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Research article

Exact explicit nonlinear wave solutions to a modified cKdV equation

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Abstract: In this paper, we study nonlinear wave solutions to a modified cKdV equation by exploiting Bifurcation method of Hamiltonian systems. We identify all possible bifurcation conditions and obtain the phase portraits of the system in different regions of the parametric space, through which, we obtain exact explicit nonlinear wave solutions, including solitary wave solutions, singular wave solutions, periodic singular wave solutions, and kink (antikink) wave solutions. Of particular interest is the appearance of the so-called V-shaped kink (antikink) wave solutions, W-shaped solitary wave solutions, and W-shaped periodic wave solutions, which were not found in previous studies.

Keywords: modified cKdV equation; bifurcation; V-shaped kink (antikink) wave solutions; W-shaped solitary wave solutions; W-shaped periodic wave solutions
Mathematics Subject Classification: 34C60, 35B3, 35C07

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1. Introduction

As is well known, the Korteweg-de Vries (KdV) equation has attracted much attention due to its significant nature in physical contexts, stratified internal waves, ion-acoustic wave, plasma physics [1]. Many generalized forms of KdV equation have been introduced, such as modified KdV (mKdV) equation and high-order KdV equation. Besides, There has been considerate studies on time-delayed KdV-related equation. Zhao and Xu [2] dealt with the existence of solitary waves for KdV equation with time delay. Liu et al. [3] studied the KdV-Burgers-Kuramoto chaotic system with distributed delay feedback and analyzed the conditions under which Hopf bifurcation occurs. Baudouin et al. [4] employed two approaches to study the stability of the nonlinear KdV equation with boundary time-delay feedback. Komornik and Pignotti [5] considered well-posedness and exponential decay estimates for a KdV-Burgers equation with time-delay.

In this paper, we focus on the following modified coupled Korteweg-de Vries (cKdV) equation,

$$\begin{cases} u_t = v_x - \frac{3}{2}uu_x + \alpha u_x, \\ v_t = \frac{1}{4}u_{xxx} - vu_x - \frac{1}{2}uv_x + \alpha v_x, \end{cases} \tag{1.1}$$

where α is a constant, which was introduced in [6]. As suggested in [6–9], Eq (1.1) is a general example of N-component systems, energy dependent schrödinger operators and bi-Hamiltonian structures for multi-component systems. The authors in [6] studied the soliton solutions to Eq (1.1), and demonstrated the soliton fission effect, kink to anti-kink transitions, and multipeaked solitons by using a class of commuting Hamiltonian systems on Riemann surfaces. Additionally, they indicated that many important equations that model physical phenomena in fluid dynamics and nonlinear optics, such as the generalized Kaup equation, the classical Boussinesq equation and the systems governing second harmonic generation (SHG), are connected to the cKdV equation through nonsingular transformations [6], which potentially enables solutions of cKdV equations to be interpreted in the context of these related equations. Besides, Wen and Wang [10] constructed some exact explicit solutions to Eq (1.1) by employing the three forms of (ω/g)-expansion method, i.e., (g'/g^2)-expansion method, (g'/g)-expansion method and (g')-expansion method.

In this paper, we further study nonlinear wave solutions to Eq (1.1) from the perspective of dynamical systems [11–28]. More preciously, we first transform Eq (1.1) into a planar system, identify all possible bifurcation conditions, and obtain the phase portraits of the system in the different regions of the parametric space. Further, we seek to obtain exact explicit expressions of nonlinear wave solutions to Eq (1.1), including solitary wave solutions, singular wave solutions, periodic singular wave solutions, and kink (antikink) wave solutions. More interestingly, we find the so-called V-shaped kink (antikink) wave solutions, W-shaped solitary wave solutions, and W-shaped periodic wave solutions. To improve the readability of the paper, we give the definitions of these kinds of solutions here. Suppose that $\varphi(\xi)$ with $\xi = x - ct$ is a traveling wave solution of the equation. Solitary wave solution $\varphi(\xi)$ means that $\varphi(\xi)$ satifies $\lim_{\xi\to\pm\infty}\varphi(\xi)$ exist and $\lim_{\xi\to+\infty}\varphi(\xi) = \lim_{\xi\to-\infty}\varphi(\xi)$. Singular wave solution $\varphi(\xi)$ means that $\varphi(\xi)$ blow up at some point, that is, there exists some point ξ_0 such that $\lim_{\xi\to\xi_0}\varphi(\xi) = \pm\infty$. Periodic singular wave solution $\varphi(\xi)$ means that $\varphi(\xi)$ blow up at some points periodically. Kink (antikink) wave solution $\varphi(\xi)$ means that $\varphi(\xi)$ satifies $\lim_{\xi\to\pm\infty}\varphi(\xi)$ exist and $\lim_{\xi\to+\infty}\varphi(\xi) > \lim_{\xi\to-\infty}\varphi(\xi)$ ($\lim_{\xi\to+\infty}\varphi(\xi) < \lim_{\xi\to-\infty}\varphi(\xi)$). V-shaped kink (antikink) wave solution $\varphi(\xi)$ means that $\varphi(\xi)$ is a kink (antikink) wave solution with one valley, such as Figure 4(b) (4(c)). W-shaped solitary wave solution $\varphi(\xi)$ means that $\varphi(\xi)$ is a solitary wave solution with one hump and two valleys, such as Figure 4(b). W-shaped periodic solitary wave solution $\varphi(\xi)$ means that $\varphi(\xi)$ is a W-shaped solitary wave solution and it is periodic, such as Figure 5(b) and 5(d).

The remaining paper is organized as follows. In section 2, we outline the procedure of identifying the bifurcation conditions and obtaining the phase portraits of the corresponding planar system in the different regions of the parametric space. In section 3, we present the main results about exact explicit of nonlinear wave solutions to Eq (1.1), especially, the V-shaped kink (antikink) wave solutions, W-shaped solitary wave solutions, and W-shaped periodic wave solutions, and the proof follows. Section 4 is devoted to numerical simulations of solutions. The paper is ended with the conclusion.

2. Bifurcation conditions and phase portraits

In this section, we identify the bifurcation conditions and derive the phase portraits corresponding to Eq (1.1).

For given constant c, substituting $u(x,t) = \mathbf{u}(\xi)$, $v(x,t) = \mathbf{v}(\xi)$ with $\xi = x - ct$ into Eq (1.1), it follows,

$$\left\{\begin{array}{l} c\mathbf{u}' + \mathbf{v}' - \frac{3}{2}\mathbf{u}\mathbf{u}' + \alpha\mathbf{u}' = 0, \\ c\mathbf{v}' + \frac{1}{4}\mathbf{u}''' - \mathbf{v}\mathbf{u}' - \frac{1}{2}\mathbf{u}\mathbf{v}' + \alpha\mathbf{v}' = 0, \end{array}\right. \tag{2.1}$$

where the prime stands for the derivative with respect to ξ.

Integrating the first equation of (2.1) once leads to

$$\mathbf{v} = \frac{3}{4}\mathbf{u}^2 - (c+\alpha)\mathbf{u} + \frac{g_1}{4}, \tag{2.2}$$

where g_1 is integral constant.

Substituting (2.2) into the second equation of (2.1) and integrating the equation, it follows that

$$\mathbf{u}'' - 2\mathbf{u}^3 + 6(c+\alpha)\mathbf{u}^2 - 4(c+\alpha)^2\mathbf{u} - g_1\mathbf{u} + g_2 = 0,$$

where g_2 is integral constant.

Letting $y = \mathbf{u}'$, we obtain a planar system

$$\left\{\begin{array}{l} \frac{d\mathbf{u}}{d\xi} = y, \\ \frac{dy}{d\xi} = 2\mathbf{u}^3 - 6(c+\alpha)\mathbf{u}^2 + (4(c+\alpha)^2 + g_1)\mathbf{u} - g_2. \end{array}\right. \tag{2.3}$$

By setting $\varphi = \mathbf{u} - (c+\alpha)$, system (2.3) becomes

$$\left\{\begin{array}{l} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = 2\varphi^3 + \left(g_1 - 2(c+\alpha)^2\right)\varphi + g_1(c+\alpha) - g_2, \end{array}\right. \tag{2.4}$$

with Hamiltonian

$$H(\varphi, y) = \frac{y^2}{2} - \frac{\varphi^4}{2} - \frac{1}{2}\left(g_1 - 2(c+\alpha)^2\right)\varphi^2 - (g_1(c+\alpha) - g_2)\varphi. \tag{2.5}$$

To study the singular points of system (2.5), let

$$f(\varphi) = 2\varphi^3 + \left(g_1 - 2(c+\alpha)^2\right)\varphi + g_1(c+\alpha) - g_2, \tag{2.6}$$

and

$$f_0(\varphi) = 2\varphi^3 + \left(g_1 - 2(c+\alpha)^2\right)\varphi. \tag{2.7}$$

Obviously, if $g_1 < 2(c+\alpha)^2$, then $f_0(\varphi)$ has three zero points,

$$\varphi = 0, \pm\varphi_1, \tag{2.8}$$

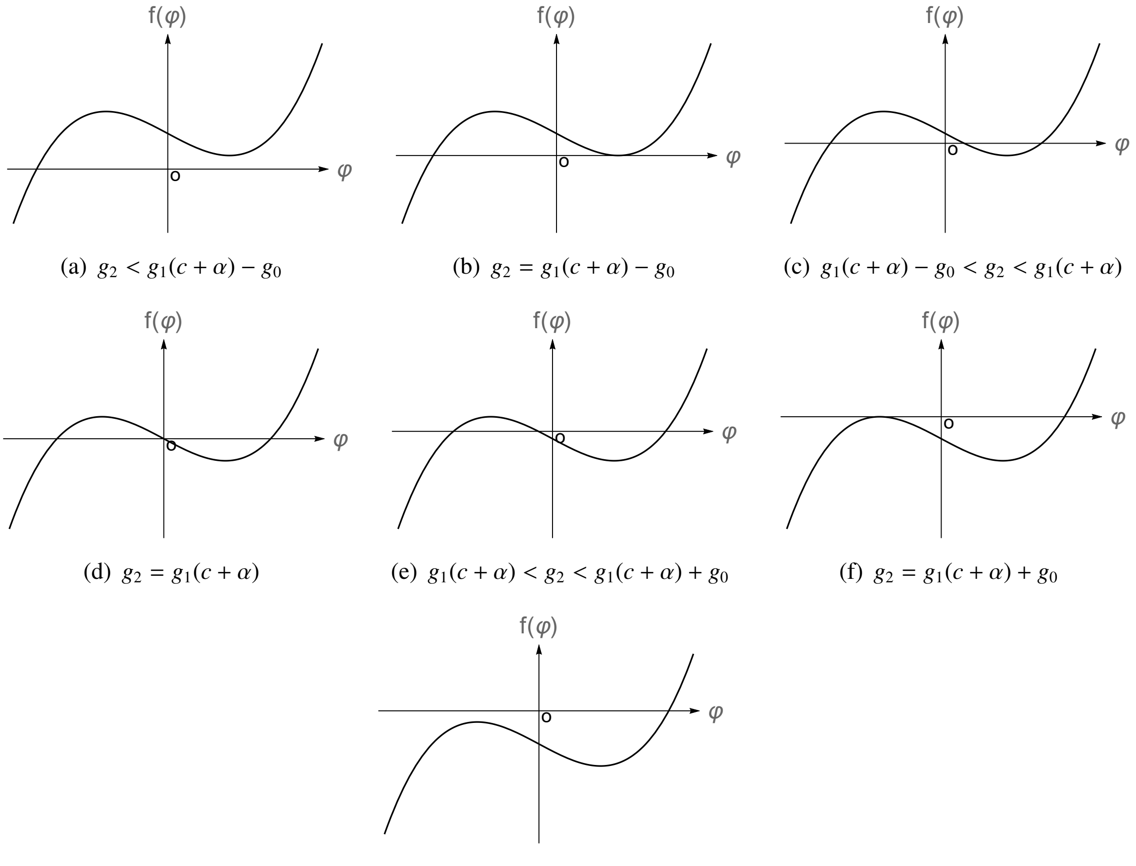
where $\varphi_1 = \sqrt{\frac{1}{2}\left(2(c+\alpha)^2 - g_1\right)}$. In addition, it is easy to obtain the two extreme points of $f(\varphi)$ as follows

$$\varphi_{\pm}^* = \sqrt{\frac{1}{6}\left(2(c+\alpha)^2 - g_1\right)}. \tag{2.9}$$

Let

$$g_0 = \left|f_0\left(\varphi_\pm^*\right)\right| = \sqrt{\frac{2}{27}\left(2(c+\alpha)^2 - g_1\right)^3}, \tag{2.10}$$

which denotes the absolute value of extreme values of $f_0(\varphi)$. Now we can easily give the profiles of $f(\varphi)$ in Figure 1.



(a) $g_2 < g_1(c+\alpha) - g_0$ (b) $g_2 = g_1(c+\alpha) - g_0$ (c) $g_1(c+\alpha) - g_0 < g_2 < g_1(c+\alpha)$

(d) $g_2 = g_1(c+\alpha)$ (e) $g_1(c+\alpha) < g_2 < g_1(c+\alpha) + g_0$ (f) $g_2 = g_1(c+\alpha) + g_0$

(g) $g_2 > g_1(c+\alpha) + g_0$

Figure 1. The profiles of $f(\varphi)$.

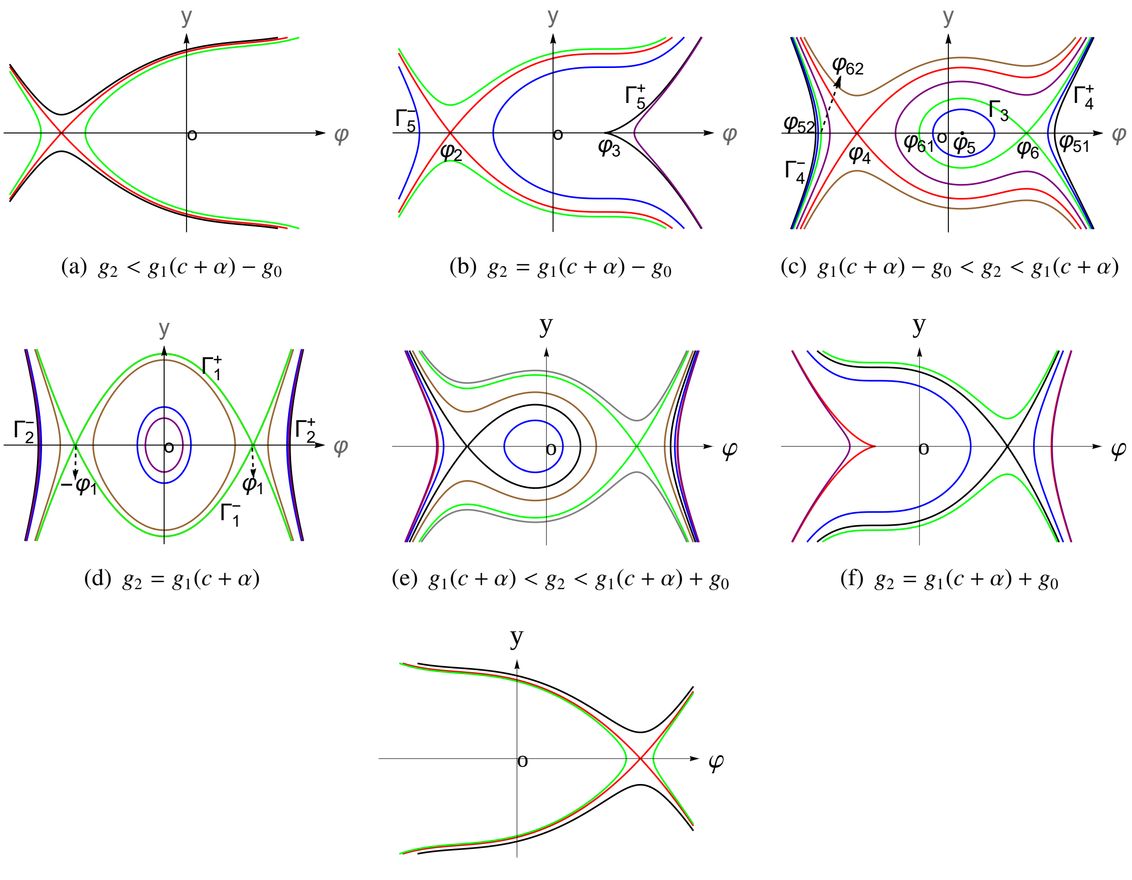
Let $(\varphi_i, 0)$ be one of the singular points of system (2.4). Then the characteristic values of the linearized system of system (2.4) at the singular point $(\varphi_i, 0)$ are

$$\lambda_\pm = \pm\sqrt{f'(\varphi_i)}. \tag{2.11}$$

From Figure 1 and Eq (2.11), we can determine the number of singular points and their dynamical behaviors (saddles, centers, etc.). Furthermore, note that when $g_2 = g_1(c+\alpha)$, we have $H(\varphi_1, 0) = H(-\varphi_1, 0)$, which indicates that the two saddles $(\pm\varphi_1, 0)$ connect.

Based on the above analysis, we obtain the phase portraits of system (2.4) in Figure 2. Note that in the phase portraits $\varphi_2 = -2\sqrt{\frac{1}{6}\left(2(c+\alpha)^2 - g_1\right)}$ and the other φ_is are given in Section 3.

Remark 1. *In the above analysis, we have supposed that* $g_1 < 2(c+\alpha)^2$*. In fact, when* $g_1 \geq 2(c+\alpha)^2$*, system (2.4) has only one saddle and the phase portrait is similar with Figure 2(a) or 2(g). Therefore, we omit the case when* $g_1 \geq 2(c+\alpha)^2$*.*



(a) $g_2 < g_1(c+\alpha) - g_0$ (b) $g_2 = g_1(c+\alpha) - g_0$ (c) $g_1(c+\alpha) - g_0 < g_2 < g_1(c+\alpha)$

(d) $g_2 = g_1(c+\alpha)$ (e) $g_1(c+\alpha) < g_2 < g_1(c+\alpha) + g_0$ (f) $g_2 = g_1(c+\alpha) + g_0$

(g) $g_2 > g_1(c+\alpha) + g_0$

Figure 2. The phase portraits of system (2.4).

3. Main results and the theoretic derivations of the main results

In this section, we state our results about solitary wave solutions, singular wave solutions, periodic singular wave solutions, and kink (antikink) wave solutions for the component $u(x,t)$ of Eq (1.1), and especially, we emphasize the results of the V-shaped kink (antikink) wave solution, W-shaped solitary wave solution, and W-shaped periodic wave solution for the component $v(x,t)$ of Eq (1.1). Note that the relation between the solutions of Eq (1.1) and the solutions of system (2.4) can be derived through the transformations $u(x,t) = \mathbf{u}(\xi)$ and $\varphi(\xi) = \mathbf{u}(\xi) - (c+\alpha)$ with $\xi = x - ct$. To relate conveniently, we also omit the expression of the component $v(x,t)$ of Eq (1.1), i.e., $v(x,t) = \frac{3}{4}u^2(x,t) - (c+\alpha)u(x,t) + \frac{g_1}{4}$, in the following theorems.

Theorem 1. *When $g_2 = g_1(c+\alpha)$, Eq (1.1) has two kink (antikink) wave solutions*

$$u(x,t) = \pm\varphi_1 \tanh\left(\varphi_1(x-ct)\right) + c + \alpha, \tag{3.1}$$

two singular wave solutions

$$u(x,t) = \pm\varphi_1 \coth\left(\varphi_1(x-ct)\right) + c + \alpha, \tag{3.2}$$

four periodic singular wave solutions

$$u(x,t) = \pm\sqrt{2}\varphi_1 \csc\left(\sqrt{2}\varphi_1(x-ct)\right) + c + \alpha, \tag{3.3}$$

$$u(x,t) = \pm\sqrt{2}\varphi_1 \sec\left(\sqrt{2}\varphi_1(x-ct)\right) + c + \alpha, \tag{3.4}$$

and a family of periodic wave solutions

$$u(x,t) = \frac{-2\alpha_1\alpha_2 sn^2\Big((x-ct)/\rho_1, \kappa_1\Big) + \alpha_1(\alpha_1+\alpha_2)}{2\alpha_1 sn^2\Big((x-ct)/\rho_1, \kappa_1\Big) - (\alpha_1+\alpha_2)}, \tag{3.5}$$

where $\rho_1 = \frac{2}{\alpha_1+\alpha_2}, \kappa_1 = \frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2}$, $\pm\alpha_1, \pm\alpha_2$ *are four roots of* $H(\varphi, y) = h, h \in (H(0,0), H(\varphi_1, 0))$*. In addition, when* $h \to H(\varphi_1, 0)$*, the periodic wave solution* (3.5) *tends to the kink (antikink) wave solutions* (3.1)*.*

Furthermore, if c, α*, and* g_1 *satisfy the condition* $-\varphi_1 < -\frac{c+\alpha}{3} < \varphi_1$*, then the component* $v(x,t) = \frac{3}{4}u^2(x,t) - (c+\alpha)u(x,t) + \frac{g_1}{4}$ *corresponding to the kink (antikink) wave solutions* (3.1) *are the V-shaped kink (antikink) wave solutions. In addition, if c,* α*,* g_1*, and h satisfy the condition* $-\alpha_1 < -\frac{c+\alpha}{3} < \alpha_1$*, then the component* $v(x,t) = \frac{3}{4}u^2(x,t) - (c+\alpha)u(x,t) + \frac{g_1}{4}$ *corresponding to the periodic wave solution* (3.5) *is the W-shaped periodic wave solution.*

Proof . *When* $g_2 = g_1(c+\alpha)$*, we consider the following three types of orbits.*

(i) First, there are two heteroclinic orbits $\Gamma_1^\pm$ *connected at two saddle points* $(\varphi_1, 0)$ *and* $(-\varphi_1, 0)$ *from Figure 2(d), which can be expressed, from (2.5),*

$$y = \pm\left(\varphi_1^2 - \varphi^2\right). \tag{3.6}$$

Substituting (3.6) into the first equation of system (2.4), and integrating along the heteroclinic orbits, it follows that

$$\int_0^{\varphi} \frac{\mathrm{d}s}{\varphi_1^2 - s^2} = |\xi|, \tag{3.7}$$

and

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{s^2 - \varphi_1^2} = |\xi|. \tag{3.8}$$

From (3.7) and (3.8), we get two kink (antikink) wave solutions (3.1) and two singular wave solutions (3.2).

(ii) Second, from Figure 2(d), there are two special orbits $\Gamma_2^\pm$*, which have the same hamiltonian with that of the center point* $(0,0)$*. In* (φ, y)*−plane, from (2.5), the expressions of these two orbits* $\Gamma_2^\pm$ *are given as*

$$y = \pm\varphi\sqrt{\varphi^2 - 2\varphi_1^2}. \tag{3.9}$$

Substituting (3.9) into the first equation of system (2.4), and integrating along the two orbits $\Gamma_2^\pm$*, it follows that*

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{s\sqrt{s^2 - 2\varphi_1^2}} = |\xi|. \tag{3.10}$$

From (3.10), we get two periodic singular wave solutions (3.3).

Further, note that if $\varphi = \varphi(\xi)$ is a solution of system , then $\varphi = \varphi(\xi + \gamma)$ is also a solution of system. Specially, we take $\gamma = \frac{\pi}{2}$, we obtain another two periodic singular solutions (3.4).

(iii) Third, from Figure 2(d) or more specifically Figure 3(a), there exist one family of periodic orbits defined by $H(\varphi, y) = h, h \in (H(0, 0), H(\varphi_1, 0))$, the expressions of which are given by

$$y = \pm\sqrt{(\alpha_2^2 - \varphi^2)(\alpha_1^2 - \varphi^2)}. \tag{3.11}$$

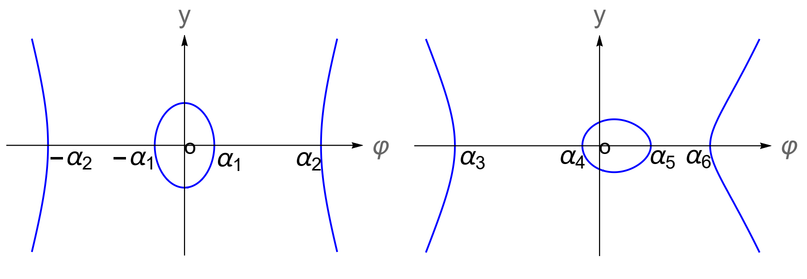
Substituting (3.11) *into the first equation of system* (2.4)*, and integrating along the families of periodic orbits, it follows that*

$$\int_{-\alpha_1}^{\varphi} \frac{1}{\sqrt{(\alpha_2^2 - s^2)(\alpha_1^2 - s^2)}} ds = |\xi|. \tag{3.12}$$

From (3.12)*, we derive the family of periodic wave solutions* (3.5) *by the elliptic integral [29].*

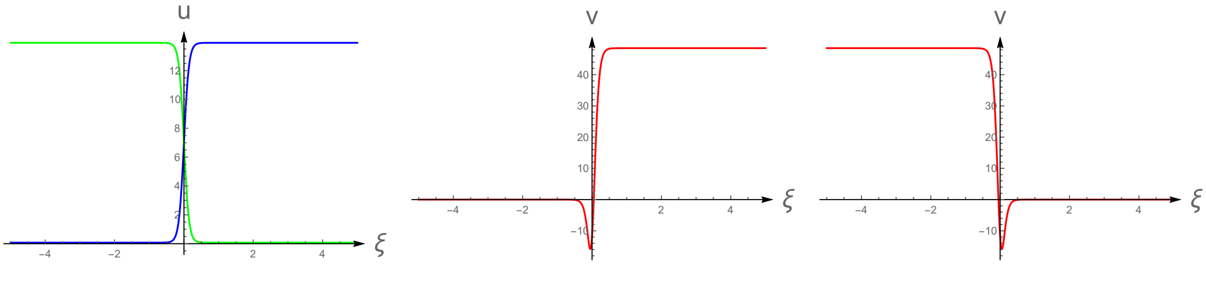
The convergence result follows from $\alpha_1 \to \varphi_1$, when $h \to H(\varphi_1, 0)$.

Furthermore, if c, α, and g_1 satisfy the condition $-\varphi_1 < -\frac{c+\alpha}{3} < \varphi_1$, then we have $-\varphi_1 + \frac{c+\alpha}{3} < 0 < \varphi_1 + \frac{c+\alpha}{3}$. Note that the component $u(x, t)$ in (3.1) *satisfies $u(x, t) \in (-\varphi_1 + c + \alpha, \varphi_1 + c + \alpha)$. It follows from the expression $v(x, t) = \frac{3}{4}u^2(x, t) - (c + \alpha)u(x, t) + \frac{g_1}{4} = \frac{3}{4}\left(u(x, t) - \frac{2}{3}(c + \alpha)\right)^2 + \frac{g_1}{4} - \frac{1}{3}(c + \alpha)^2$ that the component $v(x, t)$ corresponding to* (3.1) *is the V-shaped kink (antikink) wave solutions, the profiles of which are shown in Figure 4(b) and 4(c). In addition, if c, α, g_1, and h satisfy the condition $-\alpha_1 < -\frac{c+\alpha}{3} < \alpha_1$, then one has $-\alpha_1 + \frac{c+\alpha}{3} < 0 < \alpha_1 + \frac{c+\alpha}{3}$. Note that the component $u(x, t)$ in* (3.5) *satisfies $u(x, t) \in (-\alpha_1 + c + \alpha, \alpha_1 + c + \alpha)$. It follows from the expression $v(x, t) = \frac{3}{4}u^2(x, t) - (c + \alpha)u(x, t) + \frac{g_1}{4} = \frac{3}{4}\left(u(x, t) - \frac{2}{3}(c + \alpha)\right)^2 + \frac{g_1}{4} - \frac{1}{3}(c + \alpha)^2$ that the component $v(x, t)$ corresponding to* (3.5) *is the W-shaped periodic wave solution, the profile of which is shown in Figure 5(b).*



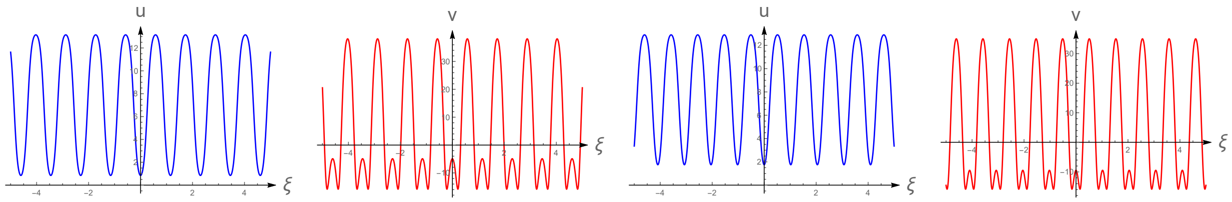
(a) $g_2 = g_1(c + \alpha)$ (b) $g_1(c + \alpha) - g_0 < g_2 < g_1(c + \alpha)$

Figure 3. The period orbits of system (2.4) under corresponding parameter conditions. (a) $g_2 = g_1(c + \alpha)$; (b) $g_1(c + \alpha) - g_0 < g_2 < g_1(c + \alpha)$.



(a) (b) (c)

Figure 4. The profiles of kink (antikink) wave solutions of Eq (1.1). (a) Kink (antikink) wave solutions (3.1); (b) V-shaped kink wave solution $v(x, t)$ corresponding to (3.1); (c) V-shaped antikink wave solution $v(x, t)$ corresponding to (3.1).



(a) $g_2 = g_1(c + \alpha)$ (b) $g_2 = g_1(c + \alpha)$ (c) $g_1(c+\alpha)-g_0 < g_2 < g_1(c+\alpha)$ (d) $g_1(c+\alpha)-g_0 < g_2 < g_1(c+\alpha)$

Figure 5. The period wave solutions of Eq (1.1) under corresponding parameter conditions. (a) Periodic wave solution (3.5); (b) W-shaped periodic wave solution $v(x, t)$ corresponding to (3.5); (c) Periodic wave solution (3.16); (d) W-shaped periodic wave solution $v(x, t)$ corresponding to (3.16).

Theorem 2. *When $g_1(c + \alpha) - g_0 < g_2 < g_1(c + \alpha)$, Eq (1.1) has one solitary wave solution*

$$u(x,t) = \varphi_6 - \frac{2\theta_1}{(\varphi_{61} - \varphi_{62}) \cosh\Big(\sqrt{\theta_1}(x-ct)\Big) + 4\varphi_6} + c + \alpha, \tag{3.13}$$

one singular wave solution

$$u(x,t) = \varphi_6 + \frac{\theta_1}{2\varphi_6 \cosh\Big(\sqrt{\theta_1}(x-ct)\Big) - \sqrt{\theta_1}\sinh\Big(\sqrt{\theta}(x-ct)\Big) - 2\varphi_6} + c + \alpha, \tag{3.14}$$

where $\theta_1 = (\varphi_6 - \varphi_{61})(\varphi_6 - \varphi_{62})$, one periodic singular wave solution

$$u(x,t) = \varphi_5 - \frac{2\theta_2}{(\varphi_{51} - \varphi_{52}) \sin\Big(\arcsin\Big(\frac{4\varphi_5}{\varphi_{51}-\varphi_{52}}\Big) - \sqrt{\theta_2}(x-ct)\Big) - 4\varphi_5} + c + \alpha, \tag{3.15}$$

where $\theta_2 = (\varphi_{51} - \varphi_5)(\varphi_5 - \varphi_{52})$, and a family of periodic wave solutions

$$u(x,t) = \frac{\alpha_3(\alpha_5 - \alpha_4)sn^2\Big((x-ct)/\rho_2, \kappa_2\Big) - \alpha_4(\alpha_5 - \alpha_3)}{(\alpha_5 - \alpha_4)sn^2\Big((x-ct)/\rho_2, \kappa_2\Big) - (\alpha_5 - \alpha_3)}, \tag{3.16}$$

where $\rho_2 = \frac{2}{\sqrt{(\alpha_6-\alpha_4)(\alpha_5-\alpha_3)}}, \kappa_2 = \sqrt{\frac{(\alpha_5-\alpha_4)(\alpha_6-\alpha_3)}{(\alpha_6-\alpha_4)(\alpha_5-\alpha_3)}}$ *and* $\alpha_3, \alpha_4, \alpha_5, \alpha_6$ *are four roots of* $H(\varphi, y) = h, h \in (H(\varphi_5, 0), H(\varphi_6, 0))$*. In addition, when* $h \to H(\varphi_6, 0)$*, the periodic wave solution* (3.16) *tends to the solitary wave solution* (3.13)*.*

Furthermore, if c*,* α*,* g_1*, and* g_2 *satisfy the condition* $\varphi_{61} < -\frac{c+\alpha}{3} < \varphi_6$*, then the component* $v(x,t) = \frac{3}{4}u^2(x,t) - (c+\alpha)u(x,t) + \frac{g_1}{4}$ *corresponding to* (3.13) *is the W-shaped solitary wave solution. In addition, if* c*,* α*,* g_1*,* g_2 *and* h *satisfy the condition* $\alpha_4 < -\frac{c+\alpha}{3} < \alpha_5$*, then the component* $v(x,t) = \frac{3}{4}u^2(x,t) - (c+\alpha)u(x,t) + \frac{g_1}{4}$ *corresponding to* (3.16) *is the W-shaped periodic wave solution.*

Proof . *When* $g_1(c+\alpha) - g_0 < g_2 < g_1(c+\alpha)$*, denote the maximum zero of* $f(\varphi)$ *as*

$$\varphi_6 = \sqrt{\frac{\lambda}{6}\left(2(c+\alpha)^2 - g_1\right)},$$

where λ *is a parameter with* $1 < \lambda < 3$*, then we can obtain the other two zeros of* $f(\varphi)$

$$\varphi_4 = -\frac{1}{2}\sqrt{\frac{\lambda}{6}\left(2(c+\alpha)^2 - g_1\right)}\left(\sqrt{\frac{12}{\lambda} - 3} + 1\right),$$

$$\varphi_5 = \frac{1}{2}\sqrt{\frac{\lambda}{6}\left(2(c+\alpha)^2 - g_1\right)}\left(\sqrt{\frac{12}{\lambda} - 3} - 1\right),$$

we have,

(i) First, there is a homoclinic orbit Γ_3*, which passes the saddle point* $(\varphi_6, 0)$ *in Figure 2(c). In* $(\varphi, y)-$*plane, from (2.5), the expressions of the homoclinic orbit are given as*

$$y = \pm\sqrt{(\varphi_6 - \varphi)^2(\varphi - \varphi_{61})(\varphi - \varphi_{62})}, \ \varphi_{62} < \varphi_{61} < \varphi < \varphi_6. \tag{3.17}$$

Substituting (3.17) into the first equation of system (2.4), and integrating along the homoclinic orbit, it follows that

$$\int_{\varphi_{61}}^{\varphi} \frac{\mathrm{d}s}{(\varphi_6 - s)\sqrt{(s - \varphi_{61})(s - \varphi_{62})}} = |\xi|, \tag{3.18}$$

and

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{(s - \varphi_6)\sqrt{(s - \varphi_{61})(s - \varphi_{62})}} = |\xi|. \tag{3.19}$$

From (3.18) and (3.19), we get the solitary wave solutions (3.13) and the singular wave solution (3.14).

(ii) Second, from Figure 2(c), there are another two special orbits $\Gamma_4^{\pm}$*, which have the same hamiltonian with that of the center point* $(\varphi_5, 0)$*. In* $(\varphi, y)-$*plane, from (2.5), the expressions of these two orbits* $\Gamma_4^{\pm}$ *are given as*

$$y = \pm\sqrt{(\varphi - \varphi_5)^2(\varphi - \varphi_{51})(\varphi - \varphi_{52})}, \ \varphi_{52} < \varphi_5 < \varphi_{51}, \tag{3.20}$$

where

$$\varphi_{51} = \sqrt{\frac{\lambda}{6}\left(2(c+\alpha)^2 - g_1\right)}\left(\sqrt{\sqrt{\frac{\lambda}{12} - 3} + 1} - \frac{1}{2}\left(\sqrt{\frac{12}{\lambda} - 3} - 1\right)\right),$$

and

$$\varphi_{52} = -\sqrt{\frac{\lambda}{6}\left(2(c+\alpha)^2 - g_1\right)}\left(\sqrt{\sqrt{\frac{\lambda}{12}-3}+1} + \frac{1}{2}\left(\sqrt{\frac{12}{\lambda}-3}-1\right)\right).$$

Substituting (3.20) into the first equation of system (2.4), and integrating along these two special orbits $\Gamma_4^{\pm}$, *it follows that*

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{(s-\varphi_5)\sqrt{(s-\varphi_{51})(s-\varphi_{52})}} = |\xi|. \tag{3.21}$$

From (3.21), we get the periodic singular wave solution (3.15).

(iii) Third, from Figure 2(c) or more specifically Figure 3(b), there exist one family of periodic orbits defined by $H(\varphi, y) = h, h \in (H(\varphi_5, 0), H(\varphi_6, 0))$, *the expressions of which are given by*

$$y = \pm\sqrt{(\alpha_6-\varphi)(\alpha_5-\varphi)(\varphi-\alpha_4)(\varphi-\alpha_3)},\ \alpha_4 < \varphi < \alpha_5. \tag{3.22}$$

Substituting (3.22) *into the first equation of* (2.4) *and integrating along the family of periodic orbits, it follows that*

$$\int_{\alpha_4}^{\varphi} \frac{1}{\sqrt{(\alpha_6-s)(\alpha_5-s)(s-\alpha_4)(s-\alpha_3)}} ds = |\xi|. \tag{3.23}$$

From (3.23), *we derive the family of periodic wave solutions* (3.16) *by the elliptic integral formula 254.00 in [29].*

The convergence result follows from $\alpha_3 \to \varphi_{62}$, $\alpha_4 \to \varphi_{61}$, $\alpha_5 \to \varphi_6$, *and* $\alpha_6 \to \varphi_6$, *when* $h \to H(\varphi_6, 0)$.

Furthermore, if c, α, g_1, *and* g_2 *satisfy the condition* $\varphi_{61} < -\frac{c+\alpha}{3} < \varphi_6$, *then we have* $\varphi_{61} + \frac{c+\alpha}{3} < 0 < \varphi_6 + \frac{c+\alpha}{3}$. *Note that the component* $u(x,t)$ *in* (3.13) *satisfies* $u(x,t) \in (\varphi_{61}+c+\alpha, \varphi_6+c+\alpha)$. *It follows from the expression* $v(x,t) = \frac{3}{4}u^2(x,t) - (c+\alpha)u(x,t) + \frac{g_1}{4} = \frac{3}{4}\left(u(x,t) - \frac{2}{3}(c+\alpha)\right)^2 + \frac{g_1}{4} - \frac{1}{3}(c+\alpha)^2$ *that the component* $v(x,t)$ *corresponding to* (3.13) *is the W-shaped solitary wave solution, the profile of which is shown in Figure 6(b). Similarly, if* c, α, g_1, g_2, *and* h *satisfy the condition* $\alpha_4 < -\frac{c+\alpha}{3} < \alpha_5$, *then we have* $\alpha_4 + \frac{c+\alpha}{3} < 0 < \alpha_5 + \frac{c+\alpha}{3}$. *Note that the component* $u(x,t)$ *in* (3.16) *satisfies* $u(x,t) \in (\alpha_4+c+\alpha, \alpha_5+c+\alpha)$. *It follows from the expression* $v(x,t) = \frac{3}{4}u^2(x,t) - (c+\alpha)u(x,t) + \frac{g_1}{4} = \frac{3}{4}\left(u(x,t) - \frac{2}{3}(c+\alpha)\right)^2 + \frac{g_1}{4} - \frac{1}{3}(c+\alpha)^2$ *that the component* $v(x,t)$ *corresponding to* (3.16) *is the W-shaped periodic wave solution, the profile of which is shown in Figure 5(d).*

Theorem 3. *When* $g_2 = g_1(c+\alpha) - g_0$, *Eq (1.1) has two singular wave solutions*

$$u(x,t) = \frac{\varphi_3\left(-3 - (2\varphi_3(x-ct)+1)^2\right)}{1 - (2\varphi_3(x-ct)+1)^2} + c + \alpha, \tag{3.24}$$

and

$$u(x,t) = \frac{\varphi_3\left(-3 - 4(\varphi_3)^2(x-ct)^2\right)}{1 - 4(\varphi_3)^2(x-ct)^2} + c + \alpha, \tag{3.25}$$

where $\varphi_3 = \sqrt{\frac{1}{6}\left(2(c+\alpha)^2 - g_1\right)}$.

Proof . *When* $g_2 = g_1(c+\alpha) - g_0$*, from Figure 2(b), there are two orbits* $\Gamma_5^{\pm}$*, which have the same hamiltonian with the degenerate saddle point* $(\varphi_3, 0)$*. In* $(\varphi, y)-$*plane, from (2.5), the expressions of these two orbits* $\Gamma_5^{\pm}$ *are given as*

$$y = \pm\sqrt{(\varphi-\varphi_3)^3\,(\varphi-\varphi_{31})}, \tag{3.26}$$

where $\varphi_{31} = -3\sqrt{\frac{1}{6}\left(2(c+\alpha)^2 - g_1\right)}$*.*

Substituting (3.26) into the first equation of system (2.4), and integrating along these two orbits $\Gamma_5^{\pm}$*, it follows that*

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{(s-\varphi_3)\,\sqrt{(s-\varphi_3)\,(s-\varphi_{31})}} = |\xi|, \tag{3.27}$$

and

$$\int_{\varphi}^{\varphi_{31}} \frac{\mathrm{d}s}{(\varphi_3-s)\,\sqrt{(\varphi_3-s)\,(\varphi_{31}-s)}} = |\xi|. \tag{3.28}$$

From (3.27) and (3.28), we get two singular wave solutions (3.24) and (3.25). □

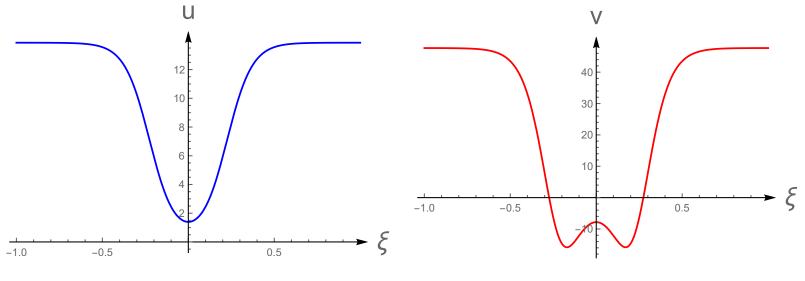
Remark 2. *In general, the homoclinic orbit corresponds to the solitary wave solution, the heteroclinic orbit corresponds to the kink (antikink) wave solution, the periodic orbit corresponds to the periodic wave solution and the unbounded orbit corresponds to the singular wave solution.*

4. Numerical simulations

Taking $\alpha = 3, c = 4, g_1 = 2, g_2 = 14, h = 1152$, which indicates that $g_2 = g_1(c+\alpha), -\varphi_1 = -4\sqrt{3} < \frac{c+\alpha}{3} = \frac{7}{3} < \varphi_1 = 4\sqrt{3}, h = H(\varphi_1)$, we illustrate the profiles of the kink (antikink) wave solutions $u(x,t)$ in (3.1) and the corresponding $v(x,t)$ in Figure 4(a), 4(b) and 4(c). Interestingly, the solutions $v(x,t)$ corresponding to the kink (antikink) wave solutions $u(x,t)$ in (3.1) are the V-shaped kink (antikink) wave solutions. In addition, If we take $\alpha = 3, c = 4, g_1 = 2, g_2 = 14, h = 1100$, then $g_2 = g_1(c+\alpha), -\alpha_1 = -6.148 < \frac{c+\alpha}{3} = \frac{7}{3} < \alpha_1 = 6.148, h \in (H(0,0), H(\varphi_1, 0)) = (0, 1152)$, and we illustrate the profiles of the periodic wave solutions $u(x,t)$ in (3.5) and the corresponding $v(x,t)$ in Figure 5(a) and 5(b). Now we find that the solution $v(x,t)$ corresponding to the periodic wave solution $u(x,t)$ in (3.5) is the W-shaped periodic wave solution.

Similarly, we choose $\alpha = 3, c = 4, g_1 = 2, g_2 = 3, h = 1176.11$, such that $g_1(c+\alpha) - g_0 < g_2 < g_1(c+\alpha), \varphi_{61} = -5.6 < \frac{c+\alpha}{3} = \frac{7}{3} < \varphi = 6.87, \lambda = 2.95 \in (1,3), h = H(\varphi_6)$. Under these parameter conditions, the profiles of the solitary wave solution $u(x,t)$ in (3.13) and the corresponding $v(x,t)$ are shown in Figure 6(a) and 6(b). The solution $v(x,t)$ corresponding to the solitary wave solution $u(x,t)$ in (3.13) is the W-shaped solitary wave solution. Additionally, if choosing $\alpha = 3, c = 4, g_1 = 2, g_2 = 3, h = 1000$, then $g_1(c+\alpha) - g_0 < g_2 < g_1(c+\alpha), \alpha_4 = -5.246 < \frac{c+\alpha}{3} = \frac{7}{3} < \alpha_5 = 5.9, \lambda = 2.95 \in (1,3), h \in (H(\varphi_5, 0), H(\varphi_6, 0)) = (-0.63, 1176.11)$, and we illustrate the profiles of the periodic wave solution $u(x,t)$ in (3.16) and the corresponding $v(x,t)$ in Figure 5(c) and 5(d). Now the solution $v(x,t)$ corresponding to the periodic wave solution $u(x,t)$ in (3.16) is the W-shaped periodic wave solution.

Remark 3. *In the above theorems, we just list the results when* $g_2 \le g_1(c+\alpha)$*, since the results when* $g_2 > g_1(c+\alpha)$ *can be derived similarly. Here, we deduce that the profiles of the solutions when* $g_2 > g_1(c+\alpha)$ *will be completely symmetric to the corresponding profiles of the solutions when* $g_2 < g_1(c+\alpha)$ *about the* ξ*-axis.*



(a) (b)

Figure 6. The solitary wave solution of Eq (1.1). (a) Solitary wave solution (3.13); (b) W-shaped solitary wave solution $v(x,t)$ corresponding to (3.13).

5. Conclusions

In this paper, by employing the bifurcation method and qualitative theory of dynamical systems, we study the nonlinear wave solutions to the modified cKdV equation (1.1), and obtain exact explicit expressions of the various types of nonlinear wave solutions, including solitary wave solutions, singular wave solutions, periodic singular wave solutions, and kink (antikink) wave solutions. Among these solutions, of particular interest is the appearance of the so-called V-shaped kink (antikink) wave solutions, W-shaped solitary wave solutions, and W-shaped periodic wave solutions, which were not found previously. These solutions may be interpreted in the context of some related equations, such as the generalized Kaup equation, the classical Boussinesq equation and the systems governing second harmonic generation (SHG), which are connected to the cKdV equation (1.1) through nonsingular transformations [6]. Additionally, in the Theorems 1 and 2, we see that if the parameters c, α, g_1 and g_2 satisfies certain conditions, the V-shaped kink (antikink) wave solutions, W-shaped solitary wave solutions, and W-shaped periodic wave solutions are found. This potentially provide a way to control the appearance of these interesting solutions. Finally, we know that time delay and perturbation play an important in modeling mathematics physics problems [30], so we may further study the solutions and their properties of the time delayed or perturbed version of Eq (1.1).

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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