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Research article

Non-lightlike Bertrand W curves: A new approach by system of differential equations for position vector

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Abstract: In this study, the characterization of position vectors belonging to non-lightlike Bertrand W curve mate with constant curvature are obtained depending on differentiable functions. The position vector of Bertrand W curve is stated by a linear combination of its Frenet frame with differentiable functions. There exist also different cases for the curve depending on the value of curvature and torsion. The relationships between Frenet apparatuas of these curves are stated separately for each case. Finally, the timelike and spacelike Bertrand curve mate visualized of given curves as examples, separately.

Keywords: W curve; Bertrand curve; Minkowski space
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1. Introduction

Offset curve is defined as locus of the points which are at a constant distant along the normal from the main curve. Offset curves play an important role in computer aided design and manufacturing (CAD/CAM). For example, offset curves are important in numerically controlled machining, where they describe for example the shape of the cut made by a round cutting piece of a two-axis machine. The shape of the cut is offset from the trajectory of the cutter by a constant distance in the direction normal to the cutter trajectory at every point [22, 15, 2].

Except in the case of a line or circle, the offset curves have a complicated mathematical structure [17]. In general, even if the main curve is rational, its offsets may not be rational [4]. For example, the offsets curves of a parabola are rational curves, but the offset curves of an ellipse or of a hyperbola are not rational [24]. On the other hand, the curves for which there exists constants A

and B, such that

$$A\kappa + B\tau = 1 \tag{1.1}$$

are also known as Bertrand curves. Here κ is the curvature and τ is the torsion of the curve. And it is well known that a curve α admits an offset curve α^* which has the same principal normal as the curve α if and only if α is a Bertrand curve in Euclidean 3-space [23].

In the literature, it is possible to reach many studies dealing with the characterizations of Bertrand curves in different kind of spaces. In some cases, it was also necessary to provide a new definition. In [1], a new type of Bertrand curve in four dimensional Minkowski space E_1^4 (Minkowski space-time) is introduced for a null curve with nonzero third curvature function, which is called $(1, 3)$ Bertrand curve. And differential geometric properties of these newly defined Bertrand curves are investigated. In study [9] , the position vector of a null Cartan curve is stated by a linear combination of its pseudo orthogonal frame with differentiable functions and Bertrand curve mate (timelike, spacelike or null) of null Cartan curves are also examined. All kind of Bertrand mate of a given null cartan curve with constant curvature function have also constant curvature functions. In some studies, Bertrand curves in two different spaces are examined together. For example, the characterizations of Bertrand mate (conjugate or couple) of the curve $\alpha : I \to M_q^3(c) \subset \mathbb{R}_v^4$ are deeply discussed where $q = 0$ or $q = 1$ (Riemann or Lorentzian, respectively) in [14]. Here $M_q^3(c)$ denotes the three dimensional space form of the index q with nonzero constant curvature c. The classical results on Bertrand curves in Minkowski 3-space are stated in [20]. It is useful to express a few of these results.

- A spatial curve is a Bertrand curve in Minkowski 3-space E_1^3 if and only if its curvature and torsion satisfy $a\kappa + b\tau = 1$ for some constants a and b.
- Let $(\gamma, \widetilde{\gamma})$ be a Bertrand mate in Minkowski 3-space E_1^3. Then $\tau(s)\widetilde{\tau}(s)$ is a constant.

Besides this, the relations between Frenet apparatus of γ and $\widetilde{\gamma}$ are given by depending on these constants a and b in [20]. Moreover, a representation formula of a Bertrand curve is obtained by using the curves in pseudo hyperbolic space H^2 in [20].

If the curvature and torsion functions of the curve $\alpha : I \to \mathbb{R}^3$ are different from zero, the curve α is called a twisted curve. If the curvature and torsion functions of the curve $\alpha : I \to \mathbb{R}^3$ are constant, then α is called W curve. The examples of W curves are circles, hyperbolas as planar W curves and helices as non-planar W curves. Some of the studies have examined curves in Euclidean space while others have examined curves in Minkowski space [16, 7, 25]. İlarslan and Arslan in their study, they stated that the Bertrand curve mate can be spacelike, timelike or null curve [11]. Erdoğdu and Yavuz studied the characterization of the position vector of the curve with constant curvatures in the Minkowski space under different conditions and obtained differentiable functions in [8, 27].

Curvature functions of a curve include the answer to many questions about the character of the curve [6]. Therefore, the cases where the curvature functions are constant have always been one of the points of interest when studying the character of the curve. For this reason, studies examining the character of curves with constant curvature are very common in the literature [21, 10, 3].

In [10], it is seen that whatever the values of curvature and torsion, it is possible to express the characterization of W curve in one way in Euclidean 3-space. But uniquely characterization of the W is not possible in Minkowski space. The relationship between curvature and torsion of the curve itself also affects characterization of the curve. Because the solution of the system of differential equation changes depending on the value $\tau^2 - \kappa^2$ or $\tau^2 + \kappa^2$. The sign of the value $\tau^2 - \kappa^2$ or $\tau^2 + \kappa^2$ actually corresponds to whether the ratio τ^2/κ^2 is greater than, equal to, or equal to 1 or -1, respectively [8, 27]. That is, the characterization of the nonlightlike W curve and its Bertrand mate also depend on the constant value of the ratio τ/κ. For this reason, we chose to consider system of differential equations for position vector as a different perspective to examine differential geometric properties of non-lightlike Bertrand W curves in our study.

The aim of this study is to obtain the characterization of the position vectors of non-lightlike Bertrand W curve mate in Minkowski space due to differentiable functions. In accordance with this scope, the position vector of a non-lightlike Bertrand W curve is stated by a linear combination of its Frenet frame with differentiable functions. There exist also different cases for the curve depending on the value of curvature and torsion. The relationships between Frenet apparatus of these curves are presented separately for each case. Finally, the timelike and spacelike Bertrand W curve mate visualized of given curves as examples, separately.

2. Preliminaries

2.1. Basic of non-lightlike curves in Minkowski space

In this section, we will give the necessary informations to understand the main subject of the study. Furthermore, the characterization of a non-lightlike curve $\alpha : I \to \mathbb{E}_1^3$ given by arc length parameter is given in terms of its curvature and torsion functions. At the same time the characterizations of the position vector of twisted spacelike curve with timelike normal vector and the spacelike normal is obtained with the differentiable functions. Similarly, characterizations of the position vector of twisted timelike curve is given with the differentiable functions in the propositions.

The *Minkowski 3-space* is a Cartesian 3-space $\mathbb{R}^3$ equipped with Lorentzian inner product

$$\langle u, v\rangle = -u_1v_1 + u_2v_2 + u_3v_3 \tag{2.1}$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{E}_1^3$.Lorentzian inner product characterizes the elements $u = (u_1, u_2, u_3)$ of $\mathbb{E}_1^3$ as follows:
if $\langle u, u\rangle > 0$ or $u = 0$ then u is called *spacelike*,
if $\langle u, u\rangle < 0$ then u is called *timelike*,
if $\langle u, u\rangle = 0$ and $u \neq 0$ then u is called *lightlike* or *null*.

The norm of $u = (u_1, u_2, u_3) \in \mathbb{E}_1^3$ is

$$\|u\| = \sqrt{|\langle u, u\rangle|}. \tag{2.2}$$

Lorentzian vector product of $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in \mathbb{E}_1^3$ is defined by

$$u \times v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{2.3}$$

For details, see [12, 18, 19, 26].

A curve α in $\mathbb{E}_1^3$ is called *timelike*, *spacelike* or *null* if and only if tangent vector field T of α is timelike, spacelike or null, respectively. Let $\alpha(s)$ be a unit speed curve in $\mathbb{E}_1^3$, *i,e.*, $\langle \alpha', \alpha'\rangle = \varepsilon_1 = \pm 1$.

The constant ε_1 is called the *causal character* of α. A unit speed curve is also called an *arclength parametrized curve.* A unit speed curve α is said to be a *Frenet curve* if $\|\alpha''\| \neq 0$. Every Frenet curve admits a Frenet frame field $\{T, N, B\}$ which is an orthonormal field along α satisfying the *Frenet-Serret equation:*

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\kappa & 0 \\ -\varepsilon_1\kappa & 0 & -\varepsilon_3\tau \\ 0 & \varepsilon_2\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.4}$$

The functions $\kappa \geq 0$ and τ are called the *curvature* and *torsion*, respectively. The vector fields $T = \alpha'$, N and B are called, *tangent, normal* and *binormal* vector fields, respectively. The constants $\varepsilon_2 = \langle N,N\rangle$ and $\varepsilon_3 = \langle B,B\rangle$ are called the *second causal character* and *third causal character* of α, respectively. Note that $\varepsilon_3 = -\varepsilon_1\varepsilon_2$, see [13, 5].

2.2. *Position vector of non-ligtlike W curves*

In this subsection, the position vectors of a non-lightlike W curves are expressed by a linear combination of their Serret Frenet Frame with differentiable functions. Since non-lightlike W curves have different kinds of frames, then we investigate the curves with respect to the Lorentzian causal characters of the frame in following propositions.

Proposition ([27]) Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3_1$ be a twisted spacelike W curve with spacelike normal vector, then the position vector $\alpha(s)$ can be written as linear combinations of their Serret-Frenet vectors as follows

$$\alpha(s) = p_0(s)T(s) + p_1(s)N(s) + p_2(s)B(s) \tag{2.5}$$

with the following differentiable functions depending on values of curvature and torsion.
a) For $\tau^2 - \kappa^2 = f^2 > 0$, we obtain

$$p_0(s) = -\tau c_0 + c_2\kappa\sinh(fs) + c_1\kappa\sinh(fs) + \frac{\tau^2}{f^2}s, \tag{2.6}$$

$$p_1(s) = -c_1 f\sinh(fs) + c_2 f\cosh(fs) + \frac{\kappa}{f^2}, \tag{2.7}$$

$$p_2(s) = \kappa c_0 + c_1\tau\cosh(fs) - c_2\tau\sinh(fs) - \frac{\kappa\tau}{f^2}s. \tag{2.8}$$

b) Differentiable functions are given for $\tau^2 - \kappa^2 = -g^2 < 0$

$$p_0(s) = -c_0\tau - c_1\kappa\cos(gs) + c_2\kappa\sin(gs) - \frac{\tau^2}{g^2}s, \tag{2.9}$$

$$p_1(s) = c_1 g\sin(gs) + c_2 g\cos(gs) - \frac{\kappa}{g^2}, \tag{2.10}$$

$$p_2(s) = c_0\kappa + c_1\tau\cos(gs) - c_2\tau\sin(gs) + \frac{\kappa\tau}{g^2}s. \tag{2.11}$$

c) For $\tau^2 - \kappa^2 = 0$, the differentiable functions are examined in two separated cases $\kappa = \tau$ and $\kappa = -\tau$.
i) For $\kappa = \tau$, the differentiable functions are obtained as follows

$$p_0(s) = -c_0\left(\frac{\kappa^2}{2}s^2 - 1\right) + c_1\kappa s - c_2\frac{\kappa^2}{2}s^2 - \frac{\kappa^2}{6}s^3 + s, \tag{2.12}$$

$$p_1(s) = -c_0\kappa s + c_1 - c_2\kappa s - \frac{\kappa}{2}s^2, \tag{2.13}$$

$$p_2(s) = \frac{1}{2}c_0\kappa^2 s^2 - c_1\kappa s + c_2(\frac{\kappa^2}{2}s^2 + 1) + \frac{\kappa^2}{6}s^3, \tag{2.14}$$

ii) For $\kappa = -\tau$, the differentiable functions are given by

$$p_0(s) = -c_0(\frac{\kappa^2}{2}s^2 - 1) + c_1\kappa s + c_2\frac{\kappa^2}{2}s^2 - \frac{\kappa^2}{6}s^3 + s, \tag{2.15}$$

$$p_1(s) = -c_0\kappa s + c_1 + c_2\kappa s - \frac{\kappa}{2}s^2, \tag{2.16}$$

$$p_2(s) = -c_0\frac{\kappa^2}{2}s^2 + c_1\kappa s + c_2(\frac{1}{2}s^2\kappa^2 + 1) - \frac{\kappa^2}{6}s^3, \tag{2.17}$$

where c_i are arbitrary constants for $0 \leq i \leq 2$.

Proposition ([27]) Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a twisted spacelike W curve with timelike normal vector, then the position vector $\alpha(s)$ is obtained with the following differentiable functions:

$$q_0(s) = -c_0\tau + c_1\kappa\cosh(ts) + c_2\kappa\sinh(ts) + \frac{\tau^2}{t^2}s, \tag{2.18}$$

$$q_1(s) = -c_1 t\sinh(ts) - c_2 t\cosh(ts) + \frac{\kappa}{t^2}, \tag{2.19}$$

$$q_2(s) = \kappa c_0 + c_1\tau\cosh(ts) + c_2\tau\sinh(ts) - \frac{\kappa\tau}{t^2}s, \tag{2.20}$$

where $\tau^2 + \kappa^2 = t^2 > 0$ and c_i are arbitrary constants for $0 \leq i \leq 2$.

Proposition ([8]) Let $\alpha : I \to \mathbb{E}_1^3$ be a timelike twisted W curve given by arc length parameter as

$$\alpha(s) = r_0(s)T(s) + r_1(s)N(s) + r_2(s)B(s) \tag{2.21}$$

where $r_0(s), r_1(s), r_2(s)$ differentiable functions depending on values of curvature and torsion as following.

a) If $\kappa^2 - \tau^2 = b^2 > 0$, then the position vector $\alpha(s)$ is stated with the following differentiable functions

$$r_0(s) = c_0\tau + c_1\kappa\cosh(bs) + c_2\kappa\sinh(bs) - \frac{\tau^2}{b^2}s \tag{2.22}$$

$$r_1(s) = -c_2 b\cosh(bs) - c_1 b\sinh(bs) + \frac{\kappa}{b^2}, \tag{2.23}$$

$$r_2(s) = c_0\kappa + c_1\tau\cosh(bs) + c_2\tau\sinh(bs) - \frac{\kappa\tau}{b^2}s, \tag{2.24}$$

where c_i are arbitrary constants for $0 \leq i \leq 2$.

b) If $\kappa^2 - \tau^2 = -a^2 < 0$, then the differentiable functions of $\alpha(s)$ are obtained

$$r_0(s) = c_0\tau + \kappa c_1 \cos(as) + \kappa c_2 \sin(as) - \frac{\tau^2}{a^2}s, \tag{2.25}$$

$$r_1(s) = c_1 a \sin(as) - c_2 a \cos(as) + \frac{\kappa}{a^2}, \tag{2.26}$$

$$r_2(s) = c_0\kappa + c_1\tau \cos(as) + c_2\tau \sin(as) - \frac{\kappa\tau}{a^2}s. \tag{2.27}$$

c) For $\tau^2 - \kappa^2 = 0$, differentiable functions are obtained in two following separated cases $\kappa = \tau$ and $\kappa = -\tau$.
i) If $\tau = \kappa$ then the position vector $\alpha(s)$ is stated with the following differentiable functions

$$r_0(s) = c_0\left(\frac{\kappa^2}{2}s^2 + 1\right) - c_1 s\kappa - c_2\frac{\kappa^2}{2}s^2 + s + \frac{\kappa^2}{6}s^3, \tag{2.28}$$

$$r_1(s) = -c_0\kappa s + c_1 + c_2\kappa s - \frac{\kappa}{2}s^2, \tag{2.29}$$

$$r_2(s) = c_0\frac{\kappa^2}{2}s^2 - c_1\kappa s - c_2\left(\frac{\kappa^2}{2}s^2 - 1\right) + \frac{\kappa^2}{6}s^3. \tag{2.30}$$

ii) If $\tau = -\kappa$ then differentiable functions are given by

$$r_0(s) = c_0\left(\frac{\kappa^2}{2}s^2 + 1\right) - c_1\kappa s + c_2\frac{\kappa^2}{2}s^2 + s + \frac{\kappa^2}{6}s^3 \tag{2.31}$$

$$r_1(s) = -c_0\kappa s + c_1 - c_2\kappa s - \frac{\kappa}{2}s^2, \tag{2.32}$$

$$r_2(s) = -c_0\frac{\kappa^2}{2}s^2 + c_1\kappa s - c_2\left(\frac{\kappa^2}{2}s^2 - 1\right) - \frac{\kappa^2}{6}s^3. \tag{2.33}$$

where c_i are arbitrary constants for $0 \le i \le 2$.

3. Relations between Frenet apparatus of Bertrand W curve mate

In this section, Frenet Frame and curvature torsion of Bertrand W curve mate are obtained in two separated subsections which are formed by causal character of Bertrand W curve couple. In the first subsection, Bertrand W curve mate for spacelike curves with spacelike normal and timelike normal vector are investigated. In second subsection, Bertrand W curve mate for timelike curves are examined in terms of its curvature and torsion functions.

3.1. Relations between Frenet apparatus of spacelike Bertrand W curve mate

Let $\alpha(s)$ be a spacelike curve with spacelike normal vector. Then $T(s)$ is a spacelike unit tangent vector, $N(s)$ is a spacelike unit normal vector and $B(s)$ is a timelike unit binormal vector of α. In this situation, the following three possibilities arise for the Frenet vectors $\widetilde{T}(s)$, $\widetilde{N}(s)$, $\widetilde{B}(s)$ of the curve $\widetilde{\alpha}(s)$.

i) $\widetilde{T}$ spacelike, $\widetilde{N}$ spacelike, $\widetilde{B}$ timelike,
ii) $\widetilde{T}$ timelike, $\widetilde{N}$ spacelike, $\widetilde{B}$ spacelike,
iii) $\widetilde{T}$ null, $\widetilde{N}$ spacelike, $\widetilde{B}$ null.

Theorem Let $\alpha : I \to \mathbb{E}_1^3$ be a unit speed spacelike W curve with spacelike principal normal, then Frenet frame, curvature and torsion of $\widetilde{\alpha} : I \to \mathbb{E}_1^3$ be a Bertrand W curve of α as follows:

$$\widetilde{T}(s) = \frac{(1-\lambda\kappa)}{\sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}} T(s) + \frac{\lambda\tau}{\sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}} B(s), \tag{3.1}$$

$$\widetilde{N}(s) = N(s), \tag{3.2}$$

$$\widetilde{B}(s) = \frac{\lambda\tau}{\sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}} T(s) + \frac{(1-\lambda\kappa)}{\sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}} B(s) \tag{3.3}$$

and

$$\widetilde{\kappa} = \left|\frac{\kappa - \lambda(\kappa^2 - \tau^2)}{(1-\lambda\kappa)^2 - \lambda^2\tau^2}\right|, \quad \widetilde{\tau} = \frac{\tau}{\lambda^2\tau^2 - (1-\lambda\kappa)^2}. \tag{3.4}$$

Proof. Assume that $\widetilde{\alpha} : I \to \mathbb{E}_1^3$ is a Bertrand W curve of α, so we can define $\widetilde{\alpha}$ as

$$\widetilde{\alpha}(s) = \alpha(s) + \lambda(s) N(s). \tag{3.5}$$

Taking derivative of the above equation according to parameter s, we find

$$\widetilde{\alpha}'(s) = T(s)(1 - \lambda(s)\kappa) + \lambda(s)\tau B(s) \tag{3.6}$$

and the norm of the Eq 3.6 is given below

$$\|\widetilde{\alpha}'(\widetilde{s})\| = \sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}. \tag{3.7}$$

Thus, we obtain

$$\widetilde{T}(s) = \frac{(1-\lambda\kappa)}{\sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}} T(s) + \frac{\lambda\tau}{\sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}} B(s). \tag{3.8}$$

We know that

$$\widetilde{N}(s) = N(s). \tag{3.9}$$

Lastly, we can obtain $\widetilde{B}(s)$ as follows:

$$\widetilde{B}(s) = \frac{\widetilde{\alpha}'(\widetilde{s}) \times \widetilde{\alpha}''(\widetilde{s})}{\|\widetilde{\alpha}'(\widetilde{s}) \times \widetilde{\alpha}''(\widetilde{s})\|}. \tag{3.10}$$

Since we have

$$\widetilde{\alpha}'(\widetilde{s}) \times \widetilde{\alpha}''(\widetilde{s}) = \left(\lambda\tau\left(-\kappa + \lambda(\kappa^2 - \tau^2)\right)\right) T(s) + (1-\lambda\kappa)\left(-\kappa + \lambda(\kappa^2 - \tau^2)\right) B(s) \tag{3.11}$$

and

$$\|\widetilde{\alpha}'(\widetilde{s}) \times \widetilde{\alpha}''(\widetilde{s})\| = \left(-\kappa + \lambda(\kappa^2 - \tau^2)\right) \sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}, \tag{3.12}$$

then we obtain

$$\widetilde{B}(s) = \frac{\lambda\tau}{\sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}}T(s) + \frac{(1-\lambda\kappa)}{\sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}}B(s). \tag{3.13}$$

By making some calculations, the curvature of the curve is obtained as follows

$$\begin{aligned}\widetilde{\kappa} &= \frac{\left(\kappa - \lambda(\kappa^2 - \tau^2)\right)\sqrt{\left|\lambda^2\tau^2 - (1-\lambda\kappa)^2\right|}}{\left(\sqrt{(1-\lambda\kappa)^2 - \lambda^2\tau^2}\right)^3} \\ &= \left|\frac{\kappa - \lambda(\kappa^2 - \tau^2)}{(1-\lambda\kappa)^2 - \lambda^2\tau^2}\right| \end{aligned} \tag{3.14}$$

and torsion of the curve is given by

$$\widetilde{\tau} = \frac{\tau}{\lambda^2\tau^2 - (1-\lambda\kappa)^2}. \tag{3.15}$$

□

Theorem Let $\alpha : I \to \mathbb{E}_1^3$ be a unit speed W spacelike curve with timelike principal normal, then the Frenet apparatus of Bertrand W curve mate $\widetilde{\alpha} : I \to \mathbb{E}_1^3$ are given by

$$\widetilde{T}(s) = \frac{(1+\lambda\kappa)}{\sqrt{(1+\lambda\kappa)^2 + \lambda^2\tau^2}}T(s) + \frac{\lambda\tau}{\sqrt{(1+\lambda\kappa)^2 + \lambda^2\tau^2}}B(s), \tag{3.16}$$

$$\widetilde{N}(s) = N(s), \tag{3.17}$$

$$\widetilde{B}(s) = \frac{-\lambda\tau}{\sqrt{\lambda^2\tau^2 + (1+\lambda\kappa)^2}}T(s) + \frac{-(1+\lambda\kappa)}{\sqrt{\lambda^2\tau^2 + (1+\lambda\kappa)^2}}B(s) \tag{3.18}$$

and

$$\widetilde{\kappa} = \frac{\kappa + \lambda(\kappa^2 + \tau^2)}{(1+\lambda\kappa)^2 + \lambda^2\tau^2}, \quad \widetilde{\tau} = \frac{\tau}{\lambda^2\tau^2 + (1+\lambda\kappa)^2}. \tag{3.19}$$

Proof. Let $\widetilde{\alpha}(s)$ be a Bertrand W curve mate of $\alpha(s)$ which is a unit speed spacelike curve with timelike principal normal. Thus following equality is satisfied

$$\widetilde{\alpha}(s) = \alpha(s) + \lambda(s)N(s). \tag{3.20}$$

Taking derivative of the Eq 3.20 according to parameter s, we have

$$\widetilde{\alpha}'(s) = T(s)(1 + \lambda(s)\kappa) + \lambda(s)\tau B(s). \tag{3.21}$$

Therefore, we calculated

$$\|\widetilde{\alpha}'(\widetilde{s})\| = \sqrt{(1+\lambda\kappa)^2 + \lambda^2\tau^2}. \tag{3.22}$$

As a result of some calculations, the tangent vector of the curve of the Bertrand W curve couple is obtained as follows

$$\widetilde{T}(s) = \frac{(1+\lambda\kappa)}{\sqrt{(1+\lambda\kappa)^2 + \lambda^2\tau^2}}T(s) + \frac{\lambda\tau}{\sqrt{(1+\lambda\kappa)^2 + \lambda^2\tau^2}}B(s). \tag{3.23}$$

We know by definition of Bertrand W curve mate that principal normal is the principal normal of another curve, so we obtained

$$\widetilde{N}(s) = N(s). \tag{3.24}$$

In order to obtain the binormal vector of Bertrand W curve couple, the vector product of the first and second derivative of the curve with respect to s is given as follows

$$\widetilde{\alpha}'(\widetilde{s}) \times \widetilde{\alpha}''(\widetilde{s}) = \left(-\lambda\tau\left(\kappa + \lambda(\kappa^2 + \tau^2)\right)\right) T(s) - (1 + \lambda\kappa)\left(\kappa + \lambda(\kappa^2 + \tau^2)\right) B(s) \tag{3.25}$$

and we get

$$\|\widetilde{\alpha}'(\widetilde{s}) \times \widetilde{\alpha}''(\widetilde{s})\| = \left(\kappa + \lambda(\kappa^2 + \tau^2)\right) \sqrt{\lambda^2\tau^2 + (1 + \lambda\kappa)^2}. \tag{3.26}$$

Thus, we have binormal vector of Bertrand W curve mate in following form

$$\widetilde{B}(s) = \frac{-\lambda\tau}{\sqrt{\lambda^2\tau^2 + (1 + \lambda\kappa)^2}} T(s) + \frac{-(1 + \lambda\kappa)}{\sqrt{\lambda^2\tau^2 + (1 + \lambda\kappa)^2}} B(s). \tag{3.27}$$

By using the formulas

$$\widetilde{\kappa} = \frac{\|\widetilde{\alpha}'(\widetilde{s}) \times \widetilde{\alpha}''(\widetilde{s})\|}{\|\widetilde{\alpha}'(\widetilde{s})\|^3}, \ \widetilde{\tau} = \frac{\langle \widetilde{\alpha}'(\widetilde{s}) \times \widetilde{\alpha}''(\widetilde{s}), \widetilde{\alpha}'''(\widetilde{s}) \rangle}{\|\widetilde{\alpha}'(\widetilde{s}) \times \widetilde{\alpha}''(\widetilde{s})\|^2} \tag{3.28}$$

we can easily find the curvature and torsion of the curve $\widetilde{\alpha}$ as follows

$$\widetilde{\kappa} = \frac{\kappa + \lambda(\kappa^2 + \tau^2)}{(1 + \lambda\kappa)^2 + \lambda^2\tau^2}, \ \widetilde{\tau} = \frac{\tau}{\lambda^2\tau^2 + (1 + \lambda\kappa)^2}. \tag{3.29}$$

□

3.2. *Relations between Frenet apparatus of timelike Bertrand W curve mate*

Theorem Let α be a unit speed timelike Bertrand W curve, then the Frenet Frame and curvature, torsion of Bertrand W curve mate $\widetilde{\alpha}$ are given by

$$\widetilde{T}(s) = \frac{(1 + \lambda\kappa)}{\sqrt{\left|-\lambda^2\tau^2 + (1 + \lambda\kappa)^2\right|}} T(s) + \frac{\lambda\tau}{\sqrt{\left|-\lambda^2\tau^2 + (1 + \lambda\kappa)^2\right|}} B(s), \tag{3.30}$$

$$\widetilde{N}(s) = N(s), \tag{3.31}$$

$$\widetilde{B}(s) = \frac{\lambda\tau}{\sqrt{\left|-\lambda^2\tau^2 + (1 + \lambda\kappa)^2\right|}} T(s) - \frac{(1 + \lambda\kappa)}{\sqrt{\left|-\lambda^2\tau^2 + (1 + \lambda\kappa)^2\right|}} B(s) \tag{3.32}$$

and

$$\widetilde{\kappa} = \left|\frac{\kappa + \lambda(\kappa^2 - \tau^2)}{\lambda^2\tau^2 - (1 + \lambda\kappa)^2}\right|, \ \widetilde{\tau} = \frac{-\tau + 2\lambda\tau\kappa}{\left|-\lambda^2\tau^2 + (1 + \lambda\kappa)^2\right|}. \tag{3.33}$$

Proof. Proof of the theorem can be obtained by making calculations similar to the proof of the previous two theorems. □

4. Characterization of Bertrand W curve mate

In this section, the situations, which are changed according to curvature and torsion values, are given as propositions. These situations arise because of the eigenvalue and eigenvector problems in the solution of the differential equations while obtaining the characterization of the curves. In this section, all results are obtained by considering these situations.

4.1. Spacelike Bertrand W curves mate

Theorem Let $\alpha : I \to \mathbb{E}_1^3$ be a twisted spacelike Bertrand W curve with spacelike normal vector and $\widetilde{\alpha}(s)$ be a Bertrand W curve mate of $\alpha(s)$ such that $\alpha(s) = p_0(s)T(s) + p_1(s)N(s) + p_2(s)B(s)$ and $\widetilde{\alpha}(s) = \widetilde{p_0}(s)\widetilde{T}(s) + \widetilde{p_1}(s)\widetilde{N}(s) + \widetilde{p_2}(s)\widetilde{B}(s)$ where $\tau^2 - \kappa^2 = f^2 > 0$ and c_i are arbitrary constants for $0 \le i \le 2$. The differentiable functions given as follows

$$\widetilde{p_0}(s) = \frac{1}{\sqrt{(1-\lambda\kappa)^2 - \lambda^2\tau^2}} \begin{bmatrix} -\tau c_0 + (c_1 + c_2)\left(\kappa \sinh(fs) + \kappa^2\lambda\right) - \\ \lambda\tau^2 \left(c_1 \cosh(fs) - c_2 \sinh(fs)\right) + \frac{\tau^2}{f^2} s \end{bmatrix}, \tag{4.1}$$

$$\widetilde{p_1}(s) = -c_1 f \sinh(fs) + c_2 f \cosh(fs) + \frac{\kappa}{f^2} + \lambda, \tag{4.2}$$

$$\widetilde{p_2}(s) = \frac{1}{\sqrt{(1-\lambda\kappa)^2 - \lambda^2\tau^2}} \begin{bmatrix} \kappa c_0 + c_1\tau \cosh(fs) - c_2\tau \sinh(fs) - \frac{\kappa\tau}{f^2} s \\ +\lambda c_0 \left(\tau^2 - \kappa^2\right) - (\lambda\tau)\left(s + c_1\kappa\left(\cosh(fs) - \sinh(fs)\right)\right) \end{bmatrix}. \tag{4.3}$$

Proof. According to assumptions, we have $\widetilde{\alpha}(s) = \widetilde{p_0}(s)\widetilde{T}(s) + \widetilde{p_1}(s)\widetilde{N}(s) + \widetilde{p_2}(s)\widetilde{B}(s)$ and $\widetilde{\alpha}(s) = \alpha(s) + \lambda(s) N(s)$. Using the relations between Frenet Frames, we obtain

$$\widetilde{p_0}(s) = \frac{1}{\sqrt{\left|(1-\lambda\kappa)^2 - \lambda^2\tau^2\right|}} \left[p_0(s)(1-\lambda\kappa) - p_2(s)\lambda\tau\right] \tag{4.4}$$

$$\widetilde{p_1}(s) = p_1(s) + \lambda \tag{4.5}$$

$$\widetilde{p_2}(s) = \frac{1}{\sqrt{\left|(1-\lambda\kappa)^2 - \lambda^2\tau^2\right|}} \left[(1-\lambda\kappa) p_2(s) - p_0(s)\lambda\tau\right] \tag{4.6}$$

If we write the $p_0(s), p_1(s), p_2(s)$ values in the Proposition 3, then we complete the proof. □

Theorem Let $\alpha : I \to \mathbb{E}_1^3$ be a twisted spacelike Bertrand W curve with spacelike normal vector, and $\widetilde{\alpha}(s)$ is a Bertrand W curve mate of $\alpha(s)$ where $\tau^2 - \kappa^2 = -g^2 < 0$ and c_i are arbitrary constants for $0 \le i \le 2$.The $\widetilde{p_0}(s), \widetilde{p_1}(s), \widetilde{p_2}(s)$ differentiable functions obtained as follows

$$\widetilde{p_0}(s) = \frac{1}{\sqrt{\left|(1-\lambda\kappa)^2 - \lambda^2\tau^2\right|}} \left((\lambda - \kappa)\left(c_1 \cos(gs) - c_2 \sin(gs)\right)\left(g^2 - \tau^2\right)\right), \tag{4.7}$$

$$\widetilde{p_1}(s) = c_1 g \sin(gs) + c_2 g \cos(gs) - \frac{\kappa}{g^2} + \lambda, \tag{4.8}$$

$$\widetilde{p_2}(s) = \frac{1}{\sqrt{\left|(1-\lambda\kappa)^2 - \lambda^2\tau^2\right|}} \left(\kappa^2 - \tau^2\right)\left((\lambda + \tau)\left(c_1 \cos(gs) - c_2 \sin(gs)\right) + c_0\kappa\right) + \kappa\tau s. \tag{4.9}$$

Proof. If the Frenet Frame of the Bertrand W curve mate obtained in Eqs 3.1–3.3, the differential functions obtained in Eqs 2.6–2.8 are written in place of Eqs 4.4–4.6, the proof of the theorem is obtained. □

Theorem Let α be a twisted spacelike curve with spacelike normal vector and $\widetilde{\alpha}(s)$ be a Bertrand W curve mate of $\alpha(s)$. The differentiable functions in the Bertrand W curve mate $\widetilde{\alpha}(s)$ are given as

$$\widetilde{p_0}(s) = \frac{1}{\sqrt{1-2\lambda\kappa}}\left(-\frac{\kappa^2}{2}s^2 - \lambda\kappa\right)(c_0+c_2) + \kappa s\,(c_1-\lambda) + c_0 - \frac{s^3\kappa^2-6s}{6}, \tag{4.10}$$

$$\widetilde{p_1}(s) = -\kappa s\,(c_0+c_2) + c_1 - \frac{\kappa}{2}s^2 + \lambda, \tag{4.11}$$

$$\widetilde{p_2}(s) = \frac{1}{\sqrt{1-2\lambda\kappa}}\left(\frac{\kappa^2}{2}s^2 - \lambda\kappa\right)(c_0+c_2) - \kappa s\,(c_1+\lambda) + \frac{s^3\kappa^2+6c_2}{6} \tag{4.12}$$

where $\kappa = \tau$ and

$$\widetilde{p_0}(s) = \frac{1}{\sqrt{1-2\lambda\kappa}}\left(\left(\frac{\kappa^2 s^2}{2} + \lambda\kappa\right)(c_2-c_0) + \kappa s\,(c_1-\lambda) + c_0 + s - \frac{\kappa^2 s^3}{6}\right), \tag{4.13}$$

$$\widetilde{p_1}(s) = -c_0\kappa s + c_1 + c_2\kappa s - \frac{\kappa}{2}s^2 + \lambda, \tag{4.14}$$

$$\widetilde{p_2}(s) = \frac{1}{\sqrt{1-2\lambda\kappa}}\left(\left(\frac{\kappa^2 s^2}{2} - \lambda\kappa\right)(c_2-c_0) + \kappa\lambda\,(c_1+s) + c_2 - \frac{\kappa^2 s^3}{6}\right) \tag{4.15}$$

and where $\kappa = -\tau$ and c_i are arbitrary constants for $0 \le i \le 2$.

Proof. Proof of the theorem can be obtained by making calculations similar to the proof of the previous two theorems. □

Theorem Let $\alpha : I \to \mathbb{E}_1^3$ be a twisted spacelike curve with timelike normal vector, and $\widetilde{\alpha}(s)$ is a Bertrand W curve mate of $\alpha(s)$ where $\tau^2 + \kappa^2 = t^2 > 0$ and c_i are arbitrary constants for $0 \le i \le 2$.The differentiable functions in the Bertrand W curve mate $\widetilde{\alpha}(s)$ are given as follows

$$\widetilde{q_0}(s) = \frac{1}{\sqrt{(1+\lambda\kappa)^2+\lambda^2\tau^2}}\left(\left(\lambda\left(\tau^2+\kappa^2\right)-\kappa\right)(c_1\cosh(ts) + c_2\sinh(ts)) - c_0\tau + \frac{\tau^2}{t^2}s\right), \tag{4.16}$$

$$\widetilde{q_1}(s) = -c_1 t\sinh(ts) - c_2 t\cosh(ts) + \frac{\kappa}{t^2} + \lambda, \tag{4.17}$$

$$\widetilde{q_2}(s) = \frac{1}{\sqrt{(1+\lambda\kappa)^2+\lambda^2\tau^2}}\left(\begin{array}{c}\lambda\left(\tau^2+\kappa^2\right)\left(c_0+\frac{\tau s}{t^2}\right)+\kappa c_0 + c_1\tau\cosh(ts)\\ +c_2\tau\sinh(ts) - \frac{\kappa\tau}{t^2}s\end{array}\right). \tag{4.18}$$

Proof. Similarly to the proof of previous theorem, we see that

$$\widetilde{q_0}(s) = \frac{1}{\sqrt{(1+\lambda\kappa)^2+\lambda^2\tau^2}}\left((1+\lambda\kappa)\,q_0(s) + (\lambda\tau)\,q_2(s)\right), \tag{4.19}$$

$$\widetilde{q_1}(s) = q_1(s) + \lambda, \tag{4.20}$$

$$\widetilde{q_2}(s) = \frac{1}{\sqrt{(1+\lambda\kappa)^2+\lambda^2\tau^2}}\left((-\lambda\tau)\,q_0(s) + (1+\lambda\kappa)\,q_2(s)\right). \tag{4.21}$$

So we rewrite functions $q_0(s)$, $q_1(s)$, $q_2(s)$ in proposition 2 to Eqs 4.19–4.21, the theorem is proved after some calculations. □

4.2. *Timelike Bertrand W curve mate*

Theorem Let α be a timelike twisted Bertrand W curve given by arc length parameter and $\widetilde{\alpha}$ be a Bertrand W curve mate of $\alpha(s)$ with $\kappa^2 - \tau^2 = b^2 > 0$. The position vector of $\widetilde{\alpha}(s)$ can be written as follows

$$\widetilde{\alpha}(s) = \widetilde{r_0}(s)\widetilde{T}(s) + \widetilde{r_1}(s)\widetilde{N}(s) + \widetilde{r_2}(s)\widetilde{B}(s)$$

such that

$$\widetilde{r_0}(s) = \frac{1}{\sqrt{\left|-\lambda^2\tau^2 + (1+\lambda\kappa)^2\right|}}\left(-\lambda\left(\kappa^2 - \tau^2\right) - \kappa\right)(c_1\cosh(bs) + c_2\sinh(bs)) + c_0\tau - \frac{\tau^2}{b^2}s, \tag{4.22}$$

$$\widetilde{r_1}(s) = -c_2 b\cosh(bs) - c_1 b\sinh(bs) + \frac{\kappa}{b^2} + \lambda, \tag{4.23}$$

$$\widetilde{r_2}(s) = \frac{1}{\sqrt{\left|-\lambda^2\tau^2 + (1+\lambda\kappa)^2\right|}}\begin{pmatrix} c_0\left(\lambda b^2 + \kappa\right) \\ +\tau\left(c_1\cosh(bs) + c_2\sinh(bs) - s\lambda - \frac{\kappa}{b^2}s\right)\end{pmatrix} \tag{4.24}$$

where c_i are arbitrary constants for $0 \le i \le 2$.

Proof. Bertrand W curve mate $\widetilde{\alpha}(s)$ can be written as linear combinations of their Serret-Frenet vectors as follows

$$\widetilde{\alpha}(s) = \widetilde{r_0}(s)\widetilde{T}(s) + \widetilde{r_1}(s)\widetilde{N}(s) + \widetilde{r_2}(s)\widetilde{B}(s) \tag{4.25}$$

and

$$\begin{aligned}\widetilde{\alpha}(s) &= \alpha(s) + \lambda N(s) \\ &= r_0(s)T(s) + (r_1(s) + \lambda)N(s) + r_2(s)B(s).\end{aligned} \tag{4.26}$$

Thus we have

$$\widetilde{r_0}(s)\widetilde{T}(s) + \widetilde{r_1}(s)\widetilde{N}(s) + \widetilde{r_2}(s)\widetilde{B}(s) = r_0(s)T(s) + (r_1(s) + \lambda)N(s) + r_2(s)B(s). \tag{4.27}$$

If we rewrite value of $\widetilde{T}(s)$, $\widetilde{N}(s)$, $\widetilde{B}(s)$ in Eqs 3.30–3.32, then we get

$$\widetilde{r_0}(s) = \frac{1}{\sqrt{\left|-\lambda^2\tau^2 + (1+\lambda\kappa)^2\right|}}\left(-(1+\lambda\kappa)r_0(s) + \lambda\tau r_2(s)\right),$$

$$\widetilde{r_1}(s) = r_1(s) + \lambda,$$

$$\widetilde{r_2}(s) = \frac{1}{\sqrt{\left|-\lambda^2\tau^2 + (1+\lambda\kappa)^2\right|}}\left(-\lambda\tau r_0(s) + (1+\lambda\kappa)r_2(s)\right).$$

We rewrite equality of $\widetilde{r_0}(s), \widetilde{r_1}(s), \widetilde{r_2}(s)$ in proposition 3, the theorem is proved. □

Theorem Let $\widetilde{\alpha}$ is a Bertrand W curve mate of timelike curve $\alpha(s)$ with $\kappa^2 - \tau^2 = -a^2 < 0$, then the position vector $\widetilde{\alpha}(s)$ is given with the following differentiable functions

$$\widetilde{r_0}(s) = \frac{1}{\sqrt{\left|-\lambda^2\tau^2 + (1+\lambda\kappa)^2\right|}} \begin{pmatrix} \left(\lambda a^2 + \kappa\right)(c_1 \cos(as) + c_2 \sin(as)) \\ +\tau\left(c_0 + \frac{\tau}{\kappa^2-\tau^2} s\right) \end{pmatrix}, \tag{4.28}$$

$$\widetilde{r_1}(s) = c_1 a \sin(as) - c_2 a \cos(as) - \frac{\kappa}{a^2} + \lambda, \tag{4.29}$$

$$\widetilde{r_2}(s) = \frac{1}{\sqrt{\left|-\lambda^2\tau^2 + (1+\lambda\kappa)^2\right|}} \begin{pmatrix} c_0\left(-\lambda a^2 + \kappa\right) \\ +\tau\left(c_1 \cosh(as) + c_2 \sinh(as) - s\lambda - \frac{\kappa}{a^2} s\right) \end{pmatrix}. \tag{4.30}$$

Proof. The proof can be done similar to the proof previous theorem. □

Theorem Let $\alpha : I \to \mathbb{E}_1^3$ be a timelike twisted Bertrand W curve given by arc length parameter and $\widetilde{\alpha}(s)$ be a Bertrand W curve mate of $\alpha(s)$ with $\kappa^2 - \tau^2 = 0$, then the position vector $\widetilde{\alpha}(s)$ is stated with the following differentiable functions

$$\widetilde{r_0}(s) = \frac{1}{\sqrt{|1+2\lambda\kappa|}} \left(\left(\frac{\kappa^2 s^2}{2} - \lambda\kappa\right)(c_0 - c_2) - s\kappa(\lambda + c_1) + c_0 + s + \frac{\kappa^2}{6}s^3\right), \tag{4.31}$$

$$\widetilde{r_1}(s) = -c_0\kappa s + c_1 + c_2\kappa s - \frac{\kappa}{2}s^2 + \lambda, \tag{4.32}$$

$$\widetilde{r_2}(s) = \frac{1}{\sqrt{|1+2\lambda\kappa|}} \left(\left(\frac{\kappa^2 s^2}{2} - \lambda\kappa\right)(c_0 - c_2) - s\kappa(\lambda + c_1) + c_0 + \frac{\kappa^2}{6}s^3\right) \tag{4.33}$$

where $\tau = \kappa$ and

$$\widetilde{r_0}(s) = \frac{1}{\sqrt{|1+2\lambda\kappa|}} \left(\left(\frac{\kappa^2 s^2}{2} - \lambda\kappa\right)(c_0 + c_2) - s\kappa(\lambda + c_1) + c_0 + s + \frac{\kappa^2}{6}s^3\right), \tag{4.34}$$

$$\widetilde{r_1}(s) = -c_0\kappa s + c_1 - c_2\kappa s - \frac{\kappa}{2}s^2 + \lambda, \tag{4.35}$$

$$\widetilde{r_2}(s) = \frac{1}{\sqrt{|1+2\lambda\kappa|}} \left(\left(\lambda\kappa - \frac{\kappa^2 s^2}{2}\right)(c_0 + c_2) + s\kappa(\lambda + c_1) + c_2 - \frac{\kappa^2}{6}s^3\right) \tag{4.36}$$

where $\tau = -\kappa$ and c_i are arbitrary constants for $0 \leq i \leq 2$.

Proof. Proof of the theorem can be obtained by making calculations similar to the proof of the previous two theorems. □

5. Some numerical examples

In this section, we give examples of timelike and spacelike Bertrand W curve mate, separately.
Example: Consider the unit speed spacelike curve $\alpha : I \to \mathbb{E}_1^3$ with the parametrization

$$\alpha(s) = (\frac{1}{2}\sinh s, \frac{1}{2}\cosh s, \frac{\sqrt{5}}{2}s). \tag{5.1}$$

We obtain the Frenet frame fields as follows:

$$T(s) = (\frac{1}{2}\cosh s, \frac{1}{2}\sinh s, \frac{\sqrt{5}}{2}), \tag{5.2}$$

$$N(s) = (\sinh s, \cosh s, 0), \tag{5.3}$$

$$B(s) = (\frac{\sqrt{5}}{2}\cosh s, \frac{\sqrt{5}}{2}\sinh s, \frac{1}{2}) \tag{5.4}$$

where the curvature and torsion of the curve α are

$$\kappa(s) = \frac{1}{2} \text{ and } \tau(s) = \frac{\sqrt{5}}{2},\ \lambda = \frac{1}{2} \tag{5.5}$$

respectively. Since $\tau^2 - \kappa^2 > 0$, then Frenet apparatus of the curve $\widetilde{\alpha}(s)$ are given by

$$\widetilde{T}(s) = (2\cosh s, 2\sinh s, \sqrt{5}) \tag{5.6}$$

$$\widetilde{N}(s) = (\sinh s, \cosh s, 0) \tag{5.7}$$

$$\widetilde{B}(s) = (\sqrt{5}\cosh s, \sqrt{5}\sinh s, 2) \tag{5.8}$$

$$\widetilde{\kappa} = 4, \widetilde{\tau} = -2\sqrt{5} \tag{5.9}$$

So we get

$$\begin{aligned}\widetilde{\alpha}(s) &= \alpha(s) + \lambda(s) N(s)\\ &= (\frac{1}{2}\sinh s, \frac{1}{2}\cosh s, \frac{\sqrt{5}}{2}s) + \frac{1}{2}(\sinh s, \cosh s, 0)\\ &= \left(\sinh s, \cosh s, \frac{\sqrt{5}}{2}s\right)\end{aligned} \tag{5.10}$$

Bertrand W curve mate $\widetilde{\alpha}(s)$ of unit speed spacelike curve α is given in Figure 1.

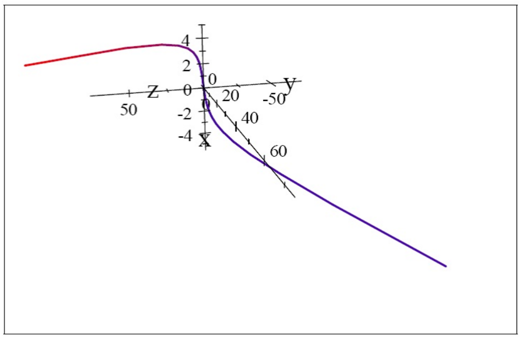

Figure 1. Bertrand W curve mate $\widetilde{\alpha}(s)$ of unit speed spacelike curve α.

Example: Let $\alpha : I \to \mathbb{E}_1^3$ be a timelike curve with following parametric expression

$$\alpha(s) = (\sqrt{2}\sinh s, \sqrt{2}\cosh s, s). \tag{5.11}$$

Then the Frenet frame is obtained

$$T(s) = (\sqrt{2}\cosh s, \sqrt{2}\sinh s, 1), \tag{5.12}$$
$$N(s) = (\sinh s, \cosh s, 0), \tag{5.13}$$
$$B(s) = (\cosh s, \sinh s, \sqrt{2}) \tag{5.14}$$

where the curvature and torsion of the curve are

$$\kappa(s) = \sqrt{2} \text{ and } \tau(s) = -1, \tag{5.15}$$

respectively. Thus we get

$$m_0(s) = -s, m_1(s) = \sqrt{2}, m_2(s) = \sqrt{2}s \tag{5.16}$$

and Frenet apparatus of curve $\widetilde{\alpha}(s)$ is obtained as follows

$$\widetilde{T}(s) = (\frac{2}{7}\sqrt{14}\cosh s, \frac{2}{7}\sqrt{14}\sinh s\frac{1}{7}\sqrt{7}), \tag{5.17}$$

$$\widetilde{N}(s) = (\sinh s, \cosh s, 0), \tag{5.18}$$

$$\widetilde{B}(s) = (-\frac{5}{7}\sqrt{7}\cosh s, -\frac{5}{7}\sqrt{7}\sinh s, -\frac{4}{7}\sqrt{2}\sqrt{7}), \tag{5.19}$$

$$\widetilde{\kappa} = \frac{1}{7}\sqrt{2}, \widetilde{\tau} = -\frac{3}{7}. \tag{5.20}$$

So by using Frenet apparatus of curve $\widetilde{\alpha}(s)$, we get

$$\widetilde{\alpha}(s) = \left(2\sqrt{2}\sinh s, 2\sqrt{2}\cosh s, s\right). \tag{5.21}$$

Bertrand W curve mate $\widetilde{\alpha}(s)$ of unit speed timelike curve α is shown in Figure 2.

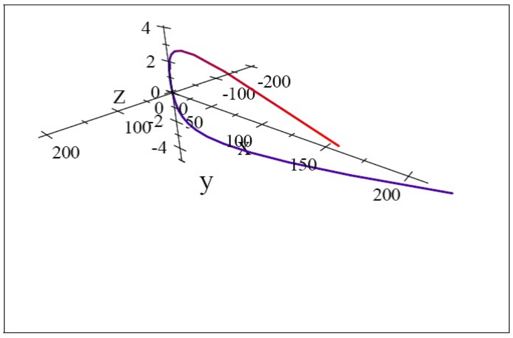

Figure 2. Bertrand W curve mate $\widetilde{\alpha}(s)$ of unit speed timelike curve α.

6. Conclusions

In this paper, authors obtained the characterization of the position vectors of non-lightlike Bertrand W curve mate in Minkowski space due to differentiable functions. In accordance with this scope, the position vector of a non-lightlike Bertrand W curve was stated by a linear combination of its Frenet frame with differentiable functions. There existed also different cases for the curve depending on the value of curvature and torsion. The relationships between Frenet apparatus of these curves were presented separately for each case. This study will accompany the scientists who will conduct new studies on similar subjects as a basic resource since it is one of the important studies on this subject.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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