Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 5627–5637.
DOI:10.3934/math.2020359
Received: 13 May 2020
Accepted: 23 June 2020
Published: 01 July 2020

Research article

Does nonuniform behavior destroy the structural stability?

Yuzhen Bai[1], **Donal O'Regan**[2], **Yong-Hui Xia**[3,*] **and Xiaoqing Yuan**[3]

[1] School of Mathematical Sciences, Qufu Normal University, Qufu, 273165, P.R.China

[2] School of Mathematics, Statistics and Applied Mathematics, National university of Ireland, Galway, Ireland

[3] Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, P.R.China

* **Correspondence:** Email: xiadoc@163.com. ORCID: 0000-0001-8918-3509. Scopus Author ID: 8703188200.

Abstract: This paper provides an answer if the nonuniform behavior can destroy the structural stability of nonlinear systems. We show that if the linear system $\dot{x}(t) = A(t)x(t)$ admits a nonuniform exponential dichotomy, then the perturbed nonautonomous system $\dot{x}(t) = A(t)x(t) + f(t, x)$ is structurally stable under suitable conditions.

Keywords: stability; exponential dichotomy; nonautonomous; nonlinear system
Mathematics Subject Classification: 37F15, 34D09, 37B55, 37D25

1. Introduction

Structural stability of systems is important since structural stable systems can resist external disturbance; we refer the reader to [10–22]. Many researchers provided sufficient conditions for structural stability of planar (2-dimension) polynomial vector fields under polynomial perturbations [30–36]. In this paper, we focus on the high-dimensional systems. Usually authors study structural stability under the assumption that the linear system has some hyperbolic property and in most papers the authors assume that the linear system admits (classical or uniform) exponential dichotomy [7, 8]. However, it is argued that (uniform) exponential dichotomy restrict the behavior of dynamical systems. For this reason, we need a more general concept of hyperbolicity. Recently, nonuniform exponential behavior and nonuniform exponential dichotomy was introduced (see e.g. [1, 3–6, 20, 21]). As a result a natural question arises: if the linear system admits a nonuniform exponential dichotomy, can structural stability of systems be destroyed by the nonuniformity? This paper gives a nonuniform version of structural stability of nonlinear systems.

2. Main results

In this section, we state our main theorem. Consider the systems

$$\dot{x}(t) = A(t)x(t), \tag{2.1}$$

$$\dot{x}(t) = A(t)x(t) + f(t, x), \tag{2.2}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^n$, $A(t)$ is a continuous matrix function, and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise continuous function. Let $T(t, s)$ be the evolution operator satisfying $x(t) = T(t, s)x(s)$, $t, s \in \mathbb{R}$, where $x(t)$ is a solution of the system (2.1).

Definition 2.1. *The linear system (2.1) is said to admit a nonuniform exponential dichotomy if there exists a projection $P(t)$ ($P^2 = P$) and constants $\alpha > 0, K > 0, \varepsilon \geq 0$, such that*

$$\begin{cases} \|T(t, s)P(s)\| \leq Ke^{-\alpha(t-s)} \cdot e^{\varepsilon|s|}, & t \geq s, \\ \|T(t, s)Q(s)\| \leq Ke^{\alpha(t-s)} \cdot e^{\varepsilon|s|}, & t \leq s, \end{cases} \tag{2.3}$$

where $P(t) + Q(t) = Id$ (identity), $T(t, s)P(s) = P(t)T(t, s), t, s \in \mathbb{R}$, and $\|\cdot\|$ is the Euclidean norm (see e.g. [1, 2, 27]).

Remark 2.1. *The nonuniform exponential dichotomy reduces to the classical (uniform) exponential dichotomy by taking $\varepsilon = 0$ in (2.3). In bad situations, an example is given in [1, 2] to show that linear system does not admit an exponential dichotomy, but it admits a nonuniform exponential dichotomy.*

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise continuous function. There exists $L_f > 0$ such that for any $x_1, x_2 \in \mathbb{R}^n, t \in \mathbb{R}$, the piecewise continuous function $f(t, x)$ satisfies (here $\varepsilon \geq 0$ as above)

(H_1) $\|f(t, x_1) - f(t, x_2)\| \leq L_f e^{-\varepsilon|t|} \cdot \|x_1 - x_2\|$.

For a small enough number $0 < \gamma < 1$, denote

$$\mathcal{S} = \{f(t, x)\Big| f(t, x) \text{ satisfies } (H_1), 2KL_f\alpha^{-1} \leq \gamma \text{ and } \sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(\sigma, 0)\|e^{\varepsilon|\sigma|}d\sigma < +\infty\}.$$

Since the conditions in $\mathcal{S}$ are used in the following proof, for sake of convenience, we denote

(H_2) $2KL_f\alpha^{-1} \leq \gamma$,

(H_3) $\sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(\sigma, 0)\|e^{\varepsilon|\sigma|}d\sigma < +\infty$.

For any $f(t, x) \in \mathcal{S}$, define $L_f = \inf\{c > 0 | \|f(t, x_1) - f(t, x_2)\| \leq ce^{-\varepsilon|t|}\|x_1 - x_2\|\}$. Taking $\|f(t, x)\|_1 = \max\{L_f | f \in \mathcal{S}\}$, then $\mathcal{S}$ is a normed linear vector space with norm $\|\cdot\|_1$. If $\dot{x}(t) = A(t)x + f(t, x)$ is topologically conjugated to $\dot{y}(t) = A(t)y + g(t, y)$, we denote it by $f \approx g$. For detailed definition of a topological conjugacy, for example, one can refer to [11, 13, 23, 26–29].

Definition 2.2. *The differential equation $\dot{x}(t) = A(t)x + f(t, x)$ is said to be structurally stable in $\mathcal{S}$, if for any $g(t, y) \in \mathcal{S}$, we have $\dot{y}(t) = A(t)y + g(t, y)$ is topologically conjugated to $\dot{x}(t) = A(t)x + f(t, x)$ (i.e. $g \approx f$).*

Theorem 2.1. *For any $f(t, x) \in \mathcal{S}$, if the linear system* (2.1) *admits a nonuniform exponential dichotomy, then system (2.2) is structurally stable in $\mathcal{S}$.*

3. Proofs of main results

To prove the main result, some preliminary lemmas are needed.
Lemma 3.1([26]) *Let $\varphi(t)$ be a non-negative locally integrable function on $\mathbb{R}$. If there exist constants $p > 0, C > 0$ such that*

$$\frac{1}{p}\int_t^{t+p} \varphi(s)ds \leq C,$$

then for any $\beta > 0$, we have

$$\int_{-\infty}^{t} \varphi(s)e^{-\beta(t-s)}ds \leq (1-e^{-\beta p})^{-1}Cp,$$

$$\int_{t}^{+\infty} \varphi(s)e^{\beta(t-s)}ds \leq (1-e^{-\beta p})^{-1}Cp.$$

Lemma 3.2 *Suppose that system* (2.1) *admits a nonuniform exponential dichotomy with the constants ε, α. If the nonlinear term $f(t,x) \in \mathcal{S}$, then the nonlinear system* (2.2) *has a unique bounded solution $y(t)$ satisfying*

$$y(t) = \int_{-\infty}^{t} T(t,s)P(s)f(s,y(s))ds - \int_{t}^{+\infty} T(t,s)Q(s)f(s,y(s))ds. \tag{3.1}$$

Proof. Now we prove this lemma in three steps.
Step 1. First, we prove that the nonlinear system (2.2) has a unique bounded solution. For this purpose, let $\sup\limits_{t\in\mathbb{R}} \int_t^{t+1} \|f(\sigma,0)\|e^{\varepsilon|\sigma|}d\sigma = M$, $x_0(t) \equiv 0$, and

$$x_1(t) = \int_{-\infty}^{t} T(t,s)P(s)f(s,x_0(s))ds - \int_{t}^{+\infty} T(t,s)Q(s)f(s,x_0(s))ds.$$

Take $t \in \mathbb{R}$. From (H_1) and Lemma 3.1, it is easy to obtain that

$$\|x_1(t)\| \leq 2KM(1-e^{-\alpha})^{-1},$$

which implies $x_1(t)$ is bounded. Assume that $x_m(t)$ is bounded. Define $x_{m+1}(t)$ as

$$x_{m+1}(t) = \int_{-\infty}^{t} T(t,s)P(s)f(s,x_m(s))ds - \int_{t}^{+\infty} T(t,s)Q(s)f(s,x_m(s))ds.$$

From (2.3) and (H_2), we have

$$\begin{aligned}
\|x_{m+1}(t)\| \leq\ & \int_{-\infty}^{t} Ke^{-\alpha(t-s)}e^{\varepsilon|s|}\cdot\Big[\delta\|x_m(s)\|e^{-\varepsilon|s|} + \|f(s,0)\|\Big]ds \\
& + \int_{t}^{+\infty} Ke^{\alpha(t-s)}e^{\varepsilon|s|}\cdot\Big[L_f\|x_m(s)\|e^{-\varepsilon|s|} + \|f(s,0)\|\Big]ds \\
=\ & \int_{-\infty}^{t} Ke^{-\alpha(t-s)}ds\cdot L_f\|x_m(s)\| + \int_{-\infty}^{t} Ke^{-\alpha(t-s)}\|f(s,0)\|e^{\varepsilon|s|}ds \\
& + \int_{t}^{+\infty} Ke^{\alpha(t-s)}\cdot L_f\|x_m(s)\| + \int_{t}^{+\infty} Ke^{\alpha(t-s)}\|f(s,0)\|e^{\varepsilon|s|}ds.
\end{aligned}$$

It follows from (H_1) and Lemma 3.1 that

$$\|x_{m+1}(t)\| \leq \frac{2KL_f}{\alpha}\|x_m(t)\| + 2KM(1-e^{-\alpha})^{-1},$$

and this implies

$$\|x_m(t)\| \leq \frac{2KL_f}{\alpha}\|x_{m-1}(t)\| + 2KM(1-e^{-\alpha})^{-1}.$$

Consequently, we have

$$\begin{aligned}
\|x_{m+1}(t)\| &\leq \frac{2KL_f}{\alpha}\Big(\frac{2KL_f}{\alpha}\|x_{m-1}(t)\| + 2KM(1-e^{-\alpha})^{-1}\Big) + 2KM(1-e^{-\alpha})^{-1} \\
&\leq \cdots \\
&\leq \Big[(\frac{2KL_f}{\alpha})^m + (\frac{2KL_f}{\alpha})^{m-1} + \cdots + \frac{2KL_f}{\alpha}\Big]\|x_1(t)\| + 2KM(1-e^{-\alpha})^{-1} \\
&\leq \frac{1-(\frac{2KL_f}{\alpha})^m}{1-\frac{2KL_f}{\alpha}} \cdot \frac{2KL_f}{\alpha}\|x_1(t)\| + 2KM(1-e^{-\alpha})^{-1}.
\end{aligned}$$

In view of (H_3), $\frac{2KL_f}{\alpha} < 1$, we obtain

$$\|x_{m+1}(t)\| \leq \frac{1}{1-\frac{2KL_f}{\alpha}} \cdot \frac{2KL_f}{\alpha} \cdot 2KM(1-e^{-\alpha})^{-1} + 2KM(1-e^{-\alpha})^{-1},$$

which implies that the sequence of function $\{x_m(t)\}$ is bounded on $\mathbb{R}$. Also,

$$\begin{aligned}
\|x_{m+1}(t) - x_m(t)\| &\leq \int_{-\infty}^{t} Ke^{-\alpha(t-s)}e^{\varepsilon|s|}\Big(L_f\|x_m(s) - x_{m-1}(s)\| \cdot e^{-\varepsilon|s|}\Big)ds \\
&\quad + \int_{t}^{+\infty} Ke^{\alpha(t-s)}e^{\varepsilon|s|}\Big(L_f\|x_m(s) - x_{m-1}(s)\| \cdot e^{-\varepsilon|s|}\Big)ds \\
&= \int_{-\infty}^{t} KL_f e^{-\alpha(t-s)}\|x_m(s) - x_{m-1}(s)\|ds \\
&\quad + \int_{t}^{+\infty} KL_f e^{\alpha(t-s)}\|x_m(s) - x_{m-1}(s)\|ds.
\end{aligned}$$

Let $T_m = \sup\limits_{t\in\mathbb{R}} \|x_m(t) - x_{m-1}(t)\|$. It follows from (H_3) that

$$\begin{aligned}
T_{m+1} &\leq \int_{-\infty}^{t} KL_f e^{-\alpha(t-s)}T_m ds + \int_{t}^{+\infty} KL_f e^{\alpha(t-s)}T_m ds \\
&\leq 2KL_f\alpha^{-1}T_m \\
&\leq \gamma T_m.
\end{aligned}$$

Since $0 < \gamma < 1$, the series $\sum\limits_{m=1}^{+\infty}\|x_m(t) - x_{m-1}(t)\|$ converges uniformly on $\mathbb{R}$. Denote $\lim\limits_{m\to\infty} x_m(t) = y(t)$, and note $y(t)$ is bounded. In addition,

$$y(t) = \int_{-\infty}^{t} T(t,s)P(s)f(s,y(s))ds - \int_{t}^{+\infty} T(t,s)Q(s)f(s,y(s))ds.$$

Step 2. We will prove that any bounded solution of system (2.2) can be expressed by formula (3.1). Now assume that system (2.2) has another bounded solution $x(t)$ satisfying $x(0) = x_0, \|x(t)\| \leq \vartheta$. We have

$$\begin{aligned} x(t) &= T(t,0)x(0) + \int_0^t T(t,s)I_{id}f(s,x(s))ds \\ &= T(t,0)x(0) + \int_0^t T(t,s)(P(s)+Q(s))f(s,x(s))ds \\ &= T(t,0)x(0) + \int_0^t T(t,s)P(s)f(s,x(s))ds + \int_0^t T(t,s)Q(s)f(s,x(s))ds \\ &= T(t,0)x(0) + \int_{-\infty}^t T(t,s)P(s)f(s,x(s))ds - \int_{-\infty}^0 T(t,s)P(s)f(s,x(s))ds \\ &\qquad + \int_0^{+\infty} T(t,s)Q(s)f(s,x(s))ds - \int_t^{+\infty} T(t,s)Q(s)f(s,x(s))ds \\ &= \int_{-\infty}^t T(t,s)P(s)f(s,x(s))ds - \int_t^{+\infty} T(t,s)Q(s)f(s,x(s))ds \\ &\quad + T(t,0)[x_0 - \int_{-\infty}^0 T(0,s)P(s)f(s,x(s))ds + \int_0^{+\infty} T(0,s)Q(s)f(s,x(s))ds]. \end{aligned} \tag{3.2}$$

From Lemma 3.1, we have

$$\begin{aligned} & \| \int_{-\infty}^t T(t,s)P(s)f(s,x(s))ds - \int_t^{+\infty} T(t,s)Q(s)f(s,x(s))ds \| \\ \leq\; & \Big| \int_{-\infty}^t Ke^{-\alpha(t-s)} \cdot e^{\varepsilon|s|}(L_f\|x(s)\| \cdot e^{-\varepsilon|s|} + \|f(s,0)\|)\Big| ds \\ & + \Big| \int_t^{+\infty} Ke^{\alpha(t-s)} \cdot e^{\varepsilon|s|}(L_f\|x(s)\| \cdot e^{-\varepsilon|s|} + \|f(s,0)\|)\Big| ds \\ \leq\; & \Big| \int_{-\infty}^t Ke^{-\alpha(t-s)}(L_f\vartheta + M)ds\Big| + \Big| \int_t^{+\infty} Ke^{\alpha(t-s)}(L_f\vartheta + M)ds\Big| \\ \leq\; & 2K(1-e^{-\alpha})^{-1}(\alpha\vartheta + M). \end{aligned}$$

Hence, we see that

$$T(t,0)[x_0 - \int_{-\infty}^0 T(0,s)P(s)f(s,x(s))ds + \int_0^{+\infty} T(0,s)Q(s)f(s,x(s))ds]$$

is bounded. In addition, the above formula is the solution of system (2.1), so it is a bounded solution. Note that the linear system has no non-trival bounded solution due to the nonuniform exponential dichotomy. Thus we have

$$T(t,0)[x_0 - \int_{-\infty}^0 T(0,s)P(s)f(s,x(s))ds + \int_0^{+\infty} T(0,s)Q(s)f(s,x(s))ds] = 0,$$

and therefore,

$$x(t) = \int_{-\infty}^t T(t,s)P(s)f(s,x(s))ds - \int_t^{+\infty} T(t,s)Q(s)f(s,x(s))ds.$$

Step 3. We prove the uniqueness of the bounded solution. From (2.3), (H_2) and (H_3), we have

$$\begin{aligned}\|y(t)-x(t)\| &\leq \int_{-\infty}^{t} Ke^{-\alpha(t-s)}\cdot e^{\varepsilon|s|}L_f\|y(s)-x(s)\|e^{-\varepsilon|s|}ds \\ &\quad + \int_{t}^{+\infty} Ke^{\alpha(t-s)}\cdot e^{\varepsilon|s|}L_f\|y(s)-x(s)\|e^{-\varepsilon|s|}ds \\ &\leq 2KL_f\alpha^{-1}\sup_{t\in\mathbb{R}}\|y(t)-x(t)\| \\ &\leq \gamma\sup_{t\in\mathbb{R}}\|y(t)-x(t)\|.\end{aligned}$$

That is, $\sup_{t\in\mathbb{R}}\|y(t)-x(t)\| \leq \gamma\sup_{t\in\mathbb{R}}\|y(t)-x(t)\|$, which implies $y(t)=x(t)$. Thus, the uniqueness is proved.

Remark 3.1 *In the proof, the function sequence $\{x_m(t)\}$ can be seen as the approximation sequence of the solution of system (2.2) and we conclude that $\{x_m(t)\}$ is bounded on $\mathbb{R}$.*

Lemma 3.3 *Suppose that the system* (2.1) *admits a nonuniform exponential dichotomy, $f_i(t,x)\in\mathcal{S}$, $(i=1,2)$ and $2KL_{f_i}\alpha^{-1}\leq\gamma$. Let $y(t,\varrho,x)$ be the bounded solution of*

$$\dot{z}(t) = A(t)z + f_1(t,x) \tag{3.3}$$

with $\varphi(\varrho,\varrho,x)=x$. Then for any $x\in\mathbb{R}^n,\varrho\in\mathbb{R}$, the following differential equation

$$\dot{z}(t) = A(t)z + f_2(t,z+\varphi(t,\varrho,x)) - f_1(t,\varphi(t,\varrho,x)) \tag{3.4}$$

has a unique bounded solution $z^{(\varrho,x)}(t)$ satisfying

$$\begin{aligned}z^{(\varrho,x)}(t) &= \int_{-\infty}^{t} T(t,s)P(s)[f_2(s,z^{(\varrho,x)}(s)+\varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))]ds \\ &\quad + \int_{t}^{+\infty} T(t,s)Q(s)[f_2(s,z^{(\varrho,x)}(s)+\varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))]ds.\end{aligned} \tag{3.5}$$

Moreover, $z^{(\varrho,x)}(\varrho)$ uniformly converges to $z^{(\varrho,x_0)}(\varrho)$ for $x\to x_0\in\mathbb{R}^n$.

Proof. For fixed $(\varrho,x)\in\mathbb{R}\times\mathbb{R}^n$, clearly, system (3.4) satisfies the conditions of Lemma 3.2. Thus, (3.4) has a unique bounded solution $z^{(\varrho,x)}(t)$ satisfying (3.5). Now we construct a sequence $\{z_m^{(\varrho,x)}(t)\}$. Let $z_0^{(\varrho,x)}(t)\equiv 0$, and

$$\begin{aligned}z_1^{(\varrho,x)}(t) &= \int_{-\infty}^{t} T(t,s)P(s)[f_2(s,z_0^{(\varrho,x)}(s)+\varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))]ds \\ &\quad - \int_{t}^{+\infty} T(t,s)Q(s)[f_2(s,z_0^{(\varrho,x)}(s)+\varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))]ds.\end{aligned}$$

Assume that $z_m^{(\varrho,x)}(t)$ is well defined. Take

$$\begin{aligned}z_{m+1}^{(\varrho,x)}(t) &= \int_{-\infty}^{t} T(t,s)P(s)[f_2(s,z_m^{(\varrho,x)}(s)+\varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))]ds \\ &\quad - \int_{t}^{+\infty} T(t,s)Q(s)[f_2(s,z_m^{(\varrho,x)}(s)+\varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))]ds.\end{aligned}$$

From Remark 3.1 and Lemma 3.2, the approximation sequence $\{z_m^{(\varrho,x)}(t)\}$ of the solution of system (3.4) uniformly converges to $z^{(\varrho,x)}(t)$ on $\mathbb{R}\times(\mathbb{R}\times\mathbb{R}^n)$.

Now we claim that for any non-negative integer m, $x_0 \in \mathbb{R}^n, h > 0$, $z_m^{(\varrho,x)}(t)$ uniformly converges to $z_m^{(\varrho,x_0)}(t)$ on $|t-\varrho| \leq h$, for $x \to x_0$.

For $m = 0$, $z_0^{(\varrho,x)}(t) = 0$, the claim is clear. Assume that the above claim holds for $m = k$. Now we consider $m = k+1$. For $x_0 \in \mathbb{R}^n, h > 0$, we prove that for any $\varepsilon > 0$, there exists a constant δ^* such that

$$\|z_{k+1}^{(\varrho,x)}(t) - z_{k+1}^{(\varrho,x_0)}(t)\| < \varepsilon, \quad |t-\varrho| \leq h,$$

where $\|x - x_0\| < \delta^*$.

Since $f_i(t,x) \in \mathcal{S}$, $i = 1,2$, let $\sup\limits_{t\in\mathbb{R}} \int_t^{t+1} \|f_i(\sigma,0)\| e^{\varepsilon|\sigma|} d\sigma = M_i$, $i = 1,2$.

From (2.3) and (H_1), we have

$$\begin{aligned}
& \|z_{k+1}^{(\varrho,x)}(t) - z_{k+1}^{(\varrho,x_0)}(t)\| \\
= & \Big\| \int_{-\infty}^t T(t,s)P(s)[f_2(s, z_k^{(\varrho,x)}(s) + \varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))]ds \\
& - \int_t^{+\infty} T(t,s)Q(s)[f_2(s, z_k^{(\varrho,x)}(s) + \varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))]ds \\
& - \int_{-\infty}^t T(t,s)P(s)[f_2(s, z_k^{(\varrho,x_0)}(s) + \varphi(s,\varrho,x_0)) - f_1(s,\varphi(s,\varrho,x_0))]ds \\
& + \int_t^{+\infty} T(t,s)Q(s)[f_2(s, z_k^{(\varrho,x_0)}(s) + \varphi(s,\varrho,x_0)) - f_1(s,\varphi(s,\varrho,x_0))]ds \Big\| \\
\leq & \int_{-\infty}^t K e^{-\alpha(t-s)} \cdot e^{\varepsilon|s|} \Big\| [f_2(s, z_k^{(\varrho,x)}(s) + \varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))] \\
& \qquad - [f_2(s, z_k^{(\varrho,x_0)}(s) + \varphi(s,\varrho,x_0)) - f_1(s, y(s,\varrho,x_0))] \Big\| ds \\
& + \int_t^{+\infty} K e^{\alpha(t-s)} \cdot e^{\varepsilon|s|} \Big\| [f_2(s, z_k^{(\varrho,x)}(s) + \varphi(s,\varrho,x)) - f_1(s,\varphi(s,\varrho,x))] \\
& \qquad - [f_2(s, z_k^{(\varrho,x_0)}(s) + \varphi(s,\varrho,x_0)) - f_1(s,\varphi(s,\varrho,x_0))] \Big\| ds \\
\triangleq & \int_{-\infty}^{t-\tau} J ds + \int_{t-\tau}^t J ds + \int_t^{t+\tau} J ds + \int_{t+\tau}^{+\infty} J ds,
\end{aligned}$$

where $\tau = \frac{1}{\alpha} |\ln \frac{\varepsilon\alpha}{8[(L_{f_1}+L_{f_2})\overline{M}+2M]}|$. From Remark 3.1 and Lemma 3.2, we see that the approximation sequence $\{z_m^{(\varrho,x)}(t)\}$ of the solution of system (3.4) is bounded on $\mathbb{R}$. Also, $\underline{y}(s,\varrho,x)$ is bounded on $\mathbb{R}$. Without loss of generality, we assume that they are all bounded above by $\overline{M}$. Since $f_i(t,x) \in \mathcal{S}, (i = 1,2)$, a standard computations lead us to

$$\begin{aligned}
& \int_{-\infty}^{t-\tau} J ds \leq \int_{-\infty}^{t-\tau} K e^{-\alpha(t-s)} \cdot e^{\varepsilon|s|} [(2L_{f_1} + 2L_{f_2})\overline{M} + 4M] e^{-\varepsilon|s|} ds \leq \frac{\varepsilon}{4}, \\
& \int_{t+\tau}^{+\infty} J ds \leq \int_{t+\tau}^{+\infty} K e^{\alpha(t-s)} \cdot e^{\varepsilon|s|} [(2L_{f_1} + 2L_{f_2})\overline{M} + 4M] e^{-\varepsilon|s|} ds \leq \frac{\varepsilon}{4}, \\
& \int_{t-\tau}^t J ds \\
\leq & \int_{t-\tau}^t K e^{-\alpha(t-s)} \cdot e^{\varepsilon|s|} [L_{f_2} \|z_k^{(\varrho,x)}(s) + \varphi(s,\varrho,x) - z_k^{(\varrho,x_0)}(s) - \varphi(s,\varrho,x_0)\| \cdot e^{-\varepsilon|s|} \\
& + L_{f_1} \|\varphi(s,\varrho,x) - \varphi(s,\varrho,x_0)\| \cdot e^{-\varepsilon|s|}] ds \\
\leq & \int_{t-\tau}^t K e^{-\alpha(t-s)} \gamma [\|z_k^{(\varrho,x)}(s) - z_k^{(\varrho,x_0)}(s)\| + 2\|\varphi(s,\varrho,x) - \varphi(s,\varrho,x_0)\|] ds
\end{aligned}$$

By assumption, for the above $\varepsilon > 0$, there exists a constant $\delta_k > 0$ such that when $\|x - x_0\| < \delta_k$, $\|z_k^{(\varrho,x)}(t) - z_k^{(\varrho,x_0)}(t)\| < \varepsilon, \quad |t-\varrho| \le h$. Since $\varphi(t,\varrho,x)$ is the solution of (2.2),

$$\varphi(t,\varrho,x) = x + \int_{\varrho}^{t} [A(s)\varphi(s,\varrho,x) + f(s,\varphi(s,\varrho,x))]ds.$$

Due to the continuity, we can assume that there is a positive constant θ such that $\|A(t)\| \le \theta$, for $|t-\varrho| \le h+\tau$. We have

$$\begin{aligned}\|\varphi(t,\varrho,x) - \varphi(t,\varrho,x_0)\| &\le \|x-x_0\| + \|\int_{\varrho}^{t} (\theta + L_f)\varphi(s,\varrho,x) - \varphi(s,\varrho,x_0)\|ds \\ &\le \|x-x_0\| + (\theta + \tfrac{\gamma}{2K\alpha})\int_{\varrho}^{t} \|\varphi(s,\varrho,x) - \varphi(s,\varrho,x_0)\|ds.\end{aligned}$$

It follows from Bellmen's inequality that

$$\|\varphi(t,\varrho,x) - \varphi(t,\varrho,x_0)\| \le \|x-x_0\|e^{(\theta+\frac{\gamma}{2K\alpha})\cdot|t-\varrho|} \le \|x-x_0\|e^{(\theta+\frac{\gamma}{2K\alpha})h}.$$

That is, for the above $\varepsilon > 0$, there exists a constant δ_0 such that

$$\|\varphi(t,\varrho,x) - \varphi(t,\varrho,x_0)\| < \varepsilon, \quad |t-\varrho| \le h,$$

where $\|x - x_0\| < \delta_0$. Consequently,

$$\int_{t-\tau}^{t} Jds \le \int_{t-\tau}^{t} Ke^{-\alpha(t-s)} \cdot \frac{\gamma}{2K\alpha} \cdot 3\varepsilon ds \le \frac{3\gamma}{2}, \ \ |t-\varrho| \le h.$$

Similarly, there exists a constant $\underline{\delta} > 0$, for $\|x - x_0\| < \underline{\delta}$, $\displaystyle\int_{t}^{t+\tau} Jds \le \frac{3\gamma}{2}$, $|t-\varrho| \le h$. Taking $\delta^* = \min\{\overline{\delta}, \underline{\delta}\}$, then for $|x - x_0| < \delta^*$, we have

$$\|z_{k+1}^{(\varrho,x)}(t) - z_{k+1}^{(\varrho,x_0)}(t)\| \le \tfrac{\varepsilon}{2} + 3\gamma\varepsilon < 4\varepsilon, \ \ |t-\varrho| \le h.$$

Therefore, for any $x_0 \in \mathbb{R}^n, h > 0$, when $x \to x_0$, $z_{k+1}^{(\varrho,x)}(t)$ uniformly converges to $z_{k+1}^{(\varrho,x_0)}(t)$ on $|t-\varrho| \le h$. From the induction principle, for any non-negative integer m, $x_0 \in \mathbb{R}^n$ and $h > 0$, if $x \to x_0$, then $z_m^{(\varrho,x)}(t)$ uniformly converges to $z_m^{(\varrho,x_0)}(t)$ on $|t-\varrho| \le h$.

In particular, taking $h = 0$, we have for any non-negative integer m, $x_0 \in \mathbb{R}^n$, if $x \to x_0$, then $z_m^{(\varrho,x)}(\varrho)$ uniformly converges to $z_m^{(\varrho,x_0)}(\varrho)$.

We finally need to prove that for $x \to x_0$, $z^{(\varrho,x)}(\varrho)$ uniformly converges to $z^{(\varrho,x_0)}(\varrho)$ on $\mathbb{R}$. In fact, for any $\widetilde{\varepsilon} > 0$, since $\{z_m^{(\varrho,x)}(\varrho)\}$ uniformly converges to $z^{(\varrho,x)}(\varrho)$ on $\mathbb{R}$, there exists a constant m_0 such that

$$\|z_{m_0}^{(\varrho,x)}(\varrho) - z^{(\varrho,x)}(\varrho)\| < \widetilde{\varepsilon}, \ \ \varrho \in \mathbb{R}, x \in \mathbb{R}^n.$$

In addition, for $x \to x_0$, since $\{z_{m_0}^{(\varrho,x)}(\varrho)\}$ uniformly converges to $z_{m_0}^{(\varrho,x_0)}(\varrho)$ on $\mathbb{R}$, there exists a constant δ^*, $\|x - x_0\| < \delta^*$ such that for the above $\widetilde{\varepsilon} > 0$,

$$|z_{m_0}^{(\varrho,x)}(\varrho) - z_{m_0}^{(\varrho,x_0)}(\varrho)| < \widetilde{\varepsilon}, \ \ \varrho \in \mathbb{R}.$$

Hence, for $|x - x_0| < \delta^*$,

$$|z^{(\varrho,x)}(\varrho) - z^{(\varrho,x_0)}(\varrho)| \leq |z^{(\varrho,x)}(\varrho) - z_{m_0}^{(\varrho,x)}(\varrho)| + |z_{m_0}^{(\varrho,x)}(\varrho) - z_{m_0}^{(\varrho,x_0)}(\varrho)| + |z_{m_0}^{(\varrho,x_0)}(\varrho) - z^{(\varrho,x_0)}(\varrho)| < 3\widetilde{\varepsilon}.$$

Therefore, for $x \to x_0$, $z^{(\varrho,x)}(\varrho)$ uniformly converges to $z^{(\varrho,x_0)}(\varrho)$ on $\mathbb{R}$. This completes the proof of Lemma 3.3.

Proof of Theorem 2.1. For any g in $\mathcal{S}$, it suffices to prove that

$$\dot{x}(t) = A(t)x + f_1(t, x). \tag{3.6}$$

is topologically conjugated to

$$\dot{x}(t) = A(t)x + f_2(t, x). \tag{3.7}$$

For any $\varrho \in \mathbb{R}$, $x \in \mathbb{R}^n$, let $y(t, \varrho, x)$ be a solution of system (2.2) and $y(\varrho, \varrho, x) = x$. From Lemma 3.3, the differential function (3.4) has a unique bounded solution $z^{(\varrho,x)}(t)$ satisfying (3.5). For $x \to x_0 \in \mathbb{R}$, $z^{(\varrho,x)}(\varrho) \to z^{(\varrho,x_0)}(\varrho)$ uniformly with respect to ϱ. Now we take

$$H(\varrho, x) = x + z^{(\varrho,x)}(\varrho).$$

Then by a similar argument as in [9] or [25, 26], it is not difficult to prove the conjugacy between system (3.6) and (3.7).

4. Conclusions

This paper provides a nonuniform version of the theorem on the structural stability of nonlinear systems. We show that if the linear system $\dot{x}(t) = A(t)x(t)$ admits a nonuniform exponential dichotomy, then the perturbed nonautonomous system $\dot{x}(t) = A(t)x(t) + f(t, x)$ is structurally stable under suitable conditions.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant (No. 11671176 and No. 11931016), Natural Science Foundation of Zhejiang Province under Grant (No. LY20A010016).

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

Yonghui Xia conceived of the study, outlined the proof, proposed the project, drafted the manuscript. Yuzhen Bai participated in the discussion, smooth the English, made the corrections and proofread the final version. Xiaoqing Yuan carried out some part of computations in the proof. Donal O'Regan participated in the discussion and help to smooth the manuscript. All authors read and approved the final manuscript.

References

1. L. Barreira, Ya. Pesin, *Nonuniform hyperbolicity*, Encyclopedia of Mathematics and Its Applications, 115, Cambridge University Press, 2007.
2. L. Barreira, C. Valls, *Stability of Nonautonomous Differential Equations*,Lecture Notes in Mathematics, vol. 1926, Springer, 2008.
3. J. Chu, *Robustness of nonuniform behavior for discrete dynamics*, Bull. Sci. Math., **137** (2013), 1031–1047.
4. J. Chu, F. Liao, S. Siegmund, et al. *Nonuniform dichotomy spectrum and reducibility for nonautonomous equations*, Bull. Sci. Math., **139** (2015), 538–557.
5. J. Zhang, M. Fan, H. Zhu, *Nonuniform (h,k,u,v)-dichotomy with applications to nonautonomous dynamical systems*, J. Math. Anal. Appl., **452** (2017), 505–551.
6. J. Zhang, M. Fan, H. Zhu, *Existence and roughness of exponential dichotomies of linear dynamic equations on time scales*, Comput. Math. Appl., **59** (2010), 2658–2675.
7. O. Perron, *Die stabilitsfrage bei differentialgleichungen*, Math. Z., **32** (1930), 703–728.
8. W. A. Coppel, *Dichotomies in stability theory*, Lecture Notes in Mathematics, Springer, Berlin, Germany, 1978.
9. Y. Gao, Y. Xia, X.Yuan, et al. *Linearization of nonautonomous impulsive system with nonuniform exponential dichotomy*, Abstr. Appl. Anal., Volume 2014, Article ID 860378, 7 pages.
10. J. Hong, R. Yuan, Z. J. Jing, *Exponential dichotomies, almost periodic structurally stable differential systems, and an example*, J. Math. Anal. Appl., **208** (1997), 71–84.
11. L. Jiang, *Generalized exponential dichotomy and global linearization*, J. Math. Anal. Appl., **315** (2006), 474–490.
12. S. G. Kryzhevich, V. A. Pliss, *Structural stability of nonautonomous systems*, Differential Equations, **39** (2003), 1395–1403.
13. F. Lin, *Exponential dichotomies*, Anhui University Press, Hefei, China, 1999 (in Chinese).
14. F. Lin, *Almost periodic structural stability of almost periodic differential equations*, Ann. Diff. Equat., **5** (1989), 35–50.
15. K. Lu, *Structural stability for scalar parabolic equations*, J. Differential Equations, **114** (1994), 253–271.
16. L. Markus, *Structurally stable differential systems*, Ann. Math., **73** (1961), 1–19.
17. K. J. Palmer, *The structurally stable linear systems on the half-line are those with exponential dichotomies*, J. Differential Equations, **33** (1979), 16–25.
18. J. Kurzweil, G. Papaschinopoulos, *Structural stability of linear discrete systems via the exponential dichotomy*, Czech. Math. J., **38** (1988), 280–284.
19. G. Papaschinopoulos, G. Schinas, *Structural stability via the density of a class of linear discrete systems*, J. Math. Anal. Appl., **127** (1987), 530–539.
20. J. Wang, M. Fěckan, Y. Zhou, *Center stable manifold for planar fractional damped equations*, Appl. Math. Comput., **296** (2017), 257–269.

21. J. Wang, M. Li, M. Fěckan, *Robustness for linear evolution equations with non-instantaneous impulsive effects*, Bull. Sci. Math., **159** (2020), 102827.

22. L. Popescu, *Topological classification and structural stability of strongly continuous groups*, Integr. Equ. Oper. Theory, **79** (2014), 355–375.

23. J. Shi, J. Zhang, *The Principle of Classification for Differential Equations*, Science Press, Beijing, 2003 (in Chinese).

24. J. C. Willems, *Topological classification and structural stability of linear systems*, J. Differential Equations, **35** (1980), 306–318.

25. Y. Xia, X. Chen, V. G. Romanovski, *On the linearization theorem of Fenner and Pinto*, J. Math. Anal. Appl., **400** (2013), 439–451.

26. Y. Xia, R. Wang, K. Kou, et al. *On the linearization theorem for nonautonomous differential equations*, Bull. Sci. Math., **139** (2015), 820–846.

27. Y. Xia, Y. Bai, D. O'Regan, *A new method to prove the nonuniform dichotomy spectrum theorem in R^n*, Proc. Amer. Math. Soc., **149** (2019), 3905–3917.

28. C. Zou, Y. Xia, M. pinto, et al. *Boundness and Linearisation of a class of differential equations with piecewise constant argument*, Qual. Theor. Dyna. Syst., **18** (2019), 495–531.

29. C. Zou, Y. Xia, M. Pinto, *Hölder Continuity of Topological Equivalence Functions of DEPCAGs*, Sci. China Math., **50** (2020), 847–872.

30. A. Algaba, N. Fuentes, E. Gamero, et al. *Structural stability of planar quasi-homogeneous vector fields*, J. Math. Anal. Appl., **468** (2018), 212–226.

31. X. Jarque, J. Llibre, *Structural stability of planar Hamiltonian polynomial vector fields*, Proc. Lond. Math. Soc., **68** (1994), 617–640.

32. J. Llibre, J. P. del Río, J. A. Rodríguez, *Structural stability of planar homogeneous polynomial vector fields: Applications to critical points and to infinity*, J. Differential Equations, **125** (1996), 490–520.

33. J. Llibre, del Río, J. A. Rodríguez, *Structural stability of planar semi-homogeneous polynomial vector fields: Applications to critical points and to infinity*, Discrete Contin. Dyn. Syst., **6** (2000), 809–828.

34. D. S. Shafer, *Structure and stability of gradient polynomial vector fields*, J. Lond. Math. Soc., **41** (1990), 109–121.

35. G. Wang, Y. L. Cao, *Dynamical spectrum in random dynamical systems*, J. Dynamic. Diff. Equat., **26** (2014), 1–20.

36. R. Oliveira, Y. Zhao, *Structural stability of planar quasi-homogeneous vector fields*, Qual. Theory Dyn. Syst., **13** (2014), 39–72.