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Research article

Some spectral sufficient conditions for a graph being pancyclic

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Abstract: Let $G(V, E)$ be a simple connected graph of order n. A graph of order n is called pancyclic if it contains all the cycles C_k for $k \in \{3, 4, \cdots, n\}$. In this paper, some new spectral sufficient conditions for the graph to be pancyclic are established in terms of the edge number, the spectral radius and the signless Laplacian spectral radius of the graph.

Keywords: pancyclic graph; edge number; spectral radius; signless Laplacian spectral radius
Mathematics Subject Classification: 05C50, 15A18

1. Introduction

In this paper, we use $G = (V(G), E(G))$ to denote a graph with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E(G)$. Let $e(G) = m = |E(G)|$ be the number of edges of the graph G. For any $v_i \in V(G)$, $d_i = d_{v_i} = d_G(v_i)$ represents the degree of v_i. Let $(d_1, d_2, \cdots, d_n)$ be the degree sequence of G, where $d_1 \leq d_2 \leq \cdots \leq d_n$. Let δ or $\delta(G)$ be the minimum degree of G, and $N_G(v)$ be the set of neighbors of a vertex v in G. The complete graph of order n is denoted by K_n, and the complete bipartite graph with the partite sets X and Y with $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. The disjoint union of G and H, denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup (H)$. In particular, if $G_1 = G_2 = \cdots = G_k$, then let $kG_1 = G_1 + G_2 + \cdots + G_k$. Similarly, we use $G \vee H$ to denote the join of G and H. The complete graph on $n - 1$ vertices together with an isolated vertex v is denoted by $K_{n-1} + v$.

Let $A(G) = [a_{ij}]$ be the adjacency matrix of G, where $a_{ij} = 1$ if v_i is adjacent to v_j, $a_{ij} = 0$ otherwise. We use $\mu(G)$ to denote the maximum eigenvalue of $A(G)$, which is called the spectral radius of G. Let $D(G)$ be the diagonal degree matrix of G, and $Q(G) = D(G) + A(G)$ be the signless Laplacian matrix of G. Let $q(G)$ be the maximum eigenvalue of $Q(G)$, which is called the signless Laplacian spectral radius.

We call the cycle (path) containing all vertices of G the Hamilton cycle (path) of G. If graph G contains a Hamiltonian cycle (path), then G is a Hamiltonian traceable graph. If for any positive integer $k(3 \leq k \leq n)$, G contains a cycle of length k, then G is called a pancyclic graph. Obviously, a bipartite graph is not a pancyclic graph. In addition, a Hamilton cycle graph is certainly Hamiltonian, but the converse is incorrect. Determining whether a given graph is a Hamilton graph or not is one of the most difficult classical problems in graph theory. In fact, it is an $NP-$complete problem.

In recent years, more and more worldwide researchers use the spectral theory of graphs to solve this problem. First, Fiedler and Nikiforov [1] established a sufficient condition on spectral radius for the existence of a graph with Hamilton paths and Hamilton cycles. Yu and Fan [2] used the spectral radius of the adjacency matrix or signless Laplacian matrix of the graph or its complement to give a sufficient condition for the graph to be Hamilton-connected. Lu et al. [3] gave the spectral sufficient conditions for a bipartite graph to be a Hamilton graph. Many scholars have studied similar problems under different spectral conditions since that time, see [4–10]. Recently, Yu et al. [11] has discussed for the first time the spectral sufficient conditions for a graph with a minimum degree $\delta(G) \geq 2$ to be a pancyclic graph. In this paper, a new edge sufficient condition for a graph with a minimum degree $\delta(G) \geq 3$ to be a pancyclic graph is established by using a similar method. Then, a sufficient condition on (signless Laplacian) spectral radius of a graph to be a pancyclic graph is given based on the edge sufficient condition.

2. Preliminary

Given a graph G with order n, a vector $X \in R^n$, is called to be defined on G, if there is a 1-1 mapping φ from $V(G)$ to vector, simply written $X_u = \varphi(u)$.

X is defined on G if X is the eigenvector of $A(G)(Q(G))$. Let X_u denote the term of X corresponding to vertex u. One can find that when λ is an eigenvalue of G corresponding to the eigenvector, X if and only if $X \neq 0$,

$$\lambda X_v = \sum_{u \in N_G(v)} X_u, \text{for each } v \in V(G). \tag{2.1}$$

The equation above is the eigen-equation of G. One can find when q is an signless Laplacian eigenvalue of G corresponding to the eigenvector X if and only if $X \neq 0$,

$$|q - d_G(v)|\lambda X_v = \sum_{u \in N_G(v)} X_u, \text{for each } v \in V(G). \tag{2.2}$$

The equation above is the *signless Laplacian* eigen-equation of G.

Lemma 2.1[12] Let G be a graph with n vertices, and its degree sequence is $d_1 \leq d_2 \leq \cdots \leq d_n$. G is a pancyclic graph or bipartite graph if for all positive integers k there is always $d_{n-k} \geq n-k$ and $d_k \leq k < n/2$.

Lemma 2.2[13] Let G be a graph with n vertices and m edges, then

$$\mu(G) \leq \sqrt{2m - n + 1},$$

and the equality holds if and only if $G = K_n$ or $G = K_{1,n-1}$.

Lemma 2.3[14] Let G be a graph with n vertices and m edges, then

$$q(G) \leq \frac{2m}{n-1} + n - 2,$$

and the equal sign is true if and only if $G = K_n$ or $G = K_{1,n-1}$ when G is connected. If G is not connected, the equal sign is true if and only if $G = K_{n-1} + v$.

Lemma 2.4[15] Let G be a Hamiltonian graph and satisfy $e(G) \geq n^2/4$, then G is either a complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ (where n is even), or a pancyclic graph.

Lemma 2.5[16] If a Hamiltonian graph G of order n satisfies

$$e(G) > \frac{(n-1)^2}{4} + 1,$$

then G is a pancyclic graph or G is a bipartite graph.

3. Main results

Theorem 3.1 Let G be a connected graph with $n(n \geq 5)$ vertices and m edges, and its minimum degree $\delta(G) \geq 3$. If

$$m \geq \binom{n-3}{2} + 9, \tag{3.1}$$

then G is a pancyclic graph or a bipartite graph or $G \in NP_1 = \{(3K_1+K_{n-6})\vee K_3, (4K_1+K_3)\vee K_4, (6K_1+K_2)\vee K_6, 9K_1\vee K_8, 8K_1\vee K_7, (K_1+K_{1,7})\vee K_6, K_{2,8}\vee K_5, K_6\vee(2K_1+K_{1,6}), K_4\vee 2K_1\vee(K_2+K_{1,6}), K_5\vee(K_1+K_{1,6}), (K_2+K_{1,5})\vee K_5, (2K_1\vee K_4)\vee 7K_1, (K_1+K_2+K_{1,4})\vee K_5, (2K_1+K_{1,5})\vee K_5, (K_1+K_2\vee 6K_1)\vee K_4, (K_{1,5}+K_2)\vee 2K_1\vee K_3, 7K_1\vee K_6, K_5\vee(2K_1+K_{1,3}+K_2), K_4\vee[K_1+K_1\vee(K_{1,4}+K_2)], K_4\vee(K_1+2K_1\vee 6K_1), K_3\vee K_{3,7}, K_5\vee(3K_1+K_{1,4}), K_3\vee 2K_1\vee(K_1+K_{1,4}+K_2), K_4\vee[K_1\vee(K_1+K_{1,5})+K_1], K_3\vee 2K_1\vee(2K_1+K_{1,5}), (5K_1+K_2)\vee K_5, K_4\vee(K_1\vee(4K_1+K_2)+K_1), 7K_1\vee K_5, (5K_1+K_2)\vee 2K_1\vee K_3, 6K_1\vee K_5, K_4\vee(K_{1,4}+K_2), K_4\vee(K_1+K_{1,5}), K_3\vee K_{2,6}, K_4\vee(K_1+K_{1,3}+K_2), K_4\vee(2K_1+K_{1,4}), K_3\vee[(K_2\vee 5K_1)+K_1], K_2\vee 2K_1\vee(K_{1,4}+K_2), K_4\vee(K_2+K_{1,2}+2K_1), K_3\vee(K_1+K_{2,5}), K_2\vee K_{3,6}, K_4\vee(K_{1,3}+3K_1), K_2\vee 2K_1\vee(K_1+K_{1,3}+K_2), [K_1\vee(K_{1,4}+K_1)+K_1]\vee K_3, K_2\vee 2K_1\vee(2K_1+K_{1,4}), (4K_1+K_2)\vee K_4, K_3\vee[K_1+K_1\vee(3K_1+K_2)], 6K_1\vee K_4, (4K_1+K_2)\vee 2K_1\vee K_2, 5K_1\vee K_4, (K_{1,3}+K_2)\vee K_3, (K_1+K_{1,4})\vee K_3, 5K_1\vee 2K_1\vee K_2, (K_1+K_{1,2}+K_2)\vee K_3, (2K_1+K_{1,3})\vee K_3, (K_{1,3}+K_2)\vee 2K_1\vee K_1\}$.

Proof: Suppose that G is neither a pancyclic graph nor a bipartite graph. According to **Lemma 2.1**, there is a positive integer k makes $3 \leq d_k \leq k < n/2$ and $d_{n-k} \leq n-k-1$ hold at the same time. Then we have

$$\begin{aligned} 2m &= \Sigma_{i=1}^{k} d_i + \Sigma_{i=k+1}^{n-k} d_i + \Sigma_{i=n-k+1}^{n} d_i \\ &\leq k^2 + (n-2k)(n-k-1) + k(n-1) \\ &= n^2 - n + 3k^2 + (1-2n)k \\ &= 2\binom{n-3}{2} + 18 - (k-3)(2n-3k-10), \end{aligned}$$

thus

$$m \leq \binom{n-3}{2} + 9 - \frac{(k-3)(2n-3k-10)}{2}. \tag{3.2}$$

Since

$$\binom{n-3}{2} + 9 \leq m \leq \binom{n-3}{2} + 9 - \frac{(k-3)(2n-3k-10)}{2}, \tag{3.3}$$

thus $(k-3)(2n-3k-10) \leq 0$. Next, the following two cases are discussed.

Case 1 Assume that $(k-3)(2n-3k-10) = 0$, i.e., $k = 3$ or $2n-3k-10 = 0$. All the above inequalities are equal and $m = \binom{n-3}{2} + 9$.

Case 1.1 If $k = 3$, then G is a graph with $d_1 = d_2 = d_3 = 3, d_4 = d_5 = \cdots = d_{n-3} = n-4, d_{n-2} = d_{n-1} = d_n = n-1$. Three vertices with degree $n-1$ must be connected to other vertices, then a K_3 is obtained. The three vertices with degree 3 are not connected with other vertices, and a $3K_1$ is obtained. The remaining $n-6$ vertices with degree $n-4$ must be connected to each other to ensure that the degree is $n-4$, so they induce a K_{n-6}. According to the above analysis, the graph G is $(3K_1 + K_{n-6}) \vee K_3$.

Case 1.2 If $2n - 3k - 10 = 0$, then we get $n \leq 19$ because $\frac{2n-10}{3} = k < \frac{n}{2}$, and hence $n = 11, k = 4$, or $n = 14, k = 6$, or $n = 17, k = 8$. The corresponding permissible graphic sequences are $(4, 4, 4, 4, 6, 6, 6, 10, 10, 10, 10)$, $(6, 6, 6, 6, 6, 6, 7, 7, 13, 13, 13, 13, 13, 13)$,$(8, 8, 8, 8, 8, 8, 8, 8, 8, 16, 16, 16, 16, 16, 16, 16, 16)$. Analysis shows that the graphs G are $(4K_1 + K_3) \vee K_4$, $(6K_1 + K_2) \vee K_6$ and $9K_1 \vee K_8$ respectively.

Case 2 Assume that $(k-3)(2n-3k-10) < 0$, i.e., $k > 3$ and $2n-3k-10 < 0$. Because $2n-10 < 3k < \frac{3}{2}n$, then, $n = 15, k = 7$ or $n = 13, k = 6$ or $n = 12, k = 5$ or $n = 11, k = 5$ or $n = 10, k = 4$ or $n = 9, k = 4$.

Case 2.1 If $n = 15$, $k = 7$. According to equation (3.3), we have $150 \leq \Sigma_{i=1}^{15} d_i \leq 154$. All possible degree sequences and their corresponding graphs are shown in $Table$ 1.

Table 1. The degree sequences and corresponding graphs of Case 2.1.

Degree sum	No.	The degree sequence	The graph
154	1	(7,7,7,7,7,7,7,7,14,14,14,14,14,14,14)	$G = 8K_1 \vee K_7$
152	2	(7,7,7,7,7,7,7,7,12,14,14,14,14,14,14)	$G = K_6 \vee (K_2 + K_{1,6})$
	3	(6,7,7,7,7,7,7,7,13,14,14,14,14,14,14)	$G = (K_1 + K_{1,7}) \vee K_6$
	4	(7,7,7,7,7,7,7,7,13,13,14,14,14,14,14)	$G = K_{2,8} \vee K_5$
150	5	(7,7,7,7,7,7,7,7,10,14,14,14,14,14,14)	$G = K_6 \vee (2K_2 + K_{1,4})$
	6	(6,7,7,7,7,7,7,7,11,14,14,14,14,14,14)	$G = (K_1 + K_{1,5} + K_2) \vee K_6$
	7	(6,6,7,7,7,7,7,7,12,14,14,14,14,14,14)	$G = K_6 \vee (2K_1 + K_{1,6})$
	8	(7,7,7,7,7,7,7,7,12,13,13,14,14,14,14)	$G = K_4 \vee 2K_1 \vee (K_2 + K_{1,6})$

Case 2.2 If $n = 13$, $k = 6$. According to equation (3.3), we have $108 \leq \Sigma_{i=1}^{13} d_i \leq 114$. All possible degree sequences and their corresponding graphs are shown in $Table$ 2.

Table 2. The degree sequences and corresponding graphs of Case 2.2.

Degree sum	No.	The degree sequence	The graph
114	1	(6,6,6,6,6,6,6,12,12,12,12,12,12)	$G = 7K_1 \vee K_6$
112	2	(5,6,6,6,6,6,6,11,12,12,12,12,12)	$G = K_5 \vee (K_1 + K_{1,6})$
	3	(6,6,6,6,6,6,6,10,12,12,12,12,12)	$G = (K_2 + K_{1,5}) \vee K_5$
	4	(6,6,6,6,6,6,6,11,11,12,12,12,12)	$G = (2K_1 \vee K_4) \vee 7K_1$
110	5	(6,6,6,6,6,6,6,8,12,12,12,12,12)	$G = (2K_2 + K_{1,3}) \vee k_5$
	6	(5,6,6,6,6,6,6,9,12,12,12,12,12)	$G = (K_1 + K_2 + K_{1,4}) \vee K_5$
	7	(5,5,6,6,6,6,6,10,12,12,12,12,12)	$G = (2K_1 + K_{1,5}) \vee K_5$
	8	(4,6,6,6,6,6,6,11,11,12,12,12,12)	$G = (K_1 + K_2 \vee 6K_1) \vee K_4$
	9	(6,6,6,6,6,6,6,10,11,11,12,12,12)	$G = (K_{1,5} + K_2) \vee 2K_1 \vee K_3$
108	10	(6,6,6,6,6,6,6,6,12,12,12,12,12)	$G = 4K_2 \vee K_5$
	11	(5,6,6,6,6,6,6,7,12,12,12,12,12)	$G = (K_1 + K_{1,2} + 2K_2) \vee K_5$
	12	(6,6,6,6,6,6,6,9,9,12,12,12,12)	$G = K_4 \vee (K_2 \vee 4K_1 + K_3)$
	13	(5,5,6,6,6,6,6,8,12,12,12,12,12)	$G = K_5 \vee (2K_1 + K_{1,3} + K_2)$
	14	(6,6,6,6,6,6,6,8,11,11,12,12,12)	$G = K_3 \vee 2K_1 \vee (K_{1,3} + 2K_2)$
	15	(4,6,6,6,6,6,6,9,11,12,12,12,12)	$G = K_4 \vee [K_1 + K_1 \vee (K_{1,4} + K_2)]$
	16	(4,6,6,6,6,6,6,10,10,12,12,12,12)	$G = K_4 \vee (K_1 + 2K_1 \vee 6K_1)$
	17	(6,6,6,6,6,6,6,10,10,10,12,12,12)	$G = K_3 \vee K_{3,7}$
	18	(5,5,5,6,6,6,6,9,12,12,12,12,12)	$G = K_5 \vee (3K_1 + K_{1,4})$
	19	(5,6,6,6,6,6,6,9,11,11,12,12,12)	$G = K_3 \vee 2K_1 \vee (K_1 + K_{1,4} + K_2)$
	20	(5,5,6,6,6,6,6,10,10,12,12,12,12)	$G = K_4 \vee (K_2 \vee 5K_1 + K_2)$
	21	(5,5,6,6,6,6,6,10,10,12,12,12,12)	$G = K_4 \vee (K_1 \vee (K_1 + K_{1,5}) + K_1)$
	22	(5,5,6,6,6,6,6,10,10,12,12,12,12)	$G = K_3 \vee 2K_1 \vee (2K_1 + K_{1,5})$

Case 2.3 If $n = 12$, $k = 5$. According to equation (3.3), we have $90 \le \Sigma_{i=1}^{12} d_i \le 92$. All possible degree sequences and their corresponding graphs are shown in $Table$ 3.

Table 3. The degree sequences and corresponding graphs of Case 2.3.

Degree sum	No.	The degree sequence	The graph
92	1	(5,5,5,5,5,6,6,11,11,11,11,11)	$G = (5K_1 + K_2) \vee K_5$
90	2	(4,5,5,5,5,6,6,10,11,11,11,11)	$G = K_4 \vee (K_1 \vee (4K_1 + K_2) + K_1)$
	3	(5,5,5,5,5,5,5,11,11,11,11,11)	$G = 7K_1 \vee K_5$
	4	(5,5,5,5,5,6,6,10,10,11,11,11)	$G = (5K_1 + K_2) \vee 2K_1 \vee K_3$

Case 2.4 If $n = 11$, $k = 5$. According to equation (3.3), we have $74 \le \Sigma_{i=1}^{11} d_i \le 80$. All possible degree sequences and their corresponding graphs are shown in $Table$ 4.

Table 4. The degree sequences and corresponding graphs of Case 2.4.

Degree sum	No.	The degree sequence	The graph
80	1	(5,5,5,5,5,5,10,10,10,10,10)	$G = 6K_1 \vee K_5$
78	2	(5,5,5,5,5,5,8,10,10,10,10)	$G = K_4 \vee (K_{1,4} + K_2)$
	3	(4,5,5,5,5,5,9,10,10,10,10)	$G = (K_1 + K_{1,5}) \vee K_4$
	4	(5,5,5,5,5,5,9,9,10,10,10)	$G = K_3 \vee K_{2,6}$
76	5	(5,5,5,5,5,5,6,10,10,10,10)	$G = K_4 \vee (K_{1,2} + 2K_2)$
	6	(4,5,5,5,5,5,7,10,10,10,10)	$G = (K_1 + K_{1,3} + K_2) \vee K_4$
	7	(4,4,5,5,5,5,8,10,10,10,10)	$G = K_4 \vee (2K_1 + K_{1,4})$
	8	(3,5,5,5,5,5,9,9,10,10,10)	$G = K_3 \vee ((K_2 \vee 5K_1) + K_1)$
	9	(5,5,5,5,5,5,8,9,9,10,10)	$G = K_2 \vee 2K_1 \vee (K_2 + K_{1,4})$
74	10	(4,5,5,5,5,5,5,10,10,10,10)	$G = (K_1 + 3K_2) \vee K_4$
	11	(4,4,5,5,5,5,6,10,10,10,10)	$G = K_4 \vee (K_2 + K_{1,2} + 2K_1)$
	12	(5,5,5,5,5,5,6,9,9,10,10)	$G = (K_{1,2} + 2K_2) \vee 2K_1 \vee K_2$
	13	(3,5,5,5,5,5,8,8,10,10,10)	$G = K_3 \vee (K_1 + K_{2,5})$
	14	(5,5,5,5,5,5,8,8,8,10,10)	$G = K_2 \vee K_{3,6}$
	15	(4,4,4,5,5,5,7,10,10,10,10)	$G = K_4 \vee (K_{1,3} + 3K_1)$
	16	(4,5,5,5,5,5,7,9,9,10,10)	$G = K_2 \vee 2K_1 \vee (K_1 + K_{1,3} + K_2)$
	17	(3,4,5,5,5,5,8,9,10,10,10)	$G = (K_1 \vee (K_{1,4} + K_1) + K_1) \vee K_3$
	18	(4,4,5,5,5,5,8,9,9,10,10)	$G = K_2 \vee 2K_1 \vee (2K_1 + K_{1,4})$

Case 2.5 If $n = 10$, $k = 4$. According to equation (3.3), we have $60 \leq \Sigma_{i=1}^{10} d_i \leq 62$. All possible degree sequences and their corresponding graphs are shown in *Table* 5.

Table 5. The degree sequences and corresponding graphs of Case 2.5.

Degree sum	No.	The degree sequence	The graph
62	1	(4,4,4,4,5,5,9,9,9,9)	$G = (4K_1 + K_2) \vee K_4$
60	2	(4,4,4,4,5,5,7,9,9,9)	$G = K_3 \vee (K_1 \vee (2K_1 + K_2) + K_2)$
	3	(3,4,4,4,5,5,8,9,9,9)	$G = K_3 \vee (K_1 + K_1 \vee (3K_1 + K_2))$
	4	(4,4,4,4,4,4,9,9,9,9)	$G = 6K_1 \vee K_4$
	5	(4,4,4,4,5,5,8,8,9,9)	$G = (4K_1 + K_2) \vee 2K_1 \vee K_2$

Case 2.6 If $n = 9$, $k = 4$. According to equation (3.3), we have $48 \leq \Sigma_{i=1}^{9} d_i \leq 52$. All possible degree sequences and their corresponding graphs are shown in *Table* 6.

Table 6. The degree sequences and corresponding graphs of Case 2.6.

Degree sum	No.	The degree sequence	The graph
52	1	(4,4,4,4,4,8,8,8,8)	$G = 5K_1 \vee K_4$
50	2	(4,4,4,4,4,6,8,8,8)	$G = (K_{1,3} + K_2) \vee K_3$
	3	(3,4,4,4,4,7,8,8,8)	$G = (K_1 + K_{1,4}) \vee K_3$
	4	(4,4,4,4,4,7,7,8,8)	$G = 5K_1 \vee 2K_1 \vee K_2$
48	5	(4,4,4,4,4,4,8,8,8)	$G = 3K_2 \vee K_3$
	6	(3,4,4,4,4,5,8,8,8)	$G = (K_1 + K_{1,2} + K_2) \vee K_3$
	7	(3,3,4,4,4,6,8,8,8)	$G = (2K_1 + K_{1,3}) \vee K_3$
	8	(4,4,4,4,4,6,7,7,8)	$G = (K_{1,3} + K_2) \vee 2K_1 \vee K_1$

Next, we consider the pancyclicity of these graphs. For convenience, we put the maximum length cycle $L(G)$ of G in the appendix. In the following description, when it comes to the description of the graphs, we will use the sequence number of the graphs in the appendix.

As can be seen from the appendix, according to **Lemma 2.5**, all the graphs of No.5-18 in the appendix are pancyclic. Others belong to NP_1 and they are neither pancyclic nor bipartite.
The proof is complete.$\diamond$

Theorem 3.2 Let G be a connected graph with $n(n \geq 5)$ vertices and m edges, and its minimum degree $\delta(G) \geq 3$. If

$$\mu(G) \geq \sqrt{n^2 - 8n + 31},$$

then G is a pancyclic graph or a bipartite graph.

Proof: Suppose that G is neither a pancyclic graph nor a bipartite graph. Because a complete graph is a pancyclic graph and $\delta(K_{1,n-1}) = 1$, then G cannot be a complete graph or $K_{1,n-1}$. In such a situation, the equation in **Lemma 2.2** is invalid. By **Lemma 2.2**, $\sqrt{n^2 - 8n + 31} \leq \mu(G) < \sqrt{2m - n + 1}$, i.e., $m > \binom{n-3}{2} + 9$.

By **Theorem 3.1**, G is a pancyclic graph or a bipartite graph or $G \in NP_1$. It can be calculated that $(3K_1 + K_{n-6}) \vee K_3$,$(4K_1 + K_3) \vee K_4$,$(6K_1 + K_2) \vee K_6$,$9K_1 \vee K_8$ satisfy $m = \binom{n-3}{2} + 9$, which is contradiction. Next, we will study whether the graphs of No.19-69 satisfy the spectral radius condition in the **Theorem 3.2**. Note that $G \in NP_2$ and NP_2 is the set of graphs of No.19-69 in the appendix.

Take $(2K_1 \vee K_4) \vee 7K_1$ as an example:

Let $\mathbf{X} = (X_1, X_2, \ldots\ldots\ldots, X_{13})^T$ be the eigenvector corresponding to $\mu(G)$, where $X_i(1 \leq i \leq 2)$ corresponds to the vertex of degree 11, $X_i(3 \leq i \leq 6)$ corresponds to the vertex of degree 12 and $X_i(7 \leq i \leq 13)$ corresponds to the vertex of degree 6. By eigen-equation (2.1), then we have

$$\begin{cases} X_1 = X_2, X_3 = X_4 = X_5 = X_6, X_7 = X_8 = X_9 = X_{10} = X_{11} = X_{12} = X_{13} \\ \mu(G)X_1 = 4X_3 + 7X_7 \\ \mu(G)X_3 = 2X_1 + 3X_3 + 7X_7 \\ \mu(G)X_7 = 2X_1 + 4X_3 \end{cases}$$

Transform the above equations into the matrix equation $(A^{'}(G) - \mu(G)I)X^{'} = 0$, where $X^{'} = (X_1, X_3, X_7)^T$ and

$$A^{'}(G) = \begin{pmatrix} 0 & 4 & 7 \\ 2 & 3 & 7 \\ 2 & 4 & 0 \end{pmatrix}.$$

Let $f(x) = |xI - A^{'}(G)| = x^3 - 3x^2 - 50x - 70$, then the maximum root of $f(x) = 0$ is $\mu(G) = 9.235$. It can be calculated that $\mu(G) < \sqrt{13^2 - 8 \times 13 + 31}$, which is contradiction. The remaining graphs are studied in the same way, and the results are shown in $Table$ 7.

From $Table$ 7, all graphs in NP_2 satisfy $\mu(G) < \sqrt{n^2 - 8n + 31}$, a contradiction. The proof is complete. ⋄

Theorem 3.3 Let G be a connected graph with $n(n \geq 5)$ vertices and minimum degree $\delta(G) \geq 3$. If

$$q(G) \geq \frac{24}{n-1} + (2n - 8), \tag{3.4}$$

then G is a pancyclic graph or a bipartite graph or $7K_1 \vee K_6$ or $6K_1 \vee K_5$ or $5K_1 \vee K_4$.

Proof: Suppose that G is neither a pancyclic graph nor a bipartite graph. Because a complete graph is a pancyclic graph and $\delta(K_{1,n-1}) = 1$, then G cannot be a complete graph or $K_{1,n-1}$. In such a situation, the equation in **Lemma 2.3** is invalid. By **Lemma 2.3**, $\frac{24}{n-1}+2n-8 \leq q(G) < \frac{2m}{n-1}+n-2$, i.e., $m > \binom{n-3}{2}+9$.

By **Theorem 3.1**, G is a pancyclic graph or a bipartite graph or $G \in NP_1$. It can be calculated that the graphs of $(3K_1 + K_{n-6}) \vee K_3$,$(4K_1 + K_3) \vee K_4$,$(6K_1 + K_2) \vee K_6$,$9K_1 \vee K_8$ satisfy $m = \binom{n-3}{2} + 9$, which is contradiction. Next, we will study whether the graphs of No.19-69 satisfy the signless Laplacian spectral radius condition in the **Theorem 3.3**. Note that $G \in NP_2$.

Take $(2K_1 \vee K_4) \vee 7K_1$ as an example:

Let $\mathbf{X} = (X_1, X_2, \ldots\ldots\ldots, X_{13})^T$ be the eigenvector corresponding to $\mu(G)$, where $X_i(1 \leq i \leq 2)$ corresponds to the vertex of degree 11, $X_i(3 \leq i \leq 6)$ corresponds to the vertex of degree 12 and $X_i(7 \leq i \leq 13)$ corresponds to the vertex of degree 6. By eigen-equation (2.2), then we have

$$\begin{cases} X_1 = X_2, X_3 = X_4 = X_5 = X_6, X_7 = X_8 = X_9 = X_{10} = X_{11} = X_{12} = X_{13} \\ (q(G) - 11)X_1 = 4X_3 + 7X_7 \\ (q(G) - 12)X_3 = 2X_1 + 3X_3 + 7X_7 \\ (q(G) - 6)X_7 = 2X_1 + 4X_3 \end{cases}$$

Transform the above equations into the matrix equation $(Q^{'}(G) - q(G)I)X^{'} = 0$, where $X^{'} = (X_1, X_3, X_7)^T$ and

$$Q^{'}(G) = \begin{pmatrix} 11 & 4 & 7 \\ 2 & 15 & 7 \\ 2 & 4 & 6 \end{pmatrix}.$$

Let $f(x) = |xI - Q^{'}(G)| = x^3 - 32x^2 + 271x - 536$, then the maximum root of $g(x) = 0$ is $q(G) = 19.528$. It can be calculated that $q(G) < \frac{24}{13-1} + 2 \times 13 - 8$, which is contradiction. The remaining graphs are studied in the same way, and the results are shown in *Table* 7.

Table 7. The spectral radius and the signless Laplacian spectral radius of G.

G	$\mu(G)$	$\sqrt{(n^2-8n+31)}$	$q(G)$	$\frac{24}{n-1}+2n-8$
No.19	11.0623	11.6619	23.4121	23.71429
No.20	10.9484	11.6619	23.2009	23.71429
No.21	10.8783	11.6619	21.9831	23.71429
No.22	10.8374	11.6619	23.0178	23.71429
No.23	10.7164	11.6619	22.6516	23.71429
No.24	9.3157	9.7980	19.7584	20
No.25	9.2660	9.7980	19.6332	20
No.26	9.2350	9.7980	19.5281	20
No.27	9.1467	9.7980	19.4616	20
No.28	9.1884	9.7980	19.5522	20
No.29	9.2049	9.7980	19.5364	20
No.30	9.0474	9.7980	19.1235	20
No.31	9.4462	9.7980	20	20
No.32	9.0325	9.7980	19.3143	20
No.33	9.0301	9.7980	19.2240	20
No.34	9.0047	9.7980	19.1492	20
No.35	8.8422	9.7980	18.7149	20
No.36	9.0652	9.7980	19.3759	20
No.37	8.9237	9.7980	18.9375	20
No.38	9.0735	9.7980	19.3203	20
No.39	8.9678	9.7980	19.0391	20
No.40	8.3530	8.888194	17.8639	18.18182
No.41	8.2145	8.888194	17.5985	18.18182
No.42	8.2450	8.888194	17.7460	18.18182
No.43	8.1124	8.888194	17.3128	18.18182
No.44	7.8310	8	16.5887	16.4
No.45	7.6196	8	16.1652	16.4
No.46	7.6779	8	16.3062	16.4
No.47	7.5826	8	16.0352	16.4
No.48	7.4816	8	15.9748	16.4
No.49	7.5284	8	16.0698	16.4
No.50	7.5528	8	16.0487	16.4
No.51	7.3589	8	15.5484	16.4
No.52	7.3507	8	15.8151	16.4
No.53	7.3171	8	15.6034	16.4
No.54	7.1207	8	15.0772	16.4
No.55	7.3840	8	15.8721	16.4
No.56	7.2137	8	15.3351	16.4

G	$\mu(G)$	$\sqrt{(n^2-8n+31)}$	$q(G)$	$\frac{24}{n-1}+2n-8$
No.57	7.3968	8	15.7967	16.4
No.58	7.2647	8	15.4474	16.4
No.59	6.7573	7.1414	14.4721	14.6667
No.60	6.5955	7.1414	14.1578	14.6667
No.61	6.6235	7.1414	14.3246	14.6667
No.62	6.4681	7.1414	13.8041	14.6667
No.63	6.2170	6.3246	13.1789	13
No.64	5.9612	6.3246	12.6769	13
No.65	6.0322	6.3246	12.8381	13
No.66	5.9150	6.3246	12.5052	13
No.67	5.7980	6.3246	12.4641	13
No.68	5.8503	6.3246	12.5601	13
No.69	5.6362	6.3246	11.8807	13

From $Table$ 7, all graphs in NP_2 except $G = 7K_1 \vee K_6$ and $G = 6K_1 \vee K_5$ and $G = 5K_1 \vee K_4$ satisfy $q(G) < \frac{24}{n-1} + 2n - 8$, a contradiction.
The proof is complete.$\diamond$

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Conflict of interest

The author declare that he has no competing interest.

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Appendix

The maximum length cycle $L(G)$ of G.

No.	G	Order	$L(G)$	$e(G)$	$\frac{(n-1)^2}{4}+1$
1	$(3K_1+K_{n-6})\vee K_3$	n	C_{n-1}	$\frac{n^2-7n+30}{2}$	$\frac{(n-1)^2}{4}+1$
2	$(4K_1+K_3)\vee K_4$	11	C_{10}	37	26
3	$(6K_1+K_2)\vee K_6$	14	C_{13}	64	43.25
4	$9K_1\vee K_8$	17	C_{16}	100	65
5	$K_6\vee(K_2+K_{1,6})$	15	C_{15}	76	50
6	$K_6\vee(2K_2+K_{1,4})$	15	C_{15}	75	50
7	$(K_1+K_{1,5}+K_2)\vee K_6$	15	C_{15}	75	50
8	$(2K_2+K_{1,3})\vee K_5$	13	C_{13}	55	37
9	$4K_2\vee K_5$	13	C_{13}	54	37
10	$(K_1+K_{1,2}+2K_2)\vee K_5$	13	C_{13}	54	37
11	$K_4\vee(K_2\vee 4K_1+K_3)$	13	C_{13}	54	37

No.	G	Order	$L(G)$	$e(G)$	$\frac{(n-1)^2}{4}+1$
12	$K_3 \vee 2K_1 \vee (K_{1,3} + 2K_2)$	13	C_{13}	54	37
13	$K_4 \vee (K_2 \vee 5K_1 + K_2)$	13	C_{13}	54	37
14	$K_4 \vee (K_{1,2} + 2K_2)$	11	C_{11}	38	26
15	$(K_1 + 3K_2) \vee K_4$	11	C_{11}	37	26
16	$(K_{1,2} + 2K_2) \vee 2K_1 \vee K_2$	11	C_{11}	37	26
17	$K_3 \vee (K_1 \vee (2K_1 + K_2) + K_2)$	10	C_{10}	30	21.25
18	$3K_2 \vee K_3$	9	C_9	24	17
19	$8K_1 \vee K_7$	15	C_{14}	77	50
20	$(K_1 + K_{1,7}) \vee K_6$	15	C_{14}	76	50
21	$K_{2,8} \vee K_5$	15	C_{14}	76	50
22	$K_6 \vee (2K_1 + K_{1,6})$	15	C_{14}	75	50
23	$K_4 \vee 2K_1 \vee (K_2 + K_{1,6})$	15	C_{11}	75	50
24	$K_5 \vee (K_1 + K_{1,6})$	13	C_{11}	56	37
25	$(K_2 + K_{1,5}) \vee K_5$	13	C_{12}	56	37
26	$(2K_1 \vee K_4) \vee 7K_1$	13	C_{12}	56	37
27	$(K_1 + K_2 + K_{1,4}) \vee K_5$	13	C_{12}	55	37
28	$(2K_1 + K_{1,5}) \vee K_5$	13	C_{11}	55	37
29	$(K_1 + K_2 \vee 6K_1) \vee K_4$	13	C_{11}	54	37
30	$(K_{1,5} + K_2) \vee 2K_1 \vee K_3$	13	C_{12}	55	37
31	$7K_1 \vee K_6$	13	C_{12}	57	37
32	$K_5 \vee (2K_1 + K_{1,3} + K_2)$	13	C_{12}	54	37
33	$K_4 \vee [K_1 + K_1 \vee (K_{1,4} + K_2)]$	13	C_{12}	54	37
34	$K_4 \vee (K_1 + 2K_1 \vee 6K_1)$	13	C_{12}	54	37
35	$K_3 \vee K_{3,7}$	13	C_{12}	54	37
36	$K_5 \vee (3K_1 + K_{1,4})$	13	C_{11}	54	37
37	$K_3 \vee 2K_1 \vee (K_1 + K_{1,4} + K_2)$	13	C_{12}	54	37
38	$K_4 \vee [K_1 \vee (K_1 + K_{1,5}) + K_1]$	13	C_{11}	54	37
39	$K_3 \vee 2K_1 \vee (2K_1 + K_{1,5})$	13	C_{11}	54	37
40	$(5K_1 + K_2) \vee K_5$	12	C_{11}	46	31.25
41	$K_4 \vee (K_1 \vee (4K_1 + K_2) + K_1)$	12	C_{11}	45	31.25
42	$7K_1 \vee K_5$	12	C_{10}	45	31.25
43	$(5K_1 + K_2) \vee 2K_1 \vee K_3$	12	C_{11}	45	31.25
44	$6K_1 \vee K_5$	11	C_{10}	40	26
45	$K_4 \vee (K_{1,4} + K_2)$	11	C_{10}	39	26
46	$K_4 \vee (K_1 + K_{1,5})$	11	C_{10}	39	26
47	$K_3 \vee K_{2,6}$	11	C_{10}	39	26
48	$K_4 \vee (K_1 + K_{1,3} + K_2)$	11	C_{10}	38	26
49	$K_4 \vee (2K_1 + K_{1,4})$	11	C_9	38	26
50	$K_3 \vee [(K_2 \vee 5K_1) + K_1]$	11	C_{10}	38	26
51	$K_2 \vee 2K_1 \vee (K_{1,4} + K_2)$	11	C_{10}	38	26
52	$K_4 \vee (K_2 + K_{1,2} + 2K_1)$	11	C_{10}	37	26

No.	G	Order	$L(G)$	$e(G)$	$\frac{(n-1)^2}{4}+1$
53	$K_3 \vee (K_1 + K_{2,5})$	11	C_{10}	37	26
54	$K_2 \vee K_{3,6}$	11	C_{10}	37	26
55	$K_4 \vee (K_{1,3} + 3K_1)$	11	C_9	37	26
56	$K_2 \vee 2K_1 \vee (K_1 + K_{1,3} + K_2)$	11	C_{10}	37	26
57	$[K_1 \vee (K_{1,4} + K_1) + K_1] \vee K_3$	11	C_9	37	26
58	$K_2 \vee 2K_1 \vee (2K_1 + K_{1,4})$	11	C_9	37	26
59	$(4K_1 + K_2) \vee K_4$	10	C_9	31	21.25
60	$K_3 \vee [K_1 + K_1 \vee (3K_1 + K_2)]$	10	C_9	30	21.25
61	$6K_1 \vee K_4$	10	C_8	30	21.25
62	$(4K_1 + K_2) \vee 2K_1 \vee K_2$	10	C_9	30	21.25
63	$5K_1 \vee K_4$	9	C_8	26	17
64	$(K_{1,3} + K_2) \vee K_3$	9	C_8	25	17
65	$(K_1 + K_{1,4}) \vee K_3$	9	C_7	25	17
66	$5K_1 \vee 2K_1 \vee K_2$	9	C_8	25	17
67	$(K_1 + K_{1,2} + K_2) \vee K_3$	9	C_8	24	17
68	$(2K_1 + K_{1,3}) \vee K_3$	9	C_7	24	17
69	$(K_{1,3} + K_2) \vee 2K_1 \vee K_1$	10	C_8	24	17