Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 5495–5509.
DOI: 10.3934/math.2020352
Received: 05 February 2020
Accepted: 18 May 2020
Published: 28 June 2020

Research article

Perturbed trapezoid inequalities for n th order differentiable convex functions and their applications

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Abstract: In this study, we introduce a new general identity for n th order differentiable functions. Also, we establish some new inequalities regarding general perturbed trapezoid inequality for the functions whose the absolute values of n th derivatives are convex. Finally, some applications for special means are provided.

Keywords: convex function; perturbed trapezoid inequalities
Mathematics Subject Classification: 39B62, 52A41

1. Introduction

Many researchers present a large number of studies to improve and generalize classical Hermite-Hadamard inequality. This double inequality suggests that the mean value of a continuous convex function $g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ lies between the value of g in the midpoint of the interval $[a, b]$ and the arithmetic mean of the values of g at the endpoints of this interval such that

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(x)\,dx \leq \frac{g(a)+g(b)}{2}. \tag{1.1}$$

In addition, each side of the mentioned inequality characterizes convexity in the sense that a real-valued continuous function g defined on an interval I is convex if its restriction to each compact subinterval $[a, b] \subset I$ hold both inequalities. If g is a concave function, then the inequality is interchanged 1.2 [3, 23]. In the literature, Hermite-Hadamard inequality is frequently preferred because of its importance in nonlinear analysis. Recently, Jain et al. [10] established some new inequalities related to Hermite-Hadamard inequality for the functions whose absolute values of second derivatives are $log-$convex. Mehrez and Agarwal [13] introduced new Hermit-Hadamard type integral inequalities for convex functions. Mo-hammed [16] presented some new Hermite-Hadamard inequalities for $MT-$convex functions. Besides, some new integral inequalities for the logarithmically $p-$preinvex functions via generalized beta function were established by Mohammed [17].

Many authors introduced a large number of studies to generalize various integral inequalities. Cerone et al. [4] presented the generalized trapezoid inequality. Ujević [27] derived some new perturbations of the trapezoid inequality. Drogomir et al. [6] improved quasi-trapezoid quadrature formula by using some well-known classical inequalities. Cerone [5] obtained explicit bounds for perturbed trapezoidal rules. Liu and Park [11] suggested some perturbed versions of generalized trapezoid inequality. On the other hand, some integral inequalities for the convex functions have been frequently investigated by many researchers. Sarikaya and Aktan [24] intoduced the generalization of some integral inequalities for convex functions. Tunç and Şanal [26] established some perturbed trapezoid inequalities for twice differentiable convex, $s-$convex and $tgs-$convex functions. Ardıc [1] presented some integral inequalities such as Hölder, Hermite-Hadamard and Jensen integral inequality for n times differentiable convex functions.

The studies in recent years have focused on the fractional integral inequalities for convex functions. Agarwal et al. [2] established some fractional integral inequalities via new Pólya-Szegö type integral inequalities. Fernandez and Mohammed [9] used the fractional integrals to obtain the Hermite-Hadamard inequality and related results. Mohammed [15] introduced some new integral inequalities by using the (k,h),(k,s) -Riemann Liouville fractional integrals. The author presented new Hermite-Hadamard's type inequalities for Riemann Liouville fractional integrals of convex function [18]. Besides, Hermite-Hadamard's type inequalities have been obtained via the fractional integrals for different type convex functions [14–28].

The main aim of the present study is to obtain some new inequalities related to general perturbed trapezoid inequality. The considered classes of functions consist of the functions whose n th derivatives of absolute values are convex.

Definition 1. *[14] A function $g : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on I if inequality*

$$g(ta + (1-t)b) \leq tg(a) + (1-t)g(b) \tag{1.2}$$

holds for all $a, b \in I$ and $t \in [0,1]$. We say that g is concave if $(-g)$ is convex. For numerical integration, the trapezoid inequality is introduced as

$$\left| \int_a^b g(x)\,dx - \frac{1}{2}(b-a)(g(a)+g(b)) \right| \leq \frac{1}{12} M_2 (b-a)^3 \tag{1.3}$$

where $g : [a,b] \to \mathbb{R}$ is supposed to be twice differentiable on the interval (a,b), with the second derivative bounded on (a,b) by $M_2 = \sup_{x\in(a,b)} |g''(x)| < +\infty$

([5–24]).

Theorem 1. ***Grüss inequality :*** *Let g and z to be two functions defined and integrable on $[a,b]$. If $k \leq g(x) \leq l$ and $m \leq z(x) \leq n$ is to be $\forall x \in [a,b]$ and for constants $k,l,m,n \in \mathbb{R}$, then*

$$\left| \frac{1}{b-a} \int_a^b g(x) z(x)\,dx - \frac{1}{b-a} \int_a^b g(x)\,dx \frac{1}{b-a} \int_a^b z(x)\,dx \right| \leq \frac{1}{4}(l-k)(n-m). \tag{1.4}$$

The above inequality is held where $\frac{1}{4}$ is the best constant [7]. For the perturbed trapezoid inequality, the inequality obtained by the application of the Grüss inequality is given as

$$\begin{aligned}&\left|\int_a^b g(x)\,dx-\frac{1}{2}(b-a)(g(a)+g(b))+\frac{1}{12}(b-a)^2\left(g'(b)-g'(a)\right)\right| \\ \leq\;& \frac{1}{32}(\Gamma_2-\gamma_2)(b-a)^3\end{aligned} \tag{1.5}$$

by Dragomir et al. [6] where g is supposed to be twice differentiable on the interval (a,b) *with the second derivative bounded on* (a,b) *by* $\Gamma_2=\sup_{x\in(a,b)}g''(x)<+\infty$ *and* $\gamma_2=\inf_{x\in(a,b)}g''(x)>-\infty$.

In the light of this information, we establish some inequalities for n th order differentiable convex functions.

Lemma 1. *[26] Let* $g: I\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I°, $a,b\in I^\circ$ *with* $a<b$. *If* $g''\in L[a,b]$, *then one obtains*

$$\begin{aligned}&\int_a^b g(x)\,dx-\frac{1}{2}(b-a)(g(a)+g(b))+\frac{5}{4}(b-a)^2\left(g'(b)-g'(a)\right) \\ =\;& \frac{(b-a)^3}{4}\int_0^1 (t+1)^2\left[g''(ta+(1-t)b)+g''(tb+(1-t)a)\right]dt.\end{aligned} \tag{1.6}$$

Theorem 2. *[12]* ***Minkowski Inequality:*** *Let* g^p, z^p *and* $(g+z)^p$ *be integrable functions on* $[a,b]$. *If* $p>1$, *then*

$$\left[\int_a^b |g(x)+z(x)|^p\,dx\right]^{\frac{1}{p}}\leq\left[\int_a^b |g(x)|^p\,dx\right]^{\frac{1}{p}}+\left[\int_a^b |z(x)|^p\,dx\right]^{\frac{1}{p}}. \tag{1.7}$$

Similarly, if $p>1$ *and* a_k,$b_k>0$, *then Minkowski sum inequality is expressed as*

$$\left[\sum_{k=1}^n |a_k+b_k|^p\right]^{\frac{1}{p}}\leq\left[\sum_{k=1}^n |a_k|^p\right]^{\frac{1}{p}}+\left[\sum_{k=1}^n |b_k|^p\right]^{\frac{1}{p}}. \tag{1.8}$$

If the sequences a_1,a_2, ... *and* b_1,b_2, ... *are proportional, the inequality is provided. We will use the following notations and conventions throughout this article. Let us consider as* $I=[0,\infty)\subset\mathbb{R}=(-\infty,+\infty)$ *and* $a,b\in I$ *with* $0<a<b$ *and* $g^{(n)}\in L[a,b]$ *and*

$$\begin{aligned}A(a,b) &= \frac{a+b}{2},\ G(a,b)=\sqrt{ab},\ H(a,b)=\frac{2ab}{a+b},\\ L(a,b) &= \frac{b-a}{\ln b-\ln a},\ a\neq b,\ L_p(a,b),\\ L_p &= L_p(a,b)=\begin{cases} a & ,\ a=b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & ,\ a\neq b\end{cases}\quad a,b\geq 0\end{aligned}$$

are the arithmetic mean, geometric mean, harmonic mean, logarithmic mean, generalized p-logarithmic mean for $a,b>0$, *respectively [8].*

In this study, we introduce some results related to the perturbed trapezoid inequality and prove some applications for special means of real numbers.

2. Main results

Lemma 2. *Let $g: I^{\circ} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be n times differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ where n is even number. If $g^{(n)} \in L[a, b]$, then the following equality is obtained:*

$$\begin{aligned} & \frac{1}{b-a} \int_{b}^{a} g(x)\, dx - \frac{g(a)+g(b)}{2} + \cdots \\ & + \frac{-(b-a)^{n-4}\left[n.(n-1)(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n} \\ & \times \left(g^{(n-4)}(a) + g^{(n-4)}(b)\right) \\ & + \frac{(b-a)^{n-3}\left[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 4.a_2\right]}{2.n!.a_n} \\ & \times \left[g^{(n-3)}(b) - g^{(n-3)}(a)\right] \\ & - \frac{(b-a)^{n-2}\left[n.a_n + \cdots + 2.a_2\right]}{2.n!.a_n} \times \left[g^{(n-2)}(a) + g^{(n-2)}(b)\right] \\ & + \frac{(b-a)^{n-1}\left[a_n + \cdots + a_1 + 2a_0\right]}{2.n!.a_n} \times \left[g^{(n-1)}(b) - g^{(n-1)}(a)\right] \\ = & \frac{(b-a)^n}{2.n!.a_n} \\ & \times \int_0^1 \left(a_n t^n + \cdots + a_1 t + a_0\right)\left[g^{(n)}(ta + (1-t)b) + g^{(n)}(tb + (1-t)a)\right] dt \end{aligned} \tag{2.1}$$

Proof. If the right-hand side of the equality is considered and the integration by parts is applied, then one obtains

$$\begin{aligned} I_1 = & \int_0^1 \left(a_n t^n + \cdots + a_1 t + a_0\right) g^{(n)}(ta + (1-t)b)\, dt \\ = & \left(a_n t^n + \cdots + a_1 t + a_0\right) \frac{g^{(n-1)}(ta + (1-t)b)}{a-b}\Bigg|_0^1 \\ & - \frac{1}{(a-b)} \int_0^1 \left(n.a_n t^{n-1} + \cdots + 2a_2 t + a_1\right) g^{(n-1)}(ta + (1-t)b)\, dt \\ = & \frac{(a_n + \cdots + a_1 + a_0)}{a-b} g^{(n-1)}(a) - \frac{a_0}{a-b} g^{(n-1)}(b) \\ & - \frac{1}{(a-b)} \int_0^1 \left(n.a_n t^{n-1} + \cdots + 2a_2 t + a_1\right) g^{(n-1)}(ta + (1-t)b)\, dt \\ = & \frac{(a_n + \cdots + a_1 + a_0)}{a-b} g^{(n-1)}(a) - \frac{a_0}{a-b} g^{(n-1)}(b) \end{aligned}$$

$$
\begin{aligned}
& -\frac{n.a_n + \cdots + 2a_2 + a_1}{(a-b)^2} g^{(n-2)}(a) + \frac{a_1}{(a-b)^2} g^{(n-2)}(b) \\
& + \frac{1}{(a-b)^2} \times \int_0^1 \left(n.(n-1).a_n t^{n-2} + \cdots + 3.2.a_3 t + 2.a_2 \right) \times g^{(n-2)}(ta + (1-t)b)\, dt \\
= \; & \frac{(a_n + \cdots + a_1 + a_0)}{a-b} g^{(n-1)}(a) - \frac{a_0}{a-b} g^{(n-1)}(b) \\
& -\frac{n.a_n + \cdots + 2a_2 + a_1}{(a-b)^2} g^{(n-2)}(a) + \frac{a_1}{(a-b)^2} g^{(n-2)}(b) \\
& + \frac{n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 2.a_2}{(a-b)^3} g^{(n-3)}(a) - \frac{2.a_2}{(a-b)^3} g^{(n-3)}(b) \\
& - \frac{n.(n-1).(n-2).a_n + \cdots + 3.2.1.a_3}{(a-b)^4} g^{(n-4)}(a) + \frac{3.2.a_3}{(a-b)^4} g^{(n-4)}(b) \\
& + \cdots - \frac{n!.a_n + (n-1)!.a_{n-1}}{(a-b)^n} g(a) + \frac{(n-1)!.a_{n-1}}{(a-b)^n} g(b) \\
& + \frac{n! a_n}{(a-b)^n} \int_0^1 g(ta + (1-t)b)\, dt
\end{aligned}
$$

$$
\begin{aligned}
I_2 = \; & \int_0^1 (a_n t^n + \cdots + a_1 t + a_0)\, g^{(n)}(tb + (1-t)a)\, dt \\
= \; & (a_n t^n + \cdots + a_1 t + a_0) \left. \frac{g^{(n-1)}(tb + (1-t)a)}{b-a} \right|_0^1 \\
& - \frac{1}{(b-a)} \int_0^1 \left(n.a_n t^{n-1} + \cdots + 2a_2 t + a_1 \right) g^{(n-1)}(tb + (1-t)a)\, dt \\
= \; & \frac{(a_n + \cdots + a_1 + a_0)}{b-a} g^{(n-1)}(b) - \frac{a_0}{b-a} g^{(n-1)}(a) \\
& - \frac{1}{(b-a)} \int_0^1 \left(n.a_n t^{n-1} + \cdots + 2a_2 t + a_1 \right) g^{(n-1)}(tb + (1-t)a)\, dt \\
= \; & \frac{(a_n + \cdots + a_1 + a_0)}{b-a} g^{(n-1)}(b) - \frac{a_0}{b-a} g^{(n-1)}(a) \\
& - \frac{n.a_n + \cdots + 2a_2 + a_1}{(b-a)^2} g^{(n-2)}(b) + \frac{a_1}{(b-a)^2} g^{(n-2)}(a) \\
& + \frac{1}{(b-a)^2} \int_0^1 \left[n.(n-1) a_n t^{n-2} + \cdots + 3.2.a_3 t + 2.a_2 \right] \times g^{(n-2)}(tb + (1-t)a)\, dt \\
= \; & \frac{(a_n + \cdots + a_1 + a_0)}{b-a} g^{(n-1)}(b) - \frac{a_0}{b-a} g^{(n-1)}(a) \\
& - \frac{n.a_n + \cdots + 2a_2 + a_1}{(b-a)^2} g^{(n-2)}(b) + \frac{a_1}{(b-a)^2} g^{(n-2)}(a) \\
& + \frac{n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 2.a_2}{(b-a)^3} g^{(n-3)}(b) - \frac{2.a_2}{(b-a)^3} g^{(n-3)}(a)
\end{aligned}
$$

$$-\frac{n.(n-1).(n-2).a_n+\cdots+3.2.a_3}{(b-a)^4}g^{(n-4)}(b)+\frac{3.2.a_3}{(b-a)^4}g^{(n-4)}(a)$$
$$+\cdots-\frac{n!.a_n+(n-1)!.a_{n-1}}{(b-a)^n}g(b)+\frac{(n-1)!.a_{n-1}}{(b-a)^n}g(a)$$
$$+\frac{n!a_n}{(b-a)^n}\int_0^1 g(tb+(1-t)a)\,dt$$

Summing I_1 and I_2, then one obtains

$$\begin{aligned}
I_1+I_2 &= \int_0^1 (a_nt^n+\cdots+a_1t+a_0)\left[g^{(n)}(ta+(1-t)b)+g^{(n)}(tb+(1-t)a)\right]dt \\
&= \frac{(a_n+\cdots+a_1+a_0)}{b-a}\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right]+\frac{a_0}{b-a}\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right] \\
&\quad -\frac{(n.a_n+\cdots+2a_2+a_1)}{(b-a)^2}\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right]+\frac{a_1}{(b-a)^2}\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right] \\
&\quad +\frac{(n.(n-1).a_n+\cdots+4.3.a_4+3.2a_3+2.a_2)}{(b-a)^3}\left[g^{(n-3)}(b)-g^{(n-3)}(a)\right] \\
&\quad +\frac{2.a_2}{(b-a)^3}\left[g^{(n-3)}(b)-g^{(n-3)}(a)\right] \\
&\quad -\frac{(n.(n-1).(n-2)a_n+\cdots+3.2a_3)}{(b-a)^4}\left[g^{(n-4)}(a)+g^{(n-4)}(b)\right] \\
&\quad +\frac{3.2.a_3}{(b-a)^4}\left[g^{(n-4)}(a)+g^{(n-4)}(b)\right]+\cdots \\
&\quad -\frac{n!.a_n+(n-1)!.a_{n-1}}{(b-a)^n}\left[g(a)+g(b)\right]+\frac{(n-1)!.a_{n-1}}{(b-a)^n}\left[g(a)+g(b)\right] \\
&\quad +\frac{(n)!.a_n}{(b-a)^n}\left[\int_0^1\left[g^{(n)}(ta+(1-t)b)+g^{(n)}(tb+(1-t)a)\right]dt\right] \\
&= \frac{a_n+\cdots+a_1+2a_0}{b-a}\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right] \\
&\quad -\frac{n.a_n+\cdots+2a_2}{(b-a)^2}\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right] \\
&\quad +\frac{n.(n-1)a_n+\cdots+4.3.a_4+3.2.a_3+4a_2}{(b-a)^3}\left[g^{(n-3)}(b)-g^{(n-3)}(a)\right] \\
&\quad -\frac{n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4}{(b-a)^4}\left[g^{(n-4)}(a)+g^{(n-4)}(b)\right] \\
&\quad +\cdots-\frac{n!.a_n}{(b-a)^n}\left[g(a)+g(b)\right]+\frac{2.n!.a_n}{(b-a)^{n+1}}\int_a^b g(x)\,dx
\end{aligned}$$

so

$$\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{g(a)+g(b)}{2}+\cdots$$
$$+\frac{-(b-a)^{n-4}\left[n.(n-1)(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[g^{(n-4)}(a)+g^{(n-4)}(b)\right]$$

$$
\begin{aligned}
&+\frac{(b-a)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4.a_2\right]}{2.n!.a_n}\left[g^{(n-3)}(b)-g^{(n-3)}(a)\right]\\
&-\frac{(b-a)^{n-2}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right]\\
&+\frac{(b-a)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right]\\
=\ &\frac{(b-a)^n}{2.n!.a_n}\int_0^1\left(a_nt^n+\cdots+a_1t+a_0\right)\times\left[g^{(n)}(ta+(1-t)b)+g^{(n)}(tb+(1-t)a)\right]dt
\end{aligned}
$$

Thus, the proof is completed. □

Remark 1. *Using the change of the variable $x=ta+(1-t)b$ where $t\in[0,1]$, Eq.(2.1) can be written as*

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{g(a)+g(b)}{2}+\cdots\\
&+\frac{-(b-a)^{n-4}\left[n.(n-1)(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\times\left[g^{(n-4)}(a)+g^{(n-4)}(b)\right]\\
&+\frac{(b-a)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}\times\left[g^{(n-3)}(b)-g^{(n-3)}(a)\right]\\
&-\frac{(b-a)^{n-2}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\times\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right]\\
&+\frac{(b-a)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\times\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right]\\
=\ &\frac{-(b-a)^{n-1}}{2.n!.a_n}\times\int_0^1\left(a_n\left(\frac{x-b}{a-b}\right)^n+\cdots+a_1\left(\frac{x-b}{a-b}\right)+a_0\right)\left[g^{(n)}(x)+g^{(n)}(a+b-x)\right]dx
\end{aligned}
$$

Theorem 3. *Let $g:I\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be n times differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$ where n is even number. If $\left|g^{(n)}\right|$ is convex on $[a,b]$, then the inequality in the following holds:*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{g(a)+g(b)}{2}+\cdots\right. \qquad (2.2)\\
&+\frac{-(b-a)^{n-4}\left[n.(n-1)(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\\
&\times\left[g^{(n-4)}(a)+g^{(n-4)}(b)\right]\\
&+\frac{(b-a)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+2.a_2\right]}{2.n!.a_n}\\
&\times\left[g^{(n-3)}(b)-g^{(n-3)}(a)\right]\\
&-\frac{(b-a)^{n-2}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\times\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right]\\
&\left.+\frac{(b-a)^{n-1}\left[a_n+\cdots+a_1+2.a_0\right]}{2.n!.a_n}\times\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right]\right|
\end{aligned}
$$

$$\leq \frac{(b-a)^n}{2.n!.\left|a_n\right|} \times \left[\sum_{k=0}^{n} \frac{\left|a_k\right|}{k+1}\right] \times \left[\left|g^{(n)}(a)\right| + \left|g^{(n)}(b)\right|\right]$$

Proof. From Lemma 2, it is concluded that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(x)\,dx - \frac{g(a)+g(b)}{2} + \cdots\right. \\
&+\frac{-(b-a)^{n-4}\left[n.(n-1)(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n} \times \left[g^{(n-4)}(a) + g^{(n-4)}(b)\right] \\
&+\frac{(b-a)^{n-3}\left[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 2.a_2\right]}{2.n!.a_n} \times \left[g^{(n-3)}(b) - g^{(n-3)}(a)\right] \\
&-\frac{(b-a)^{n-2}\left[n.a_n + \cdots + 2.a_2\right]}{2.n!.a_n} \times \left[g^{(n-2)}(a) + g^{(n-2)}(b)\right] \\
&\left.+\frac{(b-a)^{n-1}\left[a_n + ... + a_1 + 2a_0\right]}{2.n!.a_n} \times \left[g^{(n-1)}(b) - g^{(n-1)}(a)\right]\right| \\
= &\left|\frac{(b-a)^n}{2.n!.a_n} \times \left\{\int_0^1 \left(a_n t^n + \cdots + a_1 t + a_0\right)\left[g^{(n)}(ta+(1-t)b) + g^{(n)}(tb+(1-t)a)\right]dt\right\}\right| \\
\leq &\frac{(b-a)^n}{2.n!.\left|a_n\right|} \times \left\{\int_0^1 \left|a_n t^n + \cdots + a_1 t + a_0\right| \left[\left|g^{(n)}(ta+(1-t)b)\right| + \left|g^{(n)}(tb+(1-t)a)\right|\right]dt\right\} \\
\leq &\frac{(b-a)^n}{2.n!.\left|a_n\right|} \\
&\times\left\{\int_0^1 \left|a_n t^n + \cdots + a_1 t + a_0\right| \left[t\left|g^{(n)}(a)\right| + (1-t)\left|g^{(n)}(b)\right| + t\left|g^{(n)}(b)\right| + (1-t)\left|g^{(n)}(a)\right|\right]dt\right\} \\
\leq &\frac{(b-a)^n}{2.n!.\left|a_n\right|}\left[\left|g^{(n)}(a)\right| + \left|g^{(n)}(b)\right|\right] \times \int_0^1 \left[\left|a_n t^n\right| + \cdots + \left|a_1 t\right| + \left|a_0\right|\right]dt \\
\leq &\frac{(b-a)^n}{2.n!.\left|a_n\right|} \times \sum_{k=0}^{n} \frac{\left|a_k\right|}{k+1} \times \left[\left|g^{(n)}(a)\right| + \left|g^{(n)}(b)\right|\right]
\end{aligned}$$

The theorem is proved. □

Theorem 4. *Let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be n times differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $p > 1$ with $1/p + 1/q = 1$ where n is even number. If the mapping $\left|g^{(n)}\right|^q$ is convex on $[a, b]$, then we obtain:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(x)\,dx - \frac{g(a)+g(b)}{2} + \cdots\right. \\
&+\frac{-(b-a)^{n-4}\left[n.(n-1)(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n} \\
&\times\left[g^{(n-4)}(a) + g^{(n-4)}(b)\right] \\
&+\frac{(b-a)^{n-3}\left[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 2.a_2\right]}{2.n!.a_n} \\
&\times\left[g^{(n-3)}(b) - g^{(n-3)}(a)\right]
\end{aligned} \tag{2.3}$$

$$-\frac{(b-a)^{n-2}[n.a_n+\cdots+2.a_2]}{2.n!.a_n}\times\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right]$$
$$\left.+\frac{(b-a)^{n-1}[a_n+\cdots+a_1+2.a_0]}{2.n!.a_n}\times\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right]\right|$$
$$\leq\quad\frac{(b-a)^n}{n!.\,|a_n|}\times\left[\sum_{k=0}^{n}\frac{|a_k|}{(kp+1)^{1/p}}\right]\times\left[\frac{\left|g^{(n)}(a)\right|^q+\left|g^{(n)}(b)\right|^q}{2}\right]^{1/q}$$

Proof. Using Lemma 2, Hölder's integral inequality and Minkowsky's integral inequality, we establish

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{g(a)+g(b)}{2}+\cdots\right. \qquad (2.4)\\
&+\frac{-(b-a)^{n-4}[n.(n-1)(n-2).a_n+\cdots+4.3.2.a_4]}{2.n!.a_n}\\
&\times\left[g^{(n-4)}(a)+g^{(n-4)}(b)\right]\\
&+\frac{(b-a)^{n-3}[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+2.a_2]}{2.n!.a_n}\\
&\times\left[g^{(n-3)}(b)-g^{(n-3)}(a)\right]\\
&-\frac{(b-a)^{n-2}[n.a_n+\cdots+2.a_2]}{2.n!.a_n}\times\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right]\\
&\left.+\frac{(b-a)^{n-1}[a_n+\cdots+a_1+2a_0]}{2.n!.a_n}\times\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right]\right|\\
=\;&\left|\frac{(b-a)^n}{2.n!.a_n}\right.\\
&\left.\times\left\{\int_0^1(a_nt^n+\cdots+a_1t+a_0)\left[g^{(n)}(ta+(1-t)b)+g^{(n)}(tb+(1-t)a)\right]dt\right\}\right|\\
\leq\;&\frac{(b-a)^n}{2.n!.\,|a_n|}\\
&\times\left[\begin{array}{c}\int_0^1|a_nt^n+\cdots+a_1t+a_0|\left|g^{(n)}(ta+(1-t)b)\right|dt\\+\int_0^1|a_nt^n+\cdots+a_1t+a_0|\left|g^{(n)}(tb+(1-t)a)\right|dt\end{array}\right]\\
\leq\;&\frac{(b-a)^n}{2.n!.\,|a_n|}\\
&\times\left[\left(\int_0^1|a_nt^n+\cdots+a_1t+a_0|^p\,dt\right)^{\frac{1}{p}}\times\left(\int_0^1\left|g^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1|a_nt^n+\cdots+a_1t+a_0|^p\,dt\right)^{\frac{1}{p}}\times\left(\int_0^1\left|g^{(n)}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right]\\
\leq\;&\frac{(b-a)^n}{n!.\,|a_n|}\times\left[\sum_{k=0}^{n}\frac{|a_k|}{(kp+1)^{1/p}}\right]\times\left[\frac{\left|g^{(n)}(a)\right|^q+\left|g^{(n)}(b)\right|^q}{2}\right]^{1/q}
\end{aligned}$$

such that $\frac{1}{p}+\frac{1}{q}=1$. Considering the convexity of $\left|g^{(n)}\right|^q$, then we find

$$\begin{aligned} & \int_0^1 \left|g^{(n)}(ta+(1-t)b)\right|^q dt \\ \leq & \int_0^1 \left[t\left|g^{(n)}(a)\right|^q+(1-t)\left|g^{(n)}(b)\right|^q\right] dt=\frac{\left|g^{(n)}(a)\right|^q+\left|g^{(n)}(b)\right|^q}{2} \\ & \int_0^1 \left|g^{(n)}(tb+(1-t)a)\right|^q dt \\ \leq & \int_0^1 \left[t\left|g^{(n)}(b)\right|^q+(1-t)\left|g^{(n)}(a)\right|^q\right] dt=\frac{\left|g^{(n)}(a)\right|^q+\left|g^{(n)}(b)\right|^q}{2} \end{aligned} \tag{2.5}$$

Using the Theorem 2, we have

$$\begin{aligned} & \left(\int_0^1 \left|a_n t^n+\cdots+a_1 t+a_0\right|^p dt\right)^{\frac{1}{p}} \\ \leq & \left[\int_0^1 (a_n t^n)^p dt\right]^{\frac{1}{p}}+\cdots+\left[\int_0^1 (a_1 t)^p dt\right]^{\frac{1}{p}}+\left[\int_0^1 (a_0)^p dt\right]^{\frac{1}{p}} \\ = & \frac{|a_n|}{(np+1)^{1/p}}+\frac{|a_{n-1}|}{|(n-1)p+1|^{1/p}}+\cdots+\frac{|a_1|}{(p+1)^{1/p}}+|a_0| \\ = & \left[\sum_{k=0}^{n}\frac{|a_k|}{(kp+1)^{1/p}}\right] \end{aligned} \tag{2.6}$$

By using (2.5) and (2.6), the inequality (2.3) is obtained. □

Theorem 5. *Let $g: I \subseteq \mathbb{R} \to \mathbb{R}$ be n times differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $p > 1$such that $1/p + 1/q = 1$ where n is even number. If the mapping $\left|g^{(n)}\right|^p$ is convex on $[a, b]$, then the inequality in the following holds:*

$$\begin{aligned} & \left|\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{g(a)+g(b)}{2}+\cdots\right. \\ & +\frac{-(b-a)^{n-4}\left[n.(n-1)(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n} \\ & \times\left[g^{(n-4)}(a)+g^{(n-4)}(b)\right] \\ & +\frac{(b-a)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4.a_2\right]}{2.n!.a_n} \\ & \times\left[g^{(n-3)}(b)-g^{(n-3)}(a)\right] \\ & -\frac{(b-a)^{n-2}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\times\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right] \\ & \left.+\frac{(b-a)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\times\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right]\right| \end{aligned} \tag{2.7}$$

$$\leq \frac{(b-a)^n}{2.n!.\left|a_n\right|}\left[\sum_{k=0}^{n}\frac{|a_k|}{k+1}\right]^{1-\frac{1}{p}}$$
$$\times\left\{\left(\left[\sum_{k=0}^{n}\frac{|a_k|}{k+2}\right]\left|g^{(n)}(a)\right|^p+\left[\sum_{k=0}^{n}\frac{|a_k|}{(k+1)(k+2)}\right]\left|g^{(n)}(b)\right|^p\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\left[\sum_{k=0}^{n}\frac{|a_k|}{k+2}\right]\left|g^{(n)}(b)\right|^p+\left[\sum_{k=0}^{n}\frac{|a_k|}{(k+1)(k+2)}\right]\left|g^{(n)}(a)\right|^p\right)^{\frac{1}{p}}\right\}$$

Proof. Using Lemma 2 and power mean integral inequality, one obtains

$$\left|\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{g(a)+g(b)}{2}+\cdots\right.$$
$$+\frac{-(b-a)^{n-4}\left[n.(n-1)(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}$$
$$\times\left[g^{(n-4)}(a)+g^{(n-4)}(b)\right]$$
$$+\frac{(b-a)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4.a_2\right]}{2.n!.a_n}$$
$$\times\left[g^{(n-3)}(b)-g^{(n-3)}(a)\right]$$
$$-\frac{(b-a)^{n-2}\left[n.a_n+\cdots+2.a_2\right]}{2.n!.a_n}\times\left[g^{(n-2)}(a)+g^{(n-2)}(b)\right]$$
$$\left.+\frac{(b-a)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\times\left[g^{(n-1)}(b)-g^{(n-1)}(a)\right]\right|$$
$$=\left|\frac{(b-a)^n}{2.n!.a_n}\right.$$
$$\left.\times\left\{\int_0^1\left(a_nt^n+\cdots+a_1t+a_0\right)\left[g^{(n)}(ta+(1-t)b)+g^{(n)}(tb+(1-t)a)\right]dt\right\}\right|$$
$$\leq\frac{(b-a)^n}{2.n!.\left|a_n\right|}$$
$$\times\left[\begin{array}{c}\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left|g^{(n)}(ta+(1-t)b)\right|dt\\+\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left|g^{(n)}(tb+(1-t)a)\right|dt\end{array}\right]$$
$$\leq\frac{(b-a)^n}{2.n!.\left|a_n\right|}\left(\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|dt\right)^{1-\frac{1}{p}}$$
$$\times\left\{\left(\begin{array}{c}\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\\\left(t\left|g^{(n)}(a)\right|^p+(1-t)\left|g^{(n)}(b)\right|^p\right)dt\end{array}\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\begin{array}{c}\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\\\left(t\left|g^{(n)}(b)\right|^p+(1-t)\left|g^{n}(a)\right|^p\right)dt\end{array}\right)^{\frac{1}{p}}\right\}$$

$$\leq \frac{(b-a)^n}{2.n!.\,|a_n|}.\left[\sum_{k=0}^{n}\frac{|a_k|}{k+1}\right]^{1-\frac{1}{p}}$$
$$\times\left\{\left(\begin{array}{c}\left[\sum_{k=0}^{n}\frac{|a_k|}{k+2}\right]\cdot\left|g^{(n)}(a)\right|^p\\ +\left[\sum_{k=0}^{n}\frac{|a_k|}{(k+1)(k+2)}\right]\cdot\left|g^{(n)}(b)\right|^p\end{array}\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\begin{array}{c}\left[\sum_{k=0}^{n}\frac{|a_k|}{k+2}\right]\cdot\left|g^{(n)}(b)\right|^p\\ +\left[\sum_{k=0}^{n}\frac{|a_k|}{(k+1)(k+2)}\right]\cdot\left|g^{(n)}(a)\right|^p\end{array}\right)^{\frac{1}{p}}\right\}$$

The proof is completed. □

3. Applications to special means

In this section, we consider the results of Section 2 to verify the new proposed inequalities.

Proposition 1. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $n > 2$ where n is even number. Then, the inequality in the following holds:*

$$\begin{aligned}&\Big|L_n^n(a,b) - A(a^n, b^n) + \cdots\\ &+\frac{-(b-a)^{n-4}\,[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4]\left(a^4+b^4\right)}{2.4!.a_n}\\ &+\frac{(b-a)^{n-3}\,[n.(n-1).a_n + \cdots + 3.2.a_3 + 4.a_2]\left(b^3-a^3\right)}{2.3!.a_n}\\ &-\frac{(b-a)^{n-2}\,[n.a_n + \cdots + 2.a_2]\left(a^2+b^2\right)}{2.2!.a_n}\\ &+\frac{(b-a)^{n-1}\,[a_n + \cdots + 2.a_0]\,(b-a)}{2.a_n}\Big|\\ \leq\ &\frac{(b-a)^n}{|a_n|}\times\sum_{k=0}^{n}\frac{|a_k|}{k+1}\end{aligned} \tag{3.1}$$

Proof. The proof is clearly obtained from Theorem 3 for $g(x) = x^n$, $x \in \mathbb{R}$. □

Proposition 2. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $n > 2$ where n is even number. For all $p > 1$, one obtains*

$$\begin{aligned}&\Big|L_n^n(a,b) - A(a^n, b^n) + \cdots\\ &+\frac{-(b-a)^{n-4}\,[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4]\left(a^4+b^4\right)}{2.4!.a_n}\\ &+\frac{(b-a)^{n-3}\,[n.(n-1).a_n + \cdots + 3.2.a_3 + 4.a_2]\left(b^3-a^3\right)}{2.3!.a_n}\\ &-\frac{(b-a)^{n-2}\,[n.a_n + \cdots + 2.a_2]\left(a^2+b^3\right)}{2.2!.a_n}\end{aligned} \tag{3.2}$$

$$+\frac{(b-a)^{n-1}\left[a_n+\cdots+2.a_0\right](b-a)}{2.a_n}\Bigg|$$
$$\leq \quad \frac{(b-a)^n}{|a_n|}\times\sum_{k=0}^{n}\frac{|a_k|}{(kp+1)^{1/p}}$$

Proof. The proof is completed from Theorem 4 applied for $g(x)=x^n$, $x\in\mathbb{R}$. □

Proposition 3. *Let* $a,b\in\mathbb{R}$ *,* $0<a<b,\ \ n>2$ *where* n *is even number. Then, we obtain for all* $p>1$*,*

$$\begin{aligned}
&\Big|L_n^n(a,b)-A(a^n,b^n)+\cdots \\
&+\frac{-(b-a)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]\left(a^4+b^4\right)}{2.4!.a_n} \\
&+\frac{(b-a)^{n-3}\left[n.(n-1).a_n+\cdots++3.2.a_3+4.a_2\right]\left(b^3-a^3\right)}{2.3!.a_n} \\
&-\frac{(b-a)^{n-2}\left[n.a_n+\cdots+2.a_2\right]\left(a^2+b^3\right)}{2.2!.a_n} \\
&+\frac{(b-a)^{n-1}\left[a_n+...+2.a_0\right](b-a)}{2.a_n}\Bigg| \\
\leq\ & \frac{(b-a)^n}{|a_n|}\times\left(\sum_{k=0}^{n}\frac{|a_k|}{k+1}\right)^{1-1/p}\times\left(\sum_{k=0}^{n}\frac{|a_k|}{k+2}+\sum_{k=0}^{n}\frac{|a_k|}{(k+1)(k+2)}\right)^{\frac{1}{p}}
\end{aligned} \tag{3.3}$$

Proof. The proof is obtained from Theorem 5 such that $g(x)=x^n$, $x\in[a,b]$. □

Conflict of interest

The authors declare that there is no conflict of interest.

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