired result (3.6). $\square$

Theorem 3.2. *Let $-1 < \sigma < 0$ and $\lambda \neq 0$. Then*

$$c_{\lambda, \rho, -}^{\bullet} = \frac{|\Gamma(\sigma+1+\mathbf{i}\rho)|}{2\pi \lambda^{\sigma+1} \mathbf{i}^{2\sigma+1} \exp\left(\frac{\pi\rho}{2}\right)} [H_{\sigma}^{-}(\rho, \lambda) - H_{\sigma}^{+}(\rho, \lambda)]. \quad (3.13)$$

Or, equivalently,

$$c_{\lambda, \rho, -}^{\bullet} = \frac{(-1)^{1-\sigma} |\Gamma(\sigma+1+\mathbf{i}\rho)|}{\pi \lambda^{\sigma+1} \exp\left(\frac{\pi\rho}{2}\right)} F_{\sigma}(\rho, \lambda). \quad (3.14)$$

Proof. We have

$$\begin{aligned} c_{\lambda, \rho, -}^{\bullet} &= \frac{1}{2\pi} F_1(f_{\lambda}^{\bullet}, f_{-\rho, -}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1-\alpha_1^2}{2}\right)^{\sigma+\mathbf{i}\rho} \left(\frac{1+\alpha_1^2}{2} + \alpha_1\right)^{-\mathbf{i}\rho} \exp(\mathbf{i}\lambda\alpha_1) d\alpha_1 \\ &= \frac{2^{-\sigma-1}}{\pi} \left[\exp(-\mathbf{i}\lambda) \int_0^{\infty} t^{\sigma-\mathbf{i}\rho} (t+2)^{\sigma+\mathbf{i}\rho} \exp(-\mathbf{i}\lambda t) dt \right. \\ &\quad \left. + \exp(\mathbf{i}\lambda) \int_0^{\infty} t^{\sigma+\mathbf{i}\rho} (t+2)^{\sigma-\mathbf{i}\rho} \exp(\mathbf{i}\lambda t) dt \right]. \end{aligned}$$

Recalling an integral formula (see, e.g., [19, Entry 2.3.2.(3)])

$$\begin{aligned} &\int_0^{\infty} x^{\mu-1} (x+y)^{\nu-1} \exp(-sx) dx \\ &= y^{\mu+\nu-1} B(\mu, 1-\mu-\nu) {}_1F_1(\mu; \mu+\nu; sy) \\ &\quad + s^{1-\mu-\nu} \Gamma(\mu+\nu-1) {}_1F_1(1-\nu; 2-\mu-\nu; sy) \end{aligned} \quad (3.15)$$

$$(\Re(\mu) > 0, |\arg y| < \pi, \Re(s) > 0; \Re(s) = 0, \Re(\mu+\nu) < 2)$$

and using (3.9) and (1.1), we obtain

$$c_{\lambda, \rho, -}^{\bullet} = \frac{1}{2\pi} (U_+ + U_-),$$

where

$$\begin{aligned} U_{\pm} &:= \frac{B(\sigma+1 \pm \mathbf{i}\rho, -2\sigma-1) M_{\mp \mathbf{i}\rho, \sigma + \frac{1}{2}}(\mp 2\mathbf{i}\lambda) + \Gamma(2\sigma+1) M_{\mp \mathbf{i}\rho, -\sigma - \frac{1}{2}}(\mp 2\mathbf{i}\lambda)}{(\mp \mathbf{i}\lambda)^{\sigma+1}} \\ &= \frac{\Gamma(\sigma+1 \pm \mathbf{i}\rho)}{(\mp \mathbf{i}\lambda)^{\sigma+1}} \left[\frac{\Gamma(-2\sigma-1)}{\Gamma(\mathbf{i}\rho - \sigma)} M_{\mp \mathbf{i}\rho, \sigma + \frac{1}{2}}(\mp 2\mathbf{i}\lambda) + \frac{\Gamma(2\sigma+1)}{\Gamma(\pm \mathbf{i}\rho + \sigma + 1)} M_{\mp \mathbf{i}\rho, -\sigma - \frac{1}{2}}(\mp 2\mathbf{i}\lambda) \right] \\ &= \frac{\Gamma(\sigma+1 \pm \mathbf{i}\rho)}{(\mp \mathbf{i}\lambda)^{\sigma+1}} W_{\mp \mathbf{i}\rho, \sigma + \frac{1}{2}}(\mp 2\mathbf{i}\lambda) \end{aligned}$$

(see also [10, Entry 9.220.(4)]). Using (1.2), we derive (3.13). Using (3.13) and (1.4), we obtain (3.14). $\square$

4. Formulae related to the matrix $\text{diag}(-1, 1, -1)$

For any $f_\lambda^\bullet$ and any $g \in G$, let us express $T^\bullet(g)[f_\lambda^\bullet]$ as the integral operator

$$T^\bullet(g)[f_\lambda^\bullet] = \int_{-\infty}^{\infty} t_{\lambda\tilde{\lambda}}^\bullet(g) f_{\tilde{\lambda}}^\bullet d\tilde{\lambda}, \quad (4.1)$$

where $t_{\lambda\tilde{\lambda}}^\bullet$ are the matrix elements of the representation $T^\bullet$. Then

$$F_i(T^\bullet(g)[f_\lambda^\bullet], f_{\tilde{\lambda}}) = \int_{-\infty}^{\infty} t_{\lambda\tilde{\lambda}}^\bullet(g) F_i(f_\lambda^\bullet, f_{\tilde{\lambda}}) d\tilde{\lambda}.$$

Choosing here $i = 1$, in view of

$$F_1(f_\lambda^\bullet, f_{\tilde{\lambda}}) = \int_{-\infty}^{\infty} \exp(\mathbf{i}[\hat{\lambda} + \tilde{\lambda}]\alpha_1) d\alpha_1 = \delta(\hat{\lambda} + \tilde{\lambda}),$$

where $\delta(\hat{\lambda} + \tilde{\lambda})$ is the $(-\tilde{\lambda})$ -delayed Dirac delta function, we have

$$F_i(T^\bullet(g)[f_\lambda^\bullet], f_{\tilde{\lambda}}) = \int_{-\infty}^{\infty} t_{\lambda\tilde{\lambda}}^\bullet(g) \delta(\hat{\lambda} + \tilde{\lambda}) d\tilde{\lambda} = 2\pi t_{\lambda, -\tilde{\lambda}}^\bullet(g).$$

Thus

$$t_{\lambda\tilde{\lambda}}^\bullet(g) = \frac{1}{2\pi} F_i(T^\bullet(g)[f_\lambda^\bullet], f_{-\tilde{\lambda}}). \quad (4.2)$$

From (3.1) and (4.1), for any $g \in G$, we have

$$T^\bullet(g)[f_\lambda^\bullet] = \int_{-\infty}^{\infty} (\tau_+ f_{\rho,+}^\bullet + \tau_- f_{\rho,-}^\bullet) d\rho,$$

where

$$\tau_\pm = \int_{-\infty}^{\infty} t_{\lambda\tilde{\lambda}}(g) c_{\lambda,\rho,\pm}^\bullet d\tilde{\lambda}.$$

On the other hand,

$$T^\bullet(g)[f_\lambda^\bullet] = \int_{-\infty}^{\infty} (c_{\lambda,\rho,+}^\bullet T^\bullet(g)[f_{\rho,+}^\bullet] + c_{\lambda,\rho,-}^\bullet T^\bullet(g)[f_{\rho,-}^\bullet]) d\rho.$$

Since $f_{\rho,+}^\bullet$ is an eigenfunction of the linear operator $T^\bullet(\text{diag}(-1, 1, -1))$ with eigenvalue $(-1)^{\mathbf{i}\rho}$, we have

$$\int_{-\infty}^{\infty} t_{\lambda\tilde{\lambda}}(\text{diag}(-1, 1, -1)) c_{\lambda,\rho,\pm}^\bullet d\tilde{\lambda} = (-1)^{\mathbf{i}\rho} c_{\lambda,\rho,\pm}^\bullet. \quad (4.3)$$

We express the matrix elements $t_{\lambda\tilde{\lambda}}(\text{diag}(-1, 1, -1))$ in terms of Bessel-Clifford functions as in the following theorem.

Theorem 4.1. Let $-1 < \sigma < 0$. Then

$$t_{\lambda\hat{\lambda}}(\text{diag}(1, -1, -1)) = \frac{2\hat{\lambda}^{2\sigma+1} \sin(\sigma\pi)}{(-1)^\sigma \pi} \mathcal{K}_{2\sigma+1}(-\lambda\hat{\lambda}) \quad (\lambda\hat{\lambda} < 0) \quad (4.4)$$

and

$$t_{\lambda\hat{\lambda}}(\text{diag}(1, -1, -1)) = \frac{(-1)^\sigma}{2 \cos(\sigma\pi)} \left[\hat{\lambda}^{2\sigma+1} C_{2\sigma+1}(\lambda\hat{\lambda}) - \lambda^{-2\sigma-1} C_{-2\sigma-1}(\lambda\hat{\lambda}) \right] \quad (\lambda\hat{\lambda} > 0). \quad (4.5)$$

Proof. Since

$$f_\lambda^\bullet|_{\gamma_1}(\text{diag}(1, -1, -1)) = (-1)^{-\sigma-1} \alpha_1^{-2\sigma-2} \exp(-i\lambda\alpha_1^{-1})$$

and

$$f_{-\lambda}|_{\gamma_1} = \exp(-i\hat{\lambda}\alpha_1),$$

we find from (4.2) that

$$\begin{aligned} t_{\lambda\hat{\lambda}}(\text{diag}(1, -1, -1)) &= \frac{1}{2\pi} F_1(T^\bullet(\text{diag}(1, -1, -1))[f_\lambda^\bullet], f_{-\lambda}) \\ &= \frac{1}{(-1)^{\sigma+1} \pi} \int_0^\infty \alpha_1^{-2\sigma-2} \cos(\lambda\alpha_1^{-1} + \hat{\lambda}\alpha_1) d\alpha_1. \end{aligned} \quad (4.6)$$

Using known integral formulas (see [19, Entry 2.5.24.4]), we obtain the following integral formulas: For $a, b > 0$ and $|\Re(\alpha)| < 1$,

$$\int_0^\infty x^{\alpha-1} \cos\left(ax + \frac{b}{x}\right) dx = \frac{\pi}{2} \left(\frac{b}{a}\right)^{\frac{\alpha}{2}} \frac{1}{\sin\left(\frac{\alpha\pi}{2}\right)} \left[J_{-\alpha}(2\sqrt{ab}) - J_{\alpha}(2\sqrt{ab}) \right] \quad (4.7)$$

and

$$\int_0^\infty x^{\alpha-1} \cos\left(ax - \frac{b}{x}\right) dx = 2 \left(\frac{b}{a}\right)^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{2}\right) K_{\alpha}(2\sqrt{ab}). \quad (4.8)$$

Using (4.7) and (4.8) to evaluate the integral (4.6), with the aid of the relations (1.5) and (1.6), we derive the expressions (4.4) and (4.5). $\square$

Using (4.3) and Theorems 3.1 and 3.2, we obtain the following identity

$$\begin{aligned} \frac{(-1)^{\sigma+i\rho}}{\lambda^{\sigma+1}} F_{\sigma}(\rho, \lambda) &= \frac{2 \sin(\sigma\pi)}{\pi} \int_{-\infty}^0 \hat{\lambda}^{\sigma} \mathcal{K}_{2\sigma+1}(-\lambda\hat{\lambda}) F_{\sigma}(\rho, \hat{\lambda}) d\hat{\lambda} \\ &\quad + \frac{\pi \cos(\sigma\pi)}{2} \int_0^\infty \hat{\lambda}^{\sigma} C_{2\sigma+1}(\lambda\hat{\lambda}) F_{\sigma}(\rho, \hat{\lambda}) d\hat{\lambda} \\ &\quad - \frac{\pi \cos(\sigma\pi)}{2 \lambda^{2\sigma+1}} \int_0^\infty \hat{\lambda}^{-\sigma-1} C_{-2\sigma-1}(\lambda\hat{\lambda}) F_{\sigma}(\rho, \hat{\lambda}) d\hat{\lambda} \\ &\quad (\lambda \in \mathbb{R}^+, -1 < \sigma < 0). \end{aligned} \quad (4.9)$$

The identity (4.9) can be rewritten in the following theorem.

Theorem 4.2. Let $\lambda > 0$ and $-1 < \sigma < 0$. Then

$$\begin{aligned} \frac{(-1)^{\sigma+i\rho}}{\lambda^{\sigma+1}} F_{\sigma}(\rho, \lambda) &= \frac{\pi \cos(\sigma\pi)}{2} H_{2\sigma+1}^{(2)} \left[\hat{\lambda}^{-\sigma-1} F_{\sigma}(\rho, \hat{\lambda}) \right] (\lambda) \\ &\quad - \frac{\pi \cos(\sigma\pi)}{2} H_{-2\sigma-1}^{(1)} \left[\hat{\lambda}^{-\sigma-1} F_{\sigma}(\rho, \hat{\lambda}) \right] (\lambda) \\ &\quad + \frac{2(-1)^{\sigma} \sin(\sigma\pi)}{\pi} K_{2\sigma+1} \left[\hat{\lambda}^{-\sigma-1} F_{\sigma}(\rho, -\hat{\lambda}) \right] (\lambda). \end{aligned}$$

Considering the case $\lambda < 0$ in (4.3), we obtain the following similar result.

Theorem 4.3. Let $\lambda < 0$ and $-1 < \sigma < 0$. Then

$$\begin{aligned} \frac{(-1)^{i\rho}}{\lambda^{\sigma+1}} F_{\sigma}(\rho, \lambda) &= \frac{\pi \cos(\sigma\pi)}{2} H_{2\sigma+1}^{(2)} \left[\hat{\lambda}^{-\sigma-1} F_{\sigma}(\rho, -\hat{\lambda}) \right] (-\lambda) \\ &\quad - \frac{\pi \cos(\sigma\pi)}{2} H_{-2\sigma-1}^{(1)} \left[\hat{\lambda}^{-\sigma-1} F_{\sigma}(\rho, -\hat{\lambda}) \right] (-\lambda) \\ &\quad + \frac{2(-1)^{\sigma} \sin(\sigma\pi)}{\pi} K_{2\sigma+1} \left[\hat{\lambda}^{-\sigma-1} F_{\sigma}(\rho, -\hat{\lambda}) \right] (-\lambda). \end{aligned}$$

5. Formulae related to the rotation through π in the plane Ox_2x_3

Let $\mathbb{U}_n$ be the multiplicative group of all complex roots of the equation $z^n = 1$ ($n \in \mathbb{N}$). Then, for any $\varepsilon \in \mathbb{U}_2$, we have $\text{diag}(1, \varepsilon, \varepsilon) \in G$ and

$$T^{\bullet}(\text{diag}(1, \varepsilon, \varepsilon)) [f_{\rho, \pm}^{\bullet}]_{\gamma_2} = (\varepsilon)_{\pm}^{\sigma-i\rho} (\cosh \alpha_2 + \varepsilon \sinh \alpha_2)^{i\rho} = f_{\varepsilon\rho, \pm\varepsilon}^{\bullet}|_{\gamma_2}.$$

Therefore,

$$\int_{-\infty}^{\infty} t_{\lambda\hat{\lambda}}(\text{diag}(1, \varepsilon, \varepsilon)) c_{\hat{\lambda}, \rho, \pm}^{\bullet} d\hat{\lambda} = c_{\hat{\lambda}, \varepsilon\rho, \pm\varepsilon}^{\bullet}.$$

The trivial case $\varepsilon = 1$ gives $t_{\lambda\hat{\lambda}}(\text{id}) = \delta(\lambda - \hat{\lambda})$. The case $\varepsilon = -1$ yields the following theorem.

Theorem 5.1. Let $\lambda > 0$ and $-1 < \sigma < 0$. Then

$$\begin{aligned} \frac{(-1)^{\sigma} \exp(\pi\rho)}{\lambda^{\sigma+1}} F_{\sigma}(-\rho, \lambda) &= \frac{2(-1)^{\sigma+1} \sin(\pi\sigma)}{\pi} K_{2\sigma+1} \left[\hat{\lambda}^{-\sigma-1} F_{\sigma}(\rho, -\lambda) \right] (\lambda) \\ &\quad + \frac{\cos(\sigma\pi)}{2} H_{-2\sigma-1}^{(1)} \left[\hat{\lambda}^{-\sigma-1} F_{\sigma}(\rho, \lambda) \right] (\lambda) \\ &\quad - \frac{\cos(\sigma\pi)}{2} H_{2\sigma+1}^{(2)} \left[\hat{\lambda}^{-\sigma-1} F_{\sigma}(\rho, \lambda) \right] (\lambda). \end{aligned} \quad (5.1)$$

Proof. The proof of (5.1) would run parallel to that of Theorems 4.1 and 4.2. We omit the details. $\square$

Another result corresponding to (5.1) for the case $\lambda < 0$ is left to the interested reader.

6. One formula related to the rotation through $\frac{\pi}{2}$ in the plane Ox_2x_3

In this section we use the subgroup H° consisting of (circle) rotations

$$h^\circ(\xi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \xi & -\sin \xi \\ 0 & \sin \xi & \cos \xi \end{pmatrix}.$$

Note that the subgroup H° is isomorphic to the group $SO(2)$.

For any $g \in G$, let us express the function $T^\bullet(g)[f_{\rho,\pm}^\bullet]$ as linear combinations of functions belonging to the basis $B_2^\bullet$:

$$T^\bullet(g)[f_{\rho,\pm}^\bullet](x) = \int_{-\infty}^{\infty} [t_{\rho,\pm,\hat{\rho},+}^\bullet(g) f_{\hat{\rho},+}^\bullet(x) + t_{\rho,\pm,\hat{\rho},-}^\bullet(g) f_{\hat{\rho},-}^\bullet(x)] d\hat{\rho}. \quad (6.1)$$

The equation (6.1) gives that, for any $k \in \mathbb{Z}$,

$$\begin{aligned} F_i(T^\bullet(g)[f_{\rho,\pm}^\bullet], f_{\hat{\rho},\pm(-1)^k}) \\ = \int_{-\infty}^{\infty} [t_{\rho,\pm,\hat{\rho},+}^\bullet(g) F_j(f_{\hat{\rho},+}^\bullet, f_{\hat{\rho},\pm(-1)^k}) + t_{\rho,\pm,\hat{\rho},-}^\bullet(g) F_j(f_{\hat{\rho},-}^\bullet, f_{\hat{\rho},\pm(-1)^k})] d\hat{\rho}. \end{aligned}$$

Choosing here $j = 1$ and considering (3.2), we have

$$F_2(f_{\hat{\rho},\pm}^\bullet, f_{\hat{\rho},\pm(-1)^k}) = \begin{cases} 2\pi \delta(\hat{\rho} + \tilde{\rho}) & (\text{even } k) \\ 0 & (\text{odd } k). \end{cases}$$

Therefore,

$$F_i(T^\bullet(g)[f_{\rho,\pm}^\bullet], f_{\hat{\rho},\pm(-1)^k}) = 2\pi t_{\rho,\pm,-\tilde{\rho},\pm(-1)^k}^\bullet(g).$$

Or, equivalently,

$$t_{\rho,\pm,\hat{\rho},\pm(-1)^k}^\bullet(g) = \frac{1}{2\pi} F_i(T^\bullet(g)[f_{\rho,\pm}^\bullet], f_{-\hat{\rho},\pm(-1)^k}).$$

Theorem 6.1. *Let $-1 < \sigma < 0$. Then*

$$\begin{aligned} \int_{-\infty}^{\infty} \lambda^{i\hat{\rho}-1} F_{-\sigma-1}(-\rho, -\lambda) d\lambda &= \frac{2^{2\sigma+1} \exp\left(\frac{\pi\rho}{2}\right) B(1 + \sigma + i\hat{\rho}, -\sigma + i\rho)}{i^{1+\sigma+i\hat{\rho}} \Gamma(1 + \sigma - i\hat{\rho}) |\Gamma(i\rho - \sigma)|} \\ &\times {}_2F_1(-\sigma + i\rho, -\sigma - i\hat{\rho}; 1 + i(\rho + \hat{\rho}); -1). \end{aligned}$$

Proof. We find

$$\begin{aligned} t_{\rho,+, \hat{\rho},+}^\bullet\left(h^\circ\left(\frac{\pi}{2}\right)\right) &= \frac{1}{2\pi} F_1\left(T^\bullet\left(h^\circ\left(\frac{\pi}{2}\right)\right)[f_{\rho,+}^\bullet], f_{-\hat{\rho},+}\right) \\ &= \frac{2^\sigma}{\pi} \int_{-\infty}^{\infty} (\alpha_1)_+^{-\sigma-1+i\rho} \alpha_1^{2i\rho} (1 - \alpha_1^2)_+^{\sigma-i\hat{\rho}} (1 + \alpha_1)^{2i\hat{\rho}} d\alpha_1 \\ &= \frac{2^\sigma}{\pi} \int_0^1 \alpha_1^{-\sigma-1+i\rho} (1 - \alpha_1)^{\sigma-i\hat{\rho}} (1 + \alpha_1)^{\sigma+i\hat{\rho}} d\alpha_1. \end{aligned} \quad (6.2)$$

Using a known integral formula (see [19, Entry 2.2.6.(1)])

$$\begin{aligned} & \int_a^b (x-a)^{\mu-1} (b-x)^{\nu-1} (cx+d)^{\kappa} dx \\ &= (b-a)^{\mu+\nu-1} (ac+d)^{\kappa} B(\mu, \nu) {}_2F_1\left(\mu, -\kappa; \mu+\nu; \frac{c(a-b)}{ac+d}\right) \\ & \quad \left(\min\{\Re(\mu), \Re(\nu)\} > 0, \left|\arg \frac{bc+d}{ac+d}\right| < \pi\right), \end{aligned} \quad (6.3)$$

we obtain

$$\begin{aligned} t_{\rho,+, \hat{\rho},+}^{\bullet}\left(h^{\circ}\left(\frac{\pi}{2}\right)\right) &= 2^{\sigma} \pi^{-1} B(\sigma+1+\mathbf{i}\hat{\rho}, -\sigma+\mathbf{i}\rho) \\ & \quad \times {}_2F_1(-\sigma+\mathbf{i}\rho, -\sigma-\mathbf{i}\hat{\rho}; 1+\mathbf{i}(\rho+\hat{\rho}); -1). \end{aligned}$$

We also find

$$t_{\rho,+, \hat{\rho},+}^{\bullet}\left(h^{\circ}\left(\frac{\pi}{2}\right)\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c_{\rho,+, \lambda}^{\bullet} F_i\left(T^{\bullet}\left(h^{\circ}\left(\frac{\pi}{2}\right)\right)[f_{\lambda}^{\bullet}], f_{\hat{\rho},+}\right) d\alpha_1,$$

where

$$\begin{aligned} F_i\left(T^{\bullet}\left(h^{\circ}\left(\frac{\pi}{2}\right)\right)[f_{\lambda}^{\bullet}], f_{\hat{\rho},+}\right) &= F_1\left(T^{\bullet}\left(h^{\circ}\left(\frac{\pi}{2}\right)\right)[f_{\lambda}^{\bullet}], f_{\hat{\rho},+}\right) \\ &= 2^{1-\sigma} \exp(\mathbf{i}\lambda) \int_{-1}^1 (1+\alpha_1)^{\mathbf{i}\hat{\rho}-\sigma-2} (1-\alpha_1)^{\sigma-\mathbf{i}\hat{\rho}} \exp\left(\frac{-2\mathbf{i}\lambda}{\alpha_1+1}\right) d\alpha_1 \\ &= -2^{1-\sigma} \exp(\mathbf{i}\lambda) \int_{\frac{1}{2}}^{+\infty} (2u-1)^{\sigma-\mathbf{i}\hat{\rho}} \exp(-2\mathbf{i}\lambda u) du. \end{aligned} \quad (6.4)$$

Thus, $t_{\rho,+, \hat{\rho},+}^{\bullet}\left(h^{\circ}\left(\frac{\pi}{2}\right)\right)$ can be expressed in terms of Gamma functions (see, e.g., [19, Entry 2.3.4(1)]). Now, the statement of the theorem follows from (3.5), (6.2), and (6.4). $\square$

Remark 1. From Theorem 4.2 and throughout, $F_{-\sigma-1}(-\rho, -\lambda)$ can be changed by the linear combination of basis functions in $\text{Ker } \mathfrak{d}$ (see [5])

$$F_{-\sigma-1}(\rho, \lambda) = \cos \theta F_{\sigma}(\rho, \lambda) + \sin \theta G_{\sigma}(\rho, \lambda),$$

where

$$\theta := \left(\sigma + \frac{1}{2}\right)\pi + c_{-\sigma-1}(\rho) - c_{\sigma}(\rho).$$

7. Formulae related to the hyperbolic rotations in the plane $O_{x_1x_2}$

In this section we deal with the subgroup H^* of matrices

$$h^*(\tau) = \begin{pmatrix} \cosh \tau & \sinh \tau & 0 \\ \sinh \tau & \cosh \tau & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in G .

Theorem 7.1. Let $-1 < \sigma < 0$ and $\tau > 0$. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} \lambda^{-2} F_{-\sigma-1}(0, -\lambda) M_{1-i\hat{\rho}, \sigma+\frac{3}{2}}(-2\lambda e^{-\tau}) d\lambda \\ &= \frac{2^{2\sigma+4} \pi \exp(|\hat{\rho}|\pi - 2\tau\sigma) \Gamma(-\sigma - i|\hat{\rho}|)}{\Gamma(-\sigma) |\Gamma(-\sigma)| B(\sigma + 1 + i\hat{\rho}, \sigma + 1 - i\hat{\rho})} Q_{-\sigma-1}^{i|\hat{\rho}|}(\cosh \tau), \end{aligned}$$

where $Q_v^\mu(x)$ is the associated Legendre function of the second kind.

Proof. Computing the matrix elements of the subrepresentation $T^\bullet$ to H^* , we obtain

$$\begin{aligned} t_{\rho, +, \hat{\rho}, +}^\bullet(h^*(-\tau)) &= \frac{1}{2\pi} F_2(T^\bullet(h^*(-\tau))[f_{\rho, +}^\bullet], f_{\hat{\rho}, +}) \\ &= \frac{1}{2\pi} \int_{\gamma_{2, +}} T^\bullet(h^*(-\tau))[f_{\rho, +}^\bullet] f_{\hat{\rho}, +} d\gamma_{\gamma_{2, +}} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\cosh \tau + \sinh \tau \cosh \alpha_2)^{-\sigma-1-i\hat{\rho}} \\ &\quad \times (\cosh \tau \cosh \alpha_2 + \sinh \alpha_2 + \sinh \tau)^{i\hat{\rho}} \exp(-i\hat{\rho}\alpha_2) d\alpha_2. \end{aligned} \quad (7.1)$$

Since $\cosh \tau \cosh \alpha_2 + \sinh \alpha_2 + \sinh \tau > 0$ and $\sigma + 1 > 0$, using a known integral formula (see [19, Entry 2.5.48.(6)])

$$\begin{aligned} \int_0^\infty \frac{\cos bx \, dx}{(a + \cosh cx)^\nu} &= \frac{\exp\left(\frac{b\pi}{c}\right) \Gamma\left(\nu - \frac{ib}{c}\right)}{c (a^2 - 1)^{\frac{\nu}{2}} \Gamma(\nu)} Q_{\nu-1}^{\frac{ib}{c}}\left(\frac{a}{\sqrt{a^2 - 1}}\right) \\ &\quad (b \in \mathbb{R}^+, \Re(cv) > 0, a \notin [-1, 1]), \end{aligned}$$

we have

$$t_{0, +, \hat{\rho}, +}^\bullet(h^*(-\tau)) = \frac{\exp(|\hat{\rho}|\pi) \Gamma(-\sigma - i|\hat{\rho}|)}{\pi \Gamma(-\sigma)} Q_{-\sigma-1}^{i|\hat{\rho}|}(\cosh \tau).$$

Also,

$$t_{\rho, +, \hat{\rho}, +}^\bullet(h^*(-\tau)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c_{\rho, +, \lambda}^\bullet F_i(T^\bullet(h^*(-\tau))[f_\lambda^\bullet], f_{\hat{\rho}, +}) d\lambda.$$

Here

$$\begin{aligned} F_i(T^\bullet(h^*(-\tau))[f_\lambda^\bullet], f_{\hat{\rho}, +}) &= F_2(T^\bullet(h^*(-\tau))[f_\lambda^\bullet], f_{\hat{\rho}, +}) \\ &= \exp(\tau\sigma) \int_{-\infty}^{\infty} (\cosh \alpha_2 + 1)^{-\sigma-1} \exp\left(i \left[\frac{\lambda \sinh \alpha_2}{e^\tau (\cosh \alpha_2 + 1)} - \hat{\rho}\alpha_2 \right]\right) \\ &= \frac{\exp(\tau\sigma + \lambda e^{-\tau})}{2^{\sigma+1}} \int_0^1 t^{\sigma+i\hat{\rho}} (1-t)^{\sigma-i\hat{\rho}} \exp(-2\lambda e^{-\tau}t) dt, \end{aligned}$$

where the last integral is evaluated by using (3.7). Therefore we have

$$\begin{aligned} t_{\rho, +, \hat{\rho}, +}^\bullet(h^*(-\tau)) &= \frac{\exp(2\tau\sigma) B(\sigma + 1 + i\hat{\rho}, \sigma + 1 - i\hat{\rho})}{(-1)^\sigma 2^{2\sigma+4} \pi} \\ &\quad \times \int_{-\infty}^{\infty} \lambda^{-\sigma-2} c_{\rho, +, \lambda}^\bullet \exp(3\lambda e^{-\tau}) M_{1-i\hat{\rho}, \sigma+\frac{3}{2}}(-2\lambda e^{-\tau}) d\lambda. \end{aligned} \quad (7.2)$$

Finally, considering (7.1) and (7.2), we complete the proof. $\square$

Remark 2. Cylindrical (ordinary) and spherical Bessel functions can be expressed in terms of Coulomb function, respectively, by the following formulas (see, e.g., [29, Eqs. (2.1a) and (2.1c)])

$$J_\nu(\lambda) = \left(\frac{2}{\pi\lambda}\right)^{\frac{1}{2}} F_{\nu-\frac{1}{2}}(0, \lambda) \quad \text{and} \quad j_\nu(\lambda) = \frac{1}{\lambda} F_\nu(0, \lambda).$$

8. Formulae related to the product of subgroups H_1 and H^*

In this section we show that the matrix elements of the subrepresentation $T^\bullet$ to the subgroup $H_1 H^*$ yield four integral representations of the Appell function F_1 , which depend on the relations between parameters θ_1 and τ and correspond to Figures 1–4.

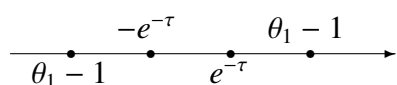


Figure 1. Theorem 8.1.

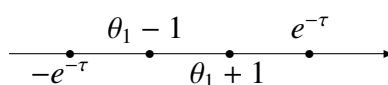


Figure 2. Theorem 8.2.

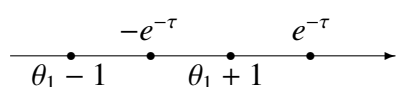


Figure 3. Theorem 8.3.

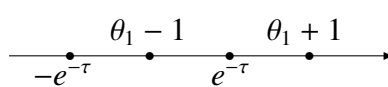


Figure 4. Theorem 8.4.

Theorem 8.1. Let $-1 < \sigma < 0$ and $e^{-\tau} < |\theta_1 \pm 1|$. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} \lambda^{2\sigma} \exp(i\theta_1 \lambda) F_{-\sigma-1}(\hat{\rho}, -\lambda) F_{-\sigma-1}(\rho, e^{-\tau} \lambda) d\lambda \\ &= \frac{\pi \exp\left(2\sigma\tau - \tau - \frac{\pi(\rho+\hat{\rho})}{2}\right)}{2^{2\sigma+2} |\Gamma(-\sigma + i\rho) \Gamma(-\sigma + i\hat{\rho})|} (1 - \theta_1 - e^{-\tau})^{\sigma-i\hat{\rho}} \\ & \quad \times (1 + \theta_1 + e^{-\tau})^{\sigma+i\hat{\rho}} B(-\sigma + i\rho, -\sigma - i\rho) \\ & \quad \times F_1\left(-\sigma + i\rho, -\sigma + i\hat{\rho}, -\sigma - i\hat{\rho}; -2\sigma; \frac{2e^{-\tau}}{e^{-\tau} + \theta_1 - 1}, \frac{2e^{-\tau}}{e^{-\tau} + \theta_1 + 1}\right). \end{aligned}$$

Proof. We show that both functionals F_1 and F_2 are invariant with respect to the pair $(T^\bullet(h^*), T(h^*))$ of representation operators. Indeed, in view of homogeneity of u and v ,

$$\begin{aligned} & F_2(T^\bullet(h^*(\tau))[u], T(h^*(\tau))[v]) = F_1(T^\bullet(h^*(\tau))[u], T(h^*(\tau))[v]) \\ &= \int_{-\infty}^{\infty} u\left(\frac{e^{-\tau} + e^{\tau} \alpha_1^2}{2}, \frac{e^{-\tau} - e^{\tau} \alpha_1^2}{2}, \alpha_1\right) \\ & \quad \times v\left(\frac{e^{-\tau} + e^{\tau} \alpha_1^2}{2}, \frac{e^{-\tau} - e^{\tau} \alpha_1^2}{2}, \alpha_1\right) d\alpha_1 \\ &= \int_{-\infty}^{\infty} u\left(\frac{1 + e^{2\tau} \alpha_1^2}{2}, \frac{1 - e^{2\tau} \alpha_1^2}{2}, e^{\tau} \alpha_1\right) \\ & \quad \times v\left(\frac{1 + e^{2\tau} \alpha_1^2}{2}, \frac{1 - e^{2\tau} \alpha_1^2}{2}, e^{\tau} \alpha_1\right) e^{\tau} d\alpha_1 \\ &= F_1(u, v) = F_2(u, v). \end{aligned}$$

Since $T^\bullet$ is a homomorphism (of G to the group $GL(\mathfrak{D}^\bullet)$ of linear operators of $\mathfrak{D}^\bullet$ with trivial kernels) and H_1 is an one-parameter subgroup (that is, $h_1^{-1}(\theta_1) = h_1(-\theta_1)$), we have

$$\begin{aligned} t_{\rho,+\hat{\rho},+}^\bullet(h_1(\theta_1)h^*(\tau)) &= \frac{1}{2\pi} F_1(T^\bullet(h_1(\theta_1)h^*(\tau))[f_{\rho,+}^\bullet], f_{-\hat{\rho},+}) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(1 - (\alpha_1 - \theta_1)^2)_+^{\sigma+i\hat{\rho}}}{(e^{-2\tau} - \alpha_1^2)_+^{\sigma+1+i\hat{\rho}}} \frac{(e^{-\tau} + \alpha_1)^{2i\hat{\rho}}}{(1 + \alpha_1 - \theta_1)^{2i\hat{\rho}}} d\alpha_1 \\ &= \frac{1}{\pi} \int_{\Omega} \frac{(1 + \alpha_1 - \theta_1)^{\sigma-i\hat{\rho}} (1 - \alpha_1 + \theta_1)^{\sigma+i\hat{\rho}}}{(e^{-\tau} + \alpha_1)^{\sigma+1-i\hat{\rho}} (e^{-\tau} - \alpha_1)^{\sigma+1+i\hat{\rho}}} d\alpha_1, \end{aligned} \quad (8.1)$$

where Ω is the intersection of the segments $|\alpha_1| \leq e^{-\tau}$ and $[\theta_1 - 1, \theta_1 + 1]$. Under the assumption $e^{-\tau} < |\theta_1 \pm 1|$ (Figure 1), this integral can be evaluated by using a known integral formula (see [10, Entry 3.211]): For $\min\{\Re(\mu), \Re(\nu)\} > 0$,

$$\int_0^1 x^{\mu-1} (1-x)^{\nu-1} (1-ux)^{-\kappa} (1-vx)^{-\omega} dx = B(\mu, \nu) F_1(\mu, \kappa, \omega; \mu + \nu; u, v).$$

Also,

$$\begin{aligned} t_{\rho,+\hat{\rho},+}^\bullet(h_1(\theta_1)h^*(\tau)) &= \frac{1}{2\pi} F_i(T^\bullet(h_1(\theta_1)h^*(\tau))[f_{\rho,+}^\bullet], f_{-\hat{\rho},+}) \\ &= \frac{1}{2\pi} F_i(T^\bullet(h^*(\tau))[f_{\rho,+}^\bullet], T(h_1^{-1}(\theta_1))[f_{-\hat{\rho},+}]) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} c_{-\hat{\rho},+,\lambda} F_1(T^\bullet(h^*(\tau))[f_{\rho,+}^\bullet], T(h_1^{-1}(\theta_1)[f_\lambda]) d\lambda. \end{aligned}$$

We thus have

$$\begin{aligned} t_{\rho,+\hat{\rho},+}^\bullet(h_1(\theta_1)h^*(\tau)) &= \frac{\exp(i\theta_1\lambda - \tau\sigma - \tau)}{2^{-\sigma}\pi} \int_{-\infty}^{\infty} c_{-\hat{\rho},+,\lambda} d\lambda \\ &\quad \times \int_{-\infty}^{\infty} (e^{-2\tau} - \alpha_1^2)_+^{-\sigma-1-i\hat{\rho}} (e^{-\tau} + \alpha_1)^{2i\hat{\rho}} \exp(-i\lambda\alpha_1) d\alpha_1 \\ &= \frac{\exp(i\theta_1\lambda - \tau\sigma - \tau)}{2^{-\sigma-1}\pi} \int_{-\infty}^{\infty} c_{-\hat{\rho},+,\lambda} \exp(i\lambda e^{-\tau}) d\lambda \\ &\quad \times \int_0^{2e^{-\tau}} t^{-\sigma-1+i\hat{\rho}} (2e^{-\tau} - t)^{-\sigma-1-i\hat{\rho}} \exp(-i\lambda t) dt. \end{aligned}$$

Using (1.1), (3.7) and (3.9), we get

$$t_{\rho,+\hat{\rho},+}^\bullet(h_1(\theta_1)h^*(\tau)) = \frac{2 \exp(i\theta_1\lambda + \frac{\pi\rho}{2})}{\pi |\Gamma(i\rho - \sigma)|^{-1}} \int_{-\infty}^{\infty} c_{-\hat{\rho},+,\lambda} F_{-\sigma-1}(\rho, \lambda e^{-\tau}) d\lambda.$$

Using (3.5), we have

$$\begin{aligned} t_{\rho,+\hat{\rho},+}^\bullet(h_1(\theta_1)h^*(\tau)) &= 2\pi^{-2} \exp\left(\frac{\pi(\rho + \hat{\rho})}{2}\right) |\Gamma(-\sigma + i\rho) \Gamma(-\sigma + i\hat{\rho})| \\ &\quad \times \int_{-\infty}^{\infty} \lambda^{2\sigma} \exp(i\theta_1\lambda) F_{-\sigma-1}(\hat{\rho}, -\lambda) F_{-\sigma-1}(\rho, e^{-\tau}\lambda) d\lambda. \end{aligned} \quad (8.2)$$

Considering (8.1) and (8.2), we complete the proof. $\square$

Employing the same proof as in Theorem 8.1 for the cases corresponding to Figures 2–4, we obtain the following results, without their proofs, which are given in Theorems 8.2, 8.3 and 8.4.

Theorem 8.2. Let $r - 1 < \sigma < 0$ and $|\theta_1 \pm 1| < e^{-\tau}$. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} \lambda^{2\sigma} \exp(\mathbf{i}\theta_1 \lambda) F_{-\sigma-1}(\hat{\rho}, -\lambda) F_{-\sigma-1}(\rho, e^{-\tau} \lambda) d\lambda \\ &= \frac{4^\sigma \pi (e^{-\tau} + \theta_1 - 1)^{-\sigma-1+\mathbf{i}\rho} B(\sigma + 1 - \mathbf{i}\hat{\rho}, \sigma + 1 + \mathbf{i}\hat{\rho})}{(e^{-\tau} + 1 - \theta_1)^{\sigma+1+\mathbf{i}\rho} \exp\left(\frac{\pi(\rho+\hat{\rho})}{2}\right) |\Gamma(-\sigma + \mathbf{i}\rho) \Gamma(-\sigma + \mathbf{i}\hat{\rho})|} \\ & \times F_1\left(\sigma + 1 - \mathbf{i}\hat{\rho}, \sigma + 1 - \mathbf{i}\rho, \sigma + 1 + \mathbf{i}\rho; 2\sigma + 2; \frac{2}{1 - \theta_1 - e^{-\tau}}, \frac{2}{1 - \theta_1 + e^{-\tau}}\right). \end{aligned}$$

Theorem 8.3. Let $-1 < \sigma < 0$, $|\theta_1 + e^{-\tau}| < 1$, and $|\theta_1 + 1| < e^{-\tau}$. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} \lambda^{2\sigma} \exp(\mathbf{i}\theta_1 \lambda) F_{-\sigma-1}(\hat{\rho}, -\lambda) F_{-\sigma-1}(\rho, e^{-\tau} \lambda) d\lambda \\ &= \frac{\pi (1 + \theta_1 + e^{-\tau})^{\mathbf{i}(\rho+\hat{\rho})} (1 - \theta_1 - e^{-\tau})^{\sigma-\mathbf{i}\hat{\rho}} B(-\sigma + \mathbf{i}\rho, \sigma + 1 + \mathbf{i}\hat{\rho})}{2^{\sigma+2+\mathbf{i}\rho} \exp\left(\frac{\pi(\rho+\hat{\rho})}{2} + \tau(1 + \sigma + \mathbf{i}\rho)\right) |\Gamma(-\sigma + \mathbf{i}\rho) \Gamma(-\sigma + \mathbf{i}\hat{\rho})|} \\ & \times F_1\left(-\sigma + \mathbf{i}\rho, -\sigma + \mathbf{i}\hat{\rho}, \sigma + 1 + \mathbf{i}\rho; 1 + \mathbf{i}(\rho + \hat{\rho}); \frac{e^{-\tau} + \theta_1 + 1}{e^{-\tau} + \theta_1 - 1}, \frac{e^{\tau}(\theta_1 + 1) + 1}{2}\right). \end{aligned}$$

Theorem 8.4. Let $-1 < \sigma < 0$, $|e^{-\tau} - \theta_1| < 1$, and $|\theta_1 - 1| < e^{-\tau}$. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} \lambda^{2\sigma} \exp(\mathbf{i}\theta_1 \lambda) F_{-\sigma-1}(\hat{\rho}, -\lambda) F_{-\sigma-1}(\rho, e^{-\tau} \lambda) d\lambda \\ &= \frac{2^{1+\mathbf{i}\hat{\rho}} \pi (e^{-\tau} + \theta_1 - 1)^{-\sigma-1-\mathbf{i}\rho} B(\sigma + 1 - \mathbf{i}\hat{\rho}, -\sigma - \mathbf{i}\rho)}{(1 + e^{-\tau} - \theta_1)^{\mathbf{i}(\rho+\hat{\rho})} \exp\left(\frac{\pi(\rho+\hat{\rho})}{2}\right) |\Gamma(-\sigma + \mathbf{i}\rho) \Gamma(-\sigma + \mathbf{i}\hat{\rho})|} \\ & \times F_1\left(\sigma + 1 - \mathbf{i}\hat{\rho}, \sigma + 1 - \mathbf{i}\rho, -\sigma - \mathbf{i}\hat{\rho}; 1 - \mathbf{i}(\rho + \hat{\rho}); \frac{1 + e^{-\tau} - \theta_1}{1 - e^{-\tau} - \theta_1}, \frac{1 + e^{\tau} - \theta_1}{2}\right). \end{aligned}$$

Remark 3. In Theorems 8.1–8.4, the Appell function F_1 can be rewritten in terms of the hypergeometric function ${}_2F_1$ by using the following relation (see, e.g., [21, Entry 7.2.4.(63)])

$$F_1(a, b, \hat{b}; b + \hat{b}; w, z) = (1 - z)^{-a} {}_2F_1\left(a, b; b + \hat{b}; \frac{w - z}{1 - z}\right).$$

9. Concluding remarks

The group theoretic approach gives a *natural* (in some sense) technique to obtain formulas for integral transforms of special functions, since they occur in relations between kernels of integral operators of representations calculated in different *ordinary* and mixed bases and matrix elements of basis transformations. For instance, by evaluating the Mellin-Laplace transform of the Coulomb Sturmian radial function, Morales [15] used series decomposition and integration term-by-term to show that the integral

$$\int_0^\infty x^{2l+j+1} \exp(-(1+\omega)x) {}_1F_1(1+l-n; 2l+2; 2x) dx \quad (9.1)$$

is equal up to multiplicative constant to

$$(1 + \omega)^{-2-2l-j} \Gamma(2 + 2l + j) {}_2F_1 \left[\begin{matrix} 1 + l - n, 2 + 2l + j \\ 2 + 2l \end{matrix} \middle| \frac{2}{1 + \omega} \right],$$

where $l, j \in \mathbb{N}_0$ and $\omega > -1$. For investigation of matrix elements of two-body Coulomb interaction in the lowest Landau level, Bentalha [2, Lemma 1] obtained the following integral formula for product of associated Laguerre polynomials ($l, m, n \in \mathbb{N}_0$) expressed in terms of Appell F_2 function:

$$\begin{aligned} & \int_0^\infty x^{l-\frac{1}{2}} \exp(-2x) L_n^l(x) L_m^l(x) dx \\ &= \frac{(n+l)! (m+l)! \Gamma\left(l + \frac{1}{2}\right)}{2^{l+\frac{1}{2}} n! m! (l!)^2} F_2 \left(l + \frac{1}{2}; -n - m; l + 1, l + 1; \frac{1}{2}, \frac{1}{2} \right), \end{aligned} \quad (9.2)$$

which may be also expressed in terms of ${}_3F_2$ (see, e.g., [20, Entry 2.19.14.(9)] and [28]). Since $L_n^{2l+1}(x)$ up to multiplicative constant coincides with ${}_1F_1(-n; 2l + 2; x)$, we have in (9.1) and (9.2) Mellin-Laplace integrals for Whittaker functions of the first kind, which are related to regular Coulomb function $F_\sigma(\rho, \lambda)$. The results in Theorems 8.1–8.4 are concerned with the Mellin-Fourier transform for Coulomb functions and, in this regard, are closely related to the examples (9.1) and (9.2). It may be said that, in some sense, Theorems 8.1–8.4 are *complexifications* of these examples.

Theorems 4.1–8.4 are based on the fact (described in Theorems 3.1 and 3.2) that the matrix elements $c_{\rho, \pm, \lambda}$ and $c_{\lambda, \rho, \pm}$ (kernels of corresponding integral operators) of the transformations $B_1 \rightarrow B_2$ and $B_2 \rightarrow B_1$ can be expressed in terms of Coulomb wave function $F_\sigma(\rho, \lambda)$. In all these theorems (except Theorem 5.1), we have used *only integrals* of the following forms

$$\int_{-\infty}^{\infty} c_{\rho, +, \lambda} \dots d\lambda \quad \text{and} \quad \int_{-\infty}^{\infty} c_{\lambda, \rho, +} \dots d\rho.$$

Analogous integral transforms to those in this paper, containing kernels $c_{\rho, -, \lambda}$ and $c_{\lambda, \rho, -}$, can yield similar results. Also these kernels were considered for two *dual* cases $\sigma = -\frac{1}{2} \pm \frac{1}{4}$ in [23].

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Conflict of interest

The authors declare that they have no conflict of interest.

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