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Research article

Fekete-Szegö problem for Bi-Bazilevič functions related to Shell-like curves

Halit Orhan[1,*], **Nanjundan Magesh**[2] **and Chinnasamy Abirami**[3]

[1] Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey
[2] Post-Graduate and Research Department of Mathematics, Government Arts College for Men, Krishnagiri 635001, Tamilnadu, India
[3] Faculty of Engineering and Technology, SRM University, Kattankulathur-603203, Tamilnadu, India

* **Correspondence:** E-mail: orhanhalit607@gmail.com.

Abstract: In the present investigation, we define a subclass of bi-univalent functions related to shell-like curves connected with Fibonacci numbers to find the estimates of second, third Taylor-Maclaurin coefficients and Fekete-Szegö inequalities. Further, certain special cases are also discussed.

Keywords: univalent functions; bi-univalent functions; shell-like function; Bazilevič function; Fibonacci number; Fekete-Szegö inequality
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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$.

For analytic functions f and g in $\mathbb{D}$, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \qquad f(z) = g(w(z)), \qquad z \in \mathbb{D}.$$

This subordination will be denoted here by

$$f \prec g, \qquad z \in \mathbb{D}$$

or, conventionally, by

$$f(z) \prec g(z), \qquad z \in \mathbb{D}.$$

In particular, when g is univalent in $\mathbb{D}$,

$$f \prec g \qquad (z \in \mathbb{D}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \qquad z \in \mathbb{D} \tag{1.2}$$

which are analytic with $\Re\{p(z)\} > 0$. Here $p(z)$ is called as Caratheodory functions [1]. It is well known that the following correspondence between the class $\mathcal{P}$ and the class of Schwarz functions w exists: $p \in \mathcal{P}$ if and only if $p(z) = 1 + w(z)/1 - w(z)$. Let $\mathcal{P}(\beta)$, $0 \le \beta < 1$, denote the class of analytic functions p in $\mathbb{D}$ with $p(0) = 1$ and $\Re\{p(z)\} > \beta$.

Recently, Sokół [2] and Dziok et al. [3] studied the classes $\mathcal{SL}(\widetilde{p})$ and $\mathcal{KSL}(\widetilde{p})$ of shell-like functions and convex shell-like functions which are characterized by $zf'/f(z) \prec \widetilde{p}(z)$ or $1 + z^2 f''/f'(z) \prec \widetilde{p}(z)$, where $\widetilde{p}(z) = (1 + \tau^2 z^2)/(1 - \tau z - \tau^2 z^2)$, $\tau = (1 - \sqrt{5})/2 \approx -0.618$ [4,5] and the function $\widetilde{p}$ is not univalent in $\mathbb{D}$, but it is univalent in the disc $|z| < (3 - \sqrt{5})/2 \approx 0.38$. For example, $\widetilde{p}(0) = \widetilde{p}(-1/2\tau) = 1$ and $\widetilde{p}(e^{\mp} \arccos(1/4)) = 1/\sqrt{5}$ and it may also be noticed that $1/|\tau| = |\tau|/1 - |\tau|$ which shows that the number $|\tau|$ divides [0, 1] such that it fulfills the golden section. The image of the unit circle $|z| = 1$ under $\widetilde{p}$ is a curve described by the equation given by $\left(10x - \sqrt{5}\right)y^2 = \left(\sqrt{5} - 2x\right)\left(\sqrt{5}x - 1\right)^2$, which is translated and revolved trisectrix of Maclaurin. The curve $\widetilde{p}\left(re^{it}\right)$ is a closed curve without any loops for $0 < r \le r_0 = (3 - \sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n as

$$\tau^n = u_n \tau + u_{n-1}.$$

Also, Raina and Sokół [5] proved that

$$\begin{aligned}
\widetilde{p}(z) &= \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \\
&= \left(t + \frac{1}{t}\right) \frac{t}{1 - t - t^2} \\
&= \frac{1}{\sqrt{5}} \left(t + \frac{1}{t}\right) \left(\frac{1}{1 - (1 - \tau)t} - \frac{1}{1 - \tau t}\right) \\
&= \left(t + \frac{1}{t}\right) \sum_{n=2}^{\infty} \frac{(1 - \tau)^n - \tau^n}{\sqrt{5}} t^n \\
&= \left(t + \frac{1}{t}\right) \sum_{n=2}^{\infty} u_n t^n
\end{aligned}$$

$$= \quad 1+\sum_{n=2}^{\infty}(u_{n-1}+u_{n+1})\,\tau^n z^n,$$

where

$$u_n=\frac{(1-\tau)^n-\tau^n}{\sqrt{5}},\qquad \tau=\frac{1-\sqrt{5}}{2},\qquad n=1,\ 2,\ \cdots.$$

This shows that the relevant connection of $\widetilde{p}$ with the sequence of Fibonacci numbers u_n, such that

$$u_0=0,\quad u_1=1,\quad u_{n+2}=u_n+u_{n+1},\qquad n=0,\ 1,\ 2,\ 3,\ \cdots.$$

Hence

$$\begin{aligned}\widetilde{p}(z) &= 1+\sum_{n=1}^{\infty}\widetilde{p}_n z^n\\ &= 1+(u_0+u_2)\,\tau z+(u_1+u_3)\,\tau^2z^2+\sum_{n=3}^{\infty}(u_{n-3}+u_{n-2}+u_{n-1}+u_n)\,\tau^n z^n\\ &= 1+\tau z+3\tau^2z^2+4\tau^3z^3+7\tau^4z^4+11\tau^5z^5+\cdots.\end{aligned}\tag{1.3}$$

We note that the function $\widetilde{p}$ belongs to the class $\mathcal{P}(\beta)$ with $\beta=\dfrac{\sqrt{5}}{10}\approx 0.2236$ [5].

It is well known that every function $f\in\mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z,\qquad z\in\mathbb{D}$$

and

$$f(f^{-1}(w))=w,\qquad |w|<r_0(f);\quad r_0(f)\geqq\frac{1}{4},$$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots.\tag{1.4}$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{D}$. Let Σ denote the class of bi-univalent functions in $\mathbb{D}$ given by (1.1) for more details one can refer [6–13] and references therein. Also the various subclasses of bi-univalent functions related to shell-like curves were studied in [14–16].

Recently, the initial coefficient estimates are found for functions in the class of bi-univalent functions defined through certain polynomials like the Faber polynomial, the Lucas polynomial, the Chebyshev polynomial, the Gegenbauer polynomial and the Meixner-Pollaczek polynomial. Motivated in this line, in the present work, we introduce the following new subclass of bi-univalent function, as follows:

Definition 1.1. A function $f\in\Sigma$ of the form (1.1) belongs to the class $\mathcal{BSL}_{\Sigma}^{\mu,\,\delta,\,\lambda}(\widetilde{p})$, $\mu\geq 0$, $\lambda\geq 1$, $\delta\geq 0$, if the following conditions are satisfied:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\xi\delta zf''(z)\prec\widetilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2},\ z\in\mathbb{D}$$

and for $g(w)=f^{-1}(w)$

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\xi\delta w g''(w)\prec\widetilde{p}(w)=\frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2},\ w\in\mathbb{D},$$

where $\tau=\dfrac{1-\sqrt{5}}{2}\approx -0.618$ and $\xi=\frac{2\lambda+\mu}{2\lambda+1}$.

By suitably specializing the values of μ, λ and δ, the class $\mathcal{BSL}_{\Sigma}^{\mu,\,\delta,\,\lambda}(\widetilde{p})$ reduces to various new subclasses, we illustrate the following subclasses:

1. For $\delta=0$, we get the class $\mathcal{BSL}_{\Sigma}^{\mu,\,0,\,\lambda}(\widetilde{p})\equiv\mathcal{NSL}_{\Sigma}^{\mu,\,\lambda}(\widetilde{p})$. A function $f\in\Sigma$ of the form (1.1) is said to be in $\mathcal{NSL}_{\Sigma}^{\mu,\,\lambda}(\widetilde{p}(z))$, if the following conditions

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\prec\widetilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2},\ z\in\mathbb{D}$$

and for $g(w)=f^{-1}(w)$

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\prec\widetilde{p}(w)=\frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2},\ w\in\mathbb{D},$$

hold, where $\tau=\dfrac{1-\sqrt{5}}{2}\approx -0.618$.

2. For $\lambda=1$ and $\delta=0$, we get the class $\mathcal{BSL}_{\Sigma}^{\mu,\,0,\,1}(\widetilde{p})\equiv\mathcal{RSL}_{\Sigma}^{\mu}(\widetilde{p})$. A function $f\in\Sigma$ of the form (1.1) is said to be in $\mathcal{RSL}_{\Sigma}^{\mu}(\widetilde{p})$, if the following conditions

$$f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\prec\widetilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2},\ z\in\mathbb{D}$$

and for $g(w)=f^{-1}(w)$

$$g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\prec\widetilde{p}(w)=\frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2},\ w\in\mathbb{D},$$

hold, where $\tau=\dfrac{1-\sqrt{5}}{2}\approx -0.618$.

3. For $\mu=1$, we get the class $\mathcal{BSL}_{\Sigma}^{1,\,\delta,\,\lambda}(\widetilde{p})\equiv\mathcal{WSL}_{\Sigma}^{\delta,\,\lambda}(\widetilde{p})$. A function $f\in\Sigma$ of the form (1.1) is said to be in $\mathcal{WSL}_{\Sigma}^{\delta,\,\lambda}(\widetilde{p})$, if the following conditions

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''(z)\prec\widetilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2},\ z\in\mathbb{D}$$

and for $g(w)=f^{-1}(w)$

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)\prec\widetilde{p}(w)=\frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2},\ w\in\mathbb{D},$$

hold, where $\tau=\dfrac{1-\sqrt{5}}{2}\approx -0.618$.

4. For $\lambda = \mu = 1$, we get the class $\mathcal{BSL}_{\Sigma}^{1,\,\delta,\,1}(\widetilde{p}) \equiv \mathcal{FSL}_{\Sigma}(\delta, \widetilde{p})$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{FSL}_{\Sigma}(\delta, \widetilde{p})$, if the following conditions

$$f'(z) + \delta z f''(z) \prec \widetilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \ z \in \mathbb{D}$$

and for $g(w) = f^{-1}(w)$

$$g'(w) + \delta w g''(w) \prec \widetilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \ w \in \mathbb{D},$$

hold, where $\tau = \dfrac{1 - \sqrt{5}}{2} \approx -0.618$.

5. For $\mu = 1$ and $\delta = 0$, we obtain the class $\mathcal{BSL}_{\Sigma}^{1,\,0,\,\lambda}(\widetilde{p}), \equiv \mathcal{BSL}_{\Sigma}(\lambda, \widetilde{p})$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{BSL}_{\Sigma}(\lambda, \widetilde{p}(z))$, if the following conditions

$$(1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \widetilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \ z \in \mathbb{D}$$

and for $g(w) = f^{-1}(w)$

$$(1 - \lambda)\frac{g(w)}{w} + \lambda g'(w) \prec \widetilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \ w \in \mathbb{D},$$

hold, where $\tau = \dfrac{1 - \sqrt{5}}{2} \approx -0.618$.

6. For $\lambda = 1, \mu = 1$ and $\delta = 0$, we have the class $\mathcal{BSL}_{\Sigma}^{1,\,0,\,1}(\widetilde{p}) \equiv \mathcal{HSL}_{\Sigma}(\widetilde{p})$. A function $f \in \Sigma$ of the form (1.1) is said to be in $\mathcal{HSL}_{\Sigma}(\widetilde{p})$, if the following conditions

$$f'(z) \prec \widetilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \ z \in \mathbb{D}$$

and for $g(w) = f^{-1}(w)$

$$g'(w) \prec \widetilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}, \ w \in \mathbb{D},$$

hold, where $\tau = \dfrac{1 - \sqrt{5}}{2} \approx -0.618$.

In order to prove our results for the functions in the class $\mathcal{BSL}_{\Sigma}^{\mu,\,\delta,\,\lambda}(\widetilde{p})$, we need the following lemma.

Lemma 1.1. *[10] If $p \in \mathcal{P}$, then $|p_i| \leqq 2$ for each i, where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{D}$, for which*

$$\Re\{p(z)\} > 0 \qquad (z \in \mathbb{D}),$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{D}).$$

In this investigation, we find the estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{BSL}_{\Sigma}^{\mu,\,\delta,\,\lambda}(\widetilde{p})$ and its special cases. Also, Fekete-Szegö inequality for functions in this subclass.

2. Coefficient estimates and Fekete-Szegö inequality

In the following theorem, we discuss coefficient estimates and Fekete-Szegö inequality for functions in the class $f \in \mathcal{BSL}_{\Sigma}^{\mu,\,\delta,\,\lambda}(\widetilde{p})$.

Theorem 2.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{BSL}_{\Sigma}^{\mu,\,\delta,\,\lambda}(\widetilde{p})$. *Then*

$$|a_2| \leq |\tau| \sqrt{\frac{2}{M}}, \qquad |a_3| \leq \frac{\left[(2\lambda+\mu)(\mu-1)\,\tau + 2(1-3\tau)(\lambda+\mu+2\xi\delta)^2\right]|\tau|}{M\,(2\lambda+\mu+6\xi\delta)}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{2\lambda+\mu+6\xi\delta} & ;\, 0 \leq |\nu-1| \leq \dfrac{M}{2\,(2\lambda+\mu+6\xi\delta)\,|\tau|} \\ \dfrac{2\,|1-\nu|\,\tau^2}{M} & ;\, |\nu-1| \geq \dfrac{M}{2\,(2\lambda+\mu+6\xi\delta)\,|\tau|}, \end{cases}$$

where

$$M = \left[\tau\left[(2\lambda+\mu)(\mu+1) + 12\xi\delta\right] + 2(1-3\tau)(\lambda+\mu+2\xi\delta)^2\right].$$

Proof. Since $f \in \mathcal{BSL}_{\Sigma}^{\mu,\,\delta,\,\lambda}(\widetilde{p})$, from the Definition 1.1, we have

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} + \xi\delta z f''(z) = \widetilde{p}(p(z)) \tag{2.1}$$

and for $g = f^{-1}$

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} + \xi\delta w g''(w) = \widetilde{p}(q(w)), \tag{2.2}$$

where $z,\ w \in \mathbb{D}$. Using the fact the function p of the form (1.2) and $p \prec \widetilde{p}$. Then there exists an analytic function p such that $|p(z)| < 1$ in $\mathbb{D}$ and $p(z) = \widetilde{p}(p(z))$. Therefore, define the function

$$h(z) = \frac{1+p(z)}{1-p(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

is in the class $\mathcal{P}$. It follows that

$$p(z) = \frac{h(z)-1}{h(z)+1} = \frac{p_1}{2}z + \left(p_2 - \frac{p_1^2}{2}\right)\frac{z^2}{2} + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right)\frac{z^3}{2} + \cdots$$

and

$$\begin{aligned} \widetilde{p}(p(z)) \;=\; & 1 + \widetilde{p}_1\left(\frac{p_1}{2}z + \left(p_2 - \frac{p_1^2}{2}\right)\frac{z^2}{2} + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right)\frac{z^3}{2} + \cdots\right) \\ & + \widetilde{p}_2\left(\frac{p_1}{2}z + \left(p_2 - \frac{p_1^2}{2}\right)\frac{z^2}{2} + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right)\frac{z^3}{2} + \cdots\right)^2 \end{aligned}$$

$$+\widetilde{p}_3\left(\frac{p_1}{2}z+\left(p_2-\frac{p_1^2}{2}\right)\frac{z^2}{2}+\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)\frac{z^3}{2}+\cdots\right)^3+\cdots$$

$$= 1+\frac{\widetilde{p}_1p_1}{2}z+\left(\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)\widetilde{p}_1+\frac{p_1^2}{4}\widetilde{p}_2\right)z^2$$

$$+\left(\frac{1}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)\widetilde{p}_1+\frac{1}{2}p_1\left(p_2-\frac{p_1^2}{2}\right)\widetilde{p}_2+\frac{p_1^3}{8}\widetilde{p}_3\right)z^3+\cdots. \qquad (2.3)$$

Similarly, there exists an analytic function v such that $|q(w)| < 1$ in $\mathbb{D}$ and $p(w) = \widetilde{p}(q(w))$. Therefore, the function

$$k(w)=\frac{1+q(w)}{1-q(w)}=1+q_1w+q_2w^2+\cdots$$

is in the class $\mathcal{P}$. It follows that

$$q(w)=\frac{k(w)-1}{k(w)+1}=\frac{q_1}{2}w+\left(q_2-\frac{q_1^2}{2}\right)\frac{w^2}{2}+\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)\frac{w^3}{2}+\cdots$$

and

$$\widetilde{p}(q(w)) = 1+\widetilde{p}_1\left(\frac{q_1}{2}w+\left(q_2-\frac{q_1^2}{2}\right)\frac{w^2}{2}+\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)\frac{w^3}{2}+\cdots\right)$$

$$+\widetilde{p}_2\left(\frac{q_1}{2}w+\left(q_2-\frac{q_1^2}{2}\right)\frac{w^2}{2}+\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)\frac{w^3}{2}+\cdots\right)^2$$

$$+\widetilde{p}_3\left(\frac{q_1}{2}w+\left(q_2-\frac{q_1^2}{2}\right)\frac{w^2}{2}+\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)\frac{w^3}{2}+\cdots\right)^3+\cdots$$

$$= 1+\frac{\widetilde{p}_1q_1}{2}w+\left(\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)\widetilde{p}_1+\frac{q_1^2}{4}\widetilde{p}_2\right)w^2$$

$$+\left(\frac{1}{2}\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)\widetilde{p}_1+\frac{1}{2}q_1\left(q_2-\frac{q_1^2}{2}\right)\widetilde{p}_2+\frac{q_1^3}{8}\widetilde{p}_3\right)w^3 \qquad (2.4)$$

$$+\cdots.$$

By virtue of (2.1), (2.2), (2.3) and (2.4), we have

$$(\lambda+\mu+2\xi\delta)\,a_2=\frac{p_1\tau}{2}, \qquad (2.5)$$

$$(2\lambda+\mu)\left[\left(\frac{\mu-1}{2}\right)a_2^2+\left(1+\frac{6\delta\xi}{2\lambda+\mu}\right)a_3\right]=\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)\tau+\frac{3p_1^2}{4}\tau^2, \qquad (2.6)$$

$$-(\lambda+\mu+2\xi\delta)\,a_2=\frac{q_1\tau}{2}, \qquad (2.7)$$

and

$$(2\lambda+\mu)\left[\left(\frac{\mu+3}{2}+\frac{12\delta\xi}{2\lambda+\mu}\right)a_2^2-\left(1+\frac{6\delta\xi}{2\lambda+\mu}\right)a_3\right]=\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)\tau+\frac{3q_1^2}{4}\tau^2. \qquad (2.8)$$

From (2.5) and (2.7), we obtain

$$p_1=-q_1,$$

and

$$
\begin{aligned}
2\left(\lambda+\mu+2\xi\delta\right)^{2}a_{2}^{2} &= \frac{(p_{1}^{2}+q_{1}^{2})\tau^{2}}{4} \\
a_{2}^{2} &= \frac{(p_{1}^{2}+q_{1}^{2})\tau^{2}}{8\left(\lambda+\mu+2\xi\delta\right)^{2}}\,. \qquad (2.9)
\end{aligned}
$$

By adding (2.6) and (2.8), we have

$$
[(2\lambda+\mu)(\mu+1)+12\xi\delta]a_{2}^{2}=\frac{1}{2}\left(p_{2}+q_{2}\right)\tau-\frac{1}{4}\left(p_{1}^{2}+q_{1}^{2}\right)\tau+\frac{3}{4}\left(p_{1}^{2}+q_{1}^{2}\right)\tau^{2}. \qquad (2.10)
$$

By substituting (2.9) in (2.10), we reduce that

$$
a_{2}^{2} = \frac{\left(p_{2}+q_{2}\right)\tau^{2}}{2\left[\tau\left[(2\lambda+\mu)(\mu+1)+12\xi\delta\right]+2(1-3\tau)(\lambda+\mu+2\xi\delta)^{2}\right]}. \qquad (2.11)
$$

Now, applying Lemma 1.1, we obtain

$$
|a_{2}| \leq \frac{\sqrt{2}\,|\tau|}{\sqrt{\tau\left[(2\lambda+\mu)(\mu+1)+12\xi\delta\right]+2(1-3\tau)(\lambda+\mu+2\xi\delta)^{2}}}. \qquad (2.12)
$$

By subtracting (2.8) from (2.6), we obtain

$$
a_{3}=\frac{\left(p_{2}-q_{2}\right)\tau}{4\left(2\lambda+\mu+6\xi\delta\right)}+a_{2}^{2}. \qquad (2.13)
$$

Hence by Lemma 1.1, we have

$$
|a_{3}| \leq \frac{\left(|p_{2}|+|q_{2}|\right)|\tau|}{4\left(2\lambda+\mu+6\xi\delta\right)}+|a_{2}|^{2}\leq\frac{|\tau|}{2\lambda+\mu+6\xi\delta}+|a_{2}|^{2}\,.
$$

Then in view of (2.12), we obtain

$$
|a_{3}|\leq\frac{|\tau|\left\{(2\,\lambda+\mu)\,(\mu-1)\,\tau+2\,\left(1-3\,\tau\right)(\lambda+\mu+2\,\delta\xi)^{2}\right\}}{(2\,\lambda+\mu+6\,\delta\,\xi)\left[\left[(2\,\lambda+\mu)\,(\mu+1)+12\,\delta\,\xi\right]\tau+2\,\left(1-3\,\tau\right)(2\,\delta\,\xi+\lambda+\mu)^{2}\right]}
$$

From (2.13), we have

$$
a_{3}-\nu a_{2}^{2}=\frac{\left(p_{2}-q_{2}\right)\tau}{4\left(2\lambda+\mu+6\xi\delta\right)}+\left(1-\nu\right)a_{2}^{2}. \qquad (2.14)
$$

By substituting (2.11) in (2.14), we have

$$
\begin{aligned}
a_{3}-\nu a_{2}^{2} &= \frac{\left(p_{2}-q_{2}\right)\tau}{4\left(2\lambda+\mu+6\xi\delta\right)}+\frac{\left(1-\nu\right)\left(p_{2}+q_{2}\right)\tau^{2}}{2\left[\tau\left[(2\lambda+\mu)(\mu+1)+12\xi\delta\right]+2(1-3\tau)(\lambda+\mu+2\xi\delta)^{2}\right]} \\
&= \left(h(\nu)+\frac{|\tau|}{4\left(2\lambda+\mu+6\xi\delta\right)}\right)p_{2}+\left(h(\nu)-\frac{|\tau|}{4\left(2\lambda+\mu+6\xi\delta\right)}\right)q_{2}, \qquad (2.15)
\end{aligned}
$$

where

$$h(\nu) = \frac{(1-\nu)\,\tau^2}{2\left[\tau\left[(2\lambda+\mu)(\mu+1)+12\xi\delta\right]+2(1-3\tau)(\lambda+\mu+2\xi\delta)^2\right]}.$$

Thus by taking modulus of (2.15), we conclude that

$$\left|a_3-\nu a_2^2\right| \le \begin{cases} \dfrac{|\tau|}{2\lambda+\mu+6\xi\delta} & ; 0 \le |h(\nu)| \le \dfrac{|\tau|}{4\,(2\lambda+\mu+6\xi\delta)} \\ 4\,|h(\nu)| & ; |h(\nu)| \ge \dfrac{|\tau|}{4\,(2\lambda+\mu+6\xi\delta)}. \end{cases}$$

□

3. Corollaries and consequences

In this section, we give coefficient estimates and Fekete-Szegö inequalities for the subclasses of $\mathcal{BSL}_{\Sigma}^{\mu,\,\delta,\,\lambda}(\widetilde{p})$.

Corollary 3.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{NSL}_{\Sigma}^{\mu,\,\lambda}(\widetilde{p})$. *Then*

$$|a_2| \le |\tau|\,\sqrt{\frac{2}{M_1}}, \qquad |a_3| \le \frac{|\tau|\left\{(2\,\lambda+\mu)\,(\mu-1)\,\tau+2\,\,(1-3\,\tau)\,(\lambda+\mu)^2\right\}}{M_1\,(2\,\lambda+\mu)}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3-\nu a_2^2\right| \le \begin{cases} \dfrac{|\tau|}{2\lambda+\mu} & ; 0 \le |\nu-1| \le \dfrac{M_1}{2\,(2\lambda+\mu)\,|\tau|} \\ \dfrac{2\,|1-\nu|\,\tau^2}{M_1} & ; |\nu-1| \ge \dfrac{M_1}{2\,(2\lambda+\mu)\,|\tau|}, \end{cases}$$

where

$$M_1 = \tau(2\lambda+\mu)(\mu+1) + 2(1-3\tau)(\lambda+\mu)^2.$$

Corollary 3.2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{RSL}_{\Sigma}^{\mu}(\widetilde{p})$. *Then*

$$|a_2| \le |\tau|\,\sqrt{\frac{2}{M_2}}, \qquad |a_3| \le \frac{|\tau|\left\{(\mu+2)\,(\mu-1)\,\tau+2(1-3\tau)(1+\mu)^2\right\}}{M_2\,(\mu+2)}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3-\nu a_2^2\right| \le \begin{cases} \dfrac{|\tau|}{2+\mu} & ; 0 \le |\nu-1| \le \dfrac{M_2}{2\,(2+\mu)\,|\tau|} \\ \dfrac{2\,|1-\nu|\,\tau^2}{M_2} & ; |\nu-1| \ge \dfrac{M_2}{2\,(2+\mu)\,|\tau|}, \end{cases}$$

where

$$M_2 = 2(1+\mu)^2 - (1+\mu)(4+5\mu)\tau.$$

Remark 3.1. For $\mu = 1$, results discussed in Corollaries 3.2 is coincides with bounds obtained in [14, Corollary 1, p.78].

Corollary 3.3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{WSL}_{\Sigma}^{\delta,\lambda}(\widetilde{p})$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{M_3}}, \qquad |a_3| \leq \frac{2\,|\tau|\,(1-3\tau)(1+\lambda+2\delta)^2}{2M_3\,(1+2\lambda+6\delta)}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{1+2\lambda+6\delta} & ; 0 \leq |\nu - 1| \leq \dfrac{M_3}{(1+2\lambda+6\delta)\,|\tau|} \\ \dfrac{|1-\nu|\,\tau^2}{M_3} & ; |h(\nu)| \geq \dfrac{M_3}{(1+2\lambda+6\delta)\,|\tau|}, \end{cases}$$

where

$$M_3 = \tau\,(1+2\lambda+6\delta) + (1-3\tau)(1+\lambda+2\delta)^2.$$

Corollary 3.4. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{FSL}_{\Sigma}(\delta, \widetilde{p})$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{M_4}}, \qquad |a_3| \leq \frac{8\,|\tau|\,(1-3\tau)(1+\delta)^2}{6M_4\,(1+2\delta)}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{3+6\delta} & ; 0 \leq |\nu - 1| \leq \dfrac{M_4}{(3+6\delta)\,|\tau|} \\ \dfrac{|1-\nu|\,\tau^2}{M_4} & ; |\nu - 1| \geq \dfrac{M_4}{(3+6\delta)\,|\tau|}, \end{cases}$$

where

$$M_4 = 3\tau\,(1+2\delta) + 4(1-3\tau)(1+\delta)^2.$$

Corollary 3.5. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{BSL}_{\Sigma}(\lambda, \widetilde{p})$. *Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{M_5}}, \quad |a_3| \leq \frac{|\tau|\,(1-3\tau)(1+\lambda)^2}{(1+2\lambda)\,M_5}$$

and for $\nu \in \mathbb{R}$,

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{1+2\lambda} & ; 0 \leq |\nu - 1| \leq \dfrac{M_5}{(1+2\lambda)\,|\tau|} \\ \dfrac{|1-\nu|\,\tau^2}{M_5} & ; |\nu - 1| \geq \dfrac{M_5}{(1+2\lambda)\,|\tau|}, \end{cases}$$

where

$$M_5 = \tau\,(1+2\lambda) + (1-3\tau)(1+\lambda)^2.$$

Corollary 3.6. *Let* $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{HSL}_\Sigma(\widetilde{p})$*. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4-9\tau}}, \qquad |a_3| \leq \frac{|\tau|\{4-12\tau\}}{3(4-9\tau)}$$

and for $\nu \in \mathbb{R}$*,*

$$\left|a_3 - \nu a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{3} & ; 0 \leq |\nu - 1| \leq \dfrac{4-9\tau}{3\,|\tau|} \\ \dfrac{|1-\nu|\,\tau^2}{4-9\tau} & ; |\nu - 1| \geq \dfrac{4-9\tau}{3\,|\tau|}. \end{cases}$$

4. Conclusions

In this investigation, we obtain upper bounds for the coefficients $|a_2|$, $|a_3|$ and Fekete-Szegö inequality $|a_3 - \nu a_2^2|$ for functions in the class $\mathcal{BSL}_\Sigma^{\mu,\,\delta,\,\lambda}(\widetilde{p})$. Also, certain special cases are also discussed.

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Conflict of interest

The authors declare that they have no conflict of interest.

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