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Research article

Solutions of an infinite system of integral equations of Volterra-Stieltjes type in the sequence spaces $\ell_p(1 < p < \infty)$ and c_0

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Abstract: In this paper, by applying the technique of measure of noncompactness and a new generalization of Darbo's theorem, we study the existence of solutions for an infinite system of integral equations in two variables. The obtained results extend and generalize some related results in previous work. Finally, some examples are included to ascertain the usefulness of the outcome.

Keywords: Banach spaces ℓ_p and c_0; Darbo's fixed point theorem; measure of noncompactness; integral equations
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

The interest for studying the theory of infinite systems of integral equations is based on the fact that the theory of infinite systems of integral equations is a branch of nonlinear analysis which has been applied in various fields of science and numerous applications. In fact, most physical and engineering problems are formed by infinite systems of integral equations, see for example [1–4]. The problem of the existence of solutions for infinite systems of integral equations plays a significant role in the investigation of these types of equations and it is important to apply original studies in our investigations (cf. [5–7]). In some papers, integral equations of Volterra type have been converted in the form of integral equations of Volterra-Stieltjes type and numerous results have been obtained on the existence of solutions of nonlinear integral equations (cf. [8, 9]). The aim of this paper is to present some results on the existence of solutions for an infinite system of integral equations of Volterra-Stieltjes type of the form

$$\begin{aligned} u_n(t,x) &= F_n\Big(t,s,f_1(t,u(t,x))\int_0^t\int_0^x g_n\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s), \\ &\quad (Tu)(t,x)\int_0^\infty V_n(t,s,u(t,x))ds\Big); \\ u(t,x) &= \Big\{u_i(t,x)\Big\}_{i=1}^\infty, \quad u_i(t,x)\in BC(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}), \end{aligned} \tag{1.1}$$

where $BC(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R})$ is the space of all real functions $u(t,x)=u:\mathbb{R}_+\times\mathbb{R}_+\longrightarrow\mathbb{R}$, which are defined, continuous and bounded on the set $\mathbb{R}_+\times\mathbb{R}_+$ with a supremum norm $\|u\|=\sup\Big\{|u(t,x)| : (t,x)\in\mathbb{R}_+\times\mathbb{R}_+\Big\}$. The obtained results extend and generalize the results of [6,8,9] in the Banach spaces c_0 and ℓ_p. In our approach, this is done by applying the measure of noncompactness and Darbo fixed point theorem.

2. Preliminaries

In future, we apply some notations, definitions and preliminary facts to obtain our main results. For a bounded subset S of a metric space X, Kuratowski [10] defined the function $\alpha(S)$ by the formula

$$\alpha(S)=\inf\Big\{\delta>0 : S=\bigcup_{i=1}^n S_i,\ diam(S_i)\le\delta \quad for \quad 1\le i\le n<\infty\Big\},$$

known as the Kuratowski measure of noncompactness. Another measure of noncompactness is the Hausdorff measure of noncompactness given by:

$$\chi(S)=\inf\Big\{\varepsilon>0 :\ S\ has\ finite\ net\ in\ X\Big\}.$$

Let E be a real Banach space with norm $\|.\|$ and zero element θ. Besides, we suppose $\overline{X}$ and Conv(X) denote the closure and convex hull of X, respectively. Moreover, let us denote by M_E the family of all nonempty and bounded subsets of E and by N_E its subfamily consisting of all relatively compact sets.

Definition 1. [11] A mapping $\mu : M_E\longrightarrow[0,\infty)$ is called a measure of noncompactness if it satisfies the following conditions:

(1) The set Ker$\mu=\{X\in M_E : \mu(X)=0\}$ is nonempty and Ker$\mu\subseteq N_E$.
(2) $X\subseteq Y\Longrightarrow\mu(X)\le\mu(Y)$.
(3) $\mu(\overline{X})=\mu(X)$.
(4) $\mu(Conv(X))=\mu(X)$.
(5) $\mu(\lambda X+(1-\lambda)Y)\le\lambda\mu(X)+(1-\lambda)\mu(Y)$ for $\lambda\in[0,1]$.
(6) If $\{X_n\}$ is a sequence of closed sets from M_E such that $X_{n+1}\subseteq X_n$ for $n=1,2,\dots$ and $\lim\limits_{n\to\infty}\mu(X_n)=0$, then $\bigcap_{n=1}^\infty X_n$ is nonempty.

We will apply the following theorem as the main tool in our investigations.
Theorem 1.(Darbo [12]) Let C be a nonempty, bounded, closed and convex subset of a Banach space

E and $T : C \to C$ be a continuous mapping. Assume that there exists a constant $K \in [0, 1)$ such that $\mu(TX) \leq K\mu(X)$ for any nonempty subset X of C, where μ is a measure of noncompactness defined in E. Then T has at least a fixed point in C.

Samadi [13] extended Darbo's fixed point theorem as follows.

Theorem 2. Let C be a nonempty bounded, closed and convex subset of a Banach space E. Assume $T : C \longrightarrow C$ be a continuous operator satisfying

$$\theta(\mu(X)) + f(\mu(T(X))) \leq f(\mu(X)) \tag{2.1}$$

for all nonempty subsets X of C, where μ is an arbitrary measure of noncompactness defined in E and $(\theta, f) \in \Delta =$. Then T has a fixed point in C.

In Theorem 2, Δ is the set of all pairs (θ, f) satisfying the following:

(Δ_1) $\theta(t_n) \nrightarrow 0$ for each strictly increasing sequence $\{t_n\}$;
(Δ_2) f is strictly increasing function;
(Δ_3) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} f(\alpha_n) = -\infty$.
(Δ_4) If $\{t_n\}$ is a decreasing sequence such that $t_n \to 0$ and $\theta(t_n) < f(t_n) - f(t_{n+1})$, then we have $\sum_{n=1}^{\infty} t_n < \infty$.

We know that the Hausdorff measure of noncompactness χ in the Banach space ℓ_p can be defined as follows:

$$\chi(B) = \lim_{n\longrightarrow\infty} \left\{ \sup_{x\in B} \left\{ \Sigma_{k\geq n} \mid x_k \mid^p \right\}^{\frac{1}{p}} \right\}, \tag{2.2}$$

where $B \in M_{\ell_p}$ and $x = (x_k) \in \ell_p$. For the Banach space $(c_0\ ,\ \|.\|_{c_0})$, the Hausdorff measure of noncompactnes χ is given by (cf. Definition 1):

$$\chi(B) = \lim_{n\longrightarrow\infty} \left\{ \sup_{u\in B} \left\{ \max_{k\geq n} \mid u_k \mid \right\} \right\}, \tag{2.3}$$

where $B \in M_{c_0}$ and $u = (u_k) \in c_0$.

Now, we recall some basic facts concerning the concept of the variation of a function (cf. [14, 15]). Assume that f is a real function defined on the interval $[a, b]$. The variation of the function f will be denoted by $\bigvee_a^b f$. If $\bigvee_a^b f$ is finite, the function f has bounded variation on the interval $[a, b]$. Similarly, if $g : [a, b] \times [c, d] \longrightarrow \mathrm{R}$ is a real function of two variables, then the variation of the function $t \longrightarrow g(t, s)$ on the interval $[p, q] \subseteq [a, b]$ will be denoted by $\bigvee_{s=p}^q g(t, s)$. Analogously, we can define $\bigvee_{t=p}^q g(t, s)$. Assume that f and g are two real functions defined on the interval $[a, b]$, then under appropriate conditions we can define the Steiltjes integral $\int_a^b f(t)dg(t)$ of the function f with respect to the function g. If the integral $\int_a^b f(t)dg(t)$ is finite, then f is Stieltjes integrable on the interval $[a, b]$.

The following lemmas will be applied in our investigations.

Lemma 1. If f is Stieltjes integrable on the interval $[a, b]$ with respect to a function g of bounded variation, then

$$\left| \int_a^b f(t)dg(t) \right| \leq \int_a^b |f(t)| d\left(\bigvee_a^t g\right).$$

Lemma 2. Let f_1 and f_2 be Stieltjes integrable functions on the interval $[a, b]$ with respect to a nondecreasing function g such that $f_1(t) \leq f_2(t)$ for $t \in [a, b]$. Then,

$$\int_a^b f_1(t)dg(t) \leq \int_a^b f_2(t)dg(t).$$

3. Existence of solutions for infinite systems of integral equations

In this section, as an application of Theorem 2, the existence of solutions for the infinite system (1.1) is studied in the spaces ℓ_p and c_0. First, we show that infinite system (1.1) has a solution that belongs to the space ℓ_p.
We consider the following conditions:

(H_1) $F_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist positive real numbers $\tau > 0$ such that

$$|F_n(t, s, x_1, y_1) - F_n(t, s, x_2, y_2)|^p \leq e^{-\tau}(|x_1 - x_2|^p + |y_1 - y_2|^p),$$

for all $t, s \in \mathbb{R}_+$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Moreover, we have

$$\lim_{i \longrightarrow \infty} \Sigma_{i=1}^{\infty} |F_i(t, s, 0, 0)|^p = 0, \quad N_1 = \Sigma_{i=1}^{\infty} |F_i(t, s, 0, 0)|^p.$$

(H_2) $f_1 : \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is continuos with $f_0 = \sup_{t \in \mathbb{R}_+} |f(t, 0)|$ and there exist positive real numbers $\tau > 0$ such that

$$\begin{aligned} |f_1(t, u(t, x)) - f_1(t, v(t, x))|^p &\leq e^{-\tau} \|u(t, x) - v(t, x)\|_{\ell_p}, \\ |f_1(t, u(t, x))|^p &\leq e^{-\tau} \|u(t, x)\|_{\ell_p}. \end{aligned}$$

for all $t, x \in \mathbb{R}_+$ and $u(t, x) = \left\{u_i(t, x)\right\}_{i=1}^{\infty}, v(t, x) = \left\{v_i(t, x)\right\}_{i=1}^{\infty} \in \ell_p$.

(H_3) $T : BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_p) \longrightarrow BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ is a continuos operator such that

$$\begin{aligned} |(Tu)(t, x) - (Tv)(t, x)| &\leq \|u(t, x) - v(t, x)\|_{\ell_p}, \\ |(Tu)(t, x)| &\leq 1. \end{aligned}$$

for all $u, v \in BC(\mathbb{R}_+ \times \mathbb{R}_+, \ell_p)$ and $t, x \in \mathbb{R}_+$.

(H_4) For any fixed $t > 0$ the function $s \longrightarrow g_i(t, s)$ has a bounded variation on the interval $[0, t]$ and the function $t \longrightarrow \bigvee_{s=0}^{t} g_i(t, s)$ is bounded over $\mathbb{R}_+$.

(H_5) $g_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is continuous and there exist continuous functions $a_n : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$|g_n(t, s, x, y, u(t, x))| \leq a_n(t, s),$$

$$\lim_{t \longrightarrow \infty} \Sigma_{n \geq 1} \int_0^t |g_n(t, s, x, y, u(t, x)) - g_n(t, s, x, y, v(t, x))| d_s \bigvee_{q=0}^{t} g_1(t, q) = 0,$$

$$\varphi_k = \sup \left\{ \Sigma_{n \geq k} \left[\left| \int_0^t \int_0^x g_n\Big(t, s, x, y, u(t, x)\Big) d_y g_2(x, y) d_s g_1(t, s) \right| \right];$$

$$t, s, x, y \in \mathbb{R}_+, u(t,x) \in \mathbb{R}^\infty \Big\}.$$

Moreover, assume that

$$\begin{aligned} A &= \sup\Big\{\Sigma_{n=1}^{\infty} \int_0^t a_n(t,s)d_s \bigvee_{p=0}^{s} g_1(t,p), \;\; t \in \mathbb{R}_+\Big\}, \\ G &= \sup\Big\{ \bigvee_{y=0}^{x} g_2(x,y); \;\; x \in \mathbb{R}_+\Big\}, \;\; \lim_{k \longrightarrow \infty} \varphi_k = 0. \end{aligned}$$

(H$_6$) $V_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is a continuous function and there exists continuous function $k : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the function $s \longrightarrow k(t,s)$ is integrable over $\mathbb{R}_+$ and the following conditions hold:

$$\begin{aligned} |V_n(t,s,u(t,x))| &\le k(t,s)|u_n(t,x)|^p, \\ |V_n(t,s,u(t,x)) - V_n(t,s,v(t,x)| &\le |u_n(t,x) - v_n(t,x)|^p k(t,s). \end{aligned}$$

for all $t, s, x \in \mathbb{R}_+$ and $u, v \in \ell_p$. Moreover, assume that

$$M = \sup_{t \in \mathbb{R}_+} \int_0^\infty k(t,s)ds.$$

(H$_7$) There exists a positive solution r_0 such that

$$2^{2p} e^{-2\tau} r_0^p (GA)^p + 2^{2p} e^{-\tau} f_0^p (GA)^p + 2^p e^{-\tau} r_0^p M^p + 2^p N_1 \le r_0^p,$$

Moreover, assume that $2^p M < 1$.

Theorem 3. Under the assumptions (H$_1$) – (H$_7$), Eq (1.1) has at least one solution $u(t,x) = \Big\{u_i(t,x)\Big\}_{i=1}^{\infty}$ in the space ℓ_p.

Proof. Let us define the operator G on $BC\Big(\mathbb{R}_+ \times \mathbb{R}_+, \ell_p\Big)$ by

$$\begin{aligned} (Gu)(t,x) &= \Big\{F_n\Big(t,s,f_1(t,u(t,x)) \int_0^t \int_0^x g_n\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s), \\ &\quad (Tu)(t,x) \int_0^\infty V_n(t,s,u(t,x))ds\Big)\Big\}. \end{aligned}$$

In view of our assumptions, for all $t, x \in \mathbb{R}_+$, we get

$$
\begin{aligned}
&\|(Gu)(t,x)\|_{\ell_p}^p \\
&= \Sigma_{i=1}^{\infty}|F_i\Big(t,s,f_1(t,u(t,x))\int_0^t\int_0^x g_i\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s), \\
&(Tu)(t,x)\int_0^{\infty}V_i(t,s,u(t,x))ds\Big)|^p \\
&\leq 2^p\Sigma_{i=1}^{\infty}|F_i\Big(t,s,f_1(t,u(t,x))\int_0^t\int_0^x g_i\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s), \\
&(Tu)(t,x)\int_0^{\infty}V_i(t,s,u(t,x))ds\Big)-F_i\Big(t,s,0,0\Big)|^p+2^p\Sigma_{i=1}^{\infty}|F_i\Big(t,s,0,0\Big)|^p \\
&\leq 2^p\Sigma_{i=1}^{\infty}\Big[e^{-\tau}|f_1(t,u(t,x))\int_0^t\int_0^x g_i\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s)|^p \\
&+e^{-\tau}|(Tu)(t,x)\int_0^{\infty}V_i(t,s,u(t,x))ds|^p\Big]+2^p\Sigma_{i=1}^{\infty}|F_i\Big(t,s,0,0\Big)|^p \\
&\leq 2^{2p}e^{-\tau}\Sigma_{i=1}^{\infty}\Big(|f_1(t,u(t,x))-f_1(t,0)|^p\Big)\times \\
&\Big(\int_0^t\int_0^x|g_i(t,s,x,y,u(t,x))|d_y\bigvee_{q=0}^{y}g_2(x,q)\bigvee_{p=0}^{s}d_s g_1(t,p)\Big)^p \\
&+2^{2p}e^{-\tau}\Sigma_{i=1}^{\infty}|f_1(t,0)|^p\Big(\int_0^t\int_0^x|g_i(t,s,x,y,u(t,x))|d_y\bigvee_{q=0}^{y}g_2(x,q)\bigvee_{p=0}^{s}d_s g_1(t,p)\Big)^p \\
&+2^p e^{-\tau}(\int_0^{\infty}k(t,s)ds)^p\Sigma_{i=1}^{\infty}|u_i(t,x)|^p+2^p\Sigma_{i=1}^{\infty}|F_i\Big(t,s,0,0\Big)|^p \\
&\leq 2^{2p}e^{-2\tau}\|u(t,x)\|_{\ell_p}^p(GA)^p+2^{2p}e^{-\tau}(f_0)^p(GA)^p+2^p e^{-\tau}M^p\|u(t,x)\|_{\ell_p}^p \\
&+2^p N_1.
\end{aligned}
\tag{3.1}
$$

Thus, by applying the last estimates and assumption (H_7) one can easily seen that G maps $\overline{B_{r_0}}$ into itself, where

$$\overline{B_{r_0}}=\Big\{u\in BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p);\|u\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)}\leq r_0\Big\}.$$

Next, we prove that the operator G is a continuous operator on the Ball $\overline{B_{r_0}}$. For this, take $\varepsilon > 0$ arbitrarily and $u(t,x)=\Big\{u_i(t,x)\Big\}_{i=1}^{\infty}$, $v(t,x)=\Big\{v_i(t,x)\Big\}_{i=1}^{\infty}\in\overline{B_{r_0}}$ with $\|u-v\|_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)}<\varepsilon$. Acordingly, taking into account our assumptions, for $(t,x)\in\mathbb{R}_+\times\mathbb{R}_+$ we have

$$
\begin{aligned}
&\|(Gu)(t,x)-(Gv)(t,x)\|_{\ell_p}^p \\
&\leq \Sigma_{i=1}^{\infty}e^{-\tau}|f_1(t,u(t,x))\int_0^t\int_0^x g_i\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s) \\
&-f_1(t,v(t,x))\int_0^t\int_0^x g_i\Big(t,s,x,y,v(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s)|^p \\
&+\Sigma_{i=1}^{\infty}e^{-\tau}|(Tu)(t,x)\int_0^{\infty}V_i(t,s,u(t,x))ds-(Tv)(t,x)\int_0^{\infty}V_i(t,s,v(t,x))ds|^p.
\end{aligned}
\tag{3.2}
$$

On the other hand, we have

$$\begin{aligned}
&|f_1(t,u(t,x))\int_0^t\int_0^x g_i\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s)\\
&-f_1(t,v(t,x))\int_0^t\int_0^x g_i\Big(t,s,x,y,v(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s)|^p\\
&\le 2^p|f_1(t,u(t,x))-f_1(t,v(t,x))|^p\\
&\times\Big(\int_0^t\int_0^x|g_n\Big(t,s,x,y,u(t,x)\Big)|d_y\bigvee_{p=0}^{y}g_2(x,p)d_s\bigvee_{q=0}^{t}g_1(t,q)\Big)^p\\
&+2^p|f_1(t,v(t,x))|^p(\int_0^t\int_0^x|g_n\Big(t,s,x,y,u(t,x)\Big)-\\
&g_n\Big(t,s,x,y,v(t,x)\Big)|d_y\bigvee_{p=0}^{y}g_2(x,p)d_s\bigvee_{q=0}^{t}g_1(t,q))^p\\
&\le e^{-\tau}2^p\|u(t,x)-v(t,x)\|_{\ell_p}\Big(\bigvee_{y=0}^{x}g_2(x,y)|\int_0^t a_n(t,s)d_s\bigvee_{q=0}^{t}g_1(t,q)\Big)^p\\
&+2^p|f_1(t,v(t,x))|^p\Big(\bigvee_{y=0}^{x}g_2(x,y)\int_0^t|g_n\Big(t,s,x,y,u(t,x)\Big)\\
&-g_n\Big(t,s,x,y,v(t,x)\Big)|d_s\bigvee_{q=0}^{t}g_1(t,q)\Big)^p\\
&\le e^{-\tau}2^p\|u(t,x)-v(t,x)\|_{\ell_p}(GA_i)^p\\
&+2^pG^p|f_1(t,v(t,x))\Big(\int_0^t|g_i\Big(t,s,x,y,u(t,x)\Big)\\
&-g_i\Big(t,s,x,y,v(t,x)\Big)|d_s\bigvee_{q=0}^{t}g_1(t,q)\Big)^p.
\end{aligned}\tag{3.3}$$

Further, by applying our assumptions, we arrive that

$$\begin{aligned}
&|(Tu)(t,x)\int_0^\infty V_i(t,s,u(t,x))ds-(Tv)(t,x)\int_0^\infty V_i(t,s,v(t,x))ds|^p\\
&\le 2^p|(Tu)(t,x)\int_0^\infty V_i(t,s,u(t,x))ds-(Tv)(t,x)\int_0^\infty V_i(t,s,u(t,x))ds|^p\\
&+2^p|(Tv)(t,x)\int_0^\infty V_i(t,s,u(t,x))ds-(Tv)(t,x)\int_0^\infty V_i(t,s,v(t,x))ds|^p\\
&\le 2^p\|u(t,x)-v(t,x)\|_{\ell_p}^p|u_i(t,x)|^pM^p+M^p|u_i(t,x)-v_i(t,x)|^p.
\end{aligned}\tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we conclude that

$$\begin{aligned}
&\|(Gu)(t,x)-(Gv)(t,x)\|_{\ell_p}^p\\
&\le\Sigma_{i=1}^\infty e^{-2\tau}2^p\|u(t,x)-v(t,x)\|_{\ell_p}^p(GA_i)^p\\
&+2^pG^pe^{-\tau}|f_1(t,v(t,x))|^p(\Sigma_{i=1}^\infty\int_0^t|g_i\Big(t,s,x,y,u(t,x)\Big)\\
&-g_i\Big(t,s,x,y,v(t,x)\Big)|d_s\bigvee_{q=0}^{t}g_1(t,q))^p\\
&+\Sigma_{i=1}^\infty|u_i(t,x)|^pe^{-\tau}2^pM^p\|u(t,x)-v(t,x)|_{\ell_p}^p\\
&+e^{-\tau}2^pM^p\Sigma_{i=1}^\infty|u_i(t,x)-v_i(t,x)|^p.
\end{aligned}\tag{3.5}$$

Using (H_5), there exists $T>0$ such that for $t>T$, we get

$$\Sigma_{i=1}^\infty\int_0^t|g_i\Big(t,s,x,y,u(t,x)\Big)-g_i\Big(t,s,x,y,v(t,x)\Big)|d_s\bigvee_{q=0}^{t}g_1(t,q)<\varepsilon.$$

Hence, by (3.5), we conclude that

$$\begin{aligned}&\|(Gu)(t,x)-(Gv)(t,x)\|^p_{\ell_p}\\&\leq 2^pe^{-2\tau}\|u-v\|^p_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)}(GA)^p+2^pG^p\varepsilon^pe^{-\tau}\|v(t,x)\|^p_{\ell_p}\\&+2^pM^pe^{-\tau}\|u-v\|^p_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)}\|u(t,x)\|^p_{\ell_p}\\&+e^{-\tau}2^pM^p\|u-v\|^p_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)}.\end{aligned}\tag{3.6}$$

For $t\in[0,T]$ we have

$$\begin{aligned}&\|(Gu)(t,x)-(Gv)(t,x)\|^p_{\ell_p}\\&\leq 2^pe^{-2\tau}\|u-v\|^p_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)}(GA)^p+\|v(t,x)\|^p_{\ell_p}2^pG^p\omega(g,\varepsilon)^pe^{-\tau}\\&+2^pM^pe^{-\tau}\|u-v\|^p_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)}\|u(t,x)\|^p_{\ell_p}\\&+e^{-\tau}2^pM^p\|u-v\|^p_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)},\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}\omega(g,\varepsilon)\quad&=\sup\Big\{\Sigma_{n=1}^{\infty}|g_n(t,s,x,,y,u)-g_n(t,s,x,y,v)|;\\&(t,s)\in\Delta_1,(x,y)\in\Delta_2,u,v\in\ell_p,\|u-v\|_{BC(\mathbb{R}_+,\mathbb{R}_+,\ell_p)}<\varepsilon\Big\},\\\Delta_1\quad&=\Big\{(t,s)\in\mathbb{R}^2;\ \ s\leq t\leq T\Big\},\ \ \Delta_2=\Big\{(x,y)\in\mathbb{R}^2;\ \ y\leq x\leq T\Big\}.\end{aligned}$$

and $\omega(g,\varepsilon)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Consequently, G is continuous on the ball $\overline{B_{r_0}}$. To finish the proof, we prove that the condition (2.1) of Theorem 2 is fulfilled. Let X be a nonempty and bounded subset of the ball $\overline{B_{r_0}}$. Assume that

$$\begin{aligned}(H_n)(u)\quad&=f_1(t,u(t,x))\int_0^t\int_0^x g_n\Big(t,s,x,y,u(t,x)\Big)d_yg_2(x,y)d_sg_1(t,s),\\(D_n)(u)\quad&=(Tu)(t,x)\int_0^{\infty}V_n(t,s,u(t,x))ds.\end{aligned}$$

Thus, by applying our assumptions, we infer that

$$\begin{aligned}&\chi_{\ell_p}\Big(G(X)\Big)(t,x)=\lim_{n\longrightarrow\infty}\Bigg[\sup_{u(t,x)\in X}\Big\{\Sigma_{k\geq n}|F_k\Big(t,s,(H_k)(u),(D_k)(u)\Big)|^p\Big\}^{\frac{1}{p}}\Bigg]\\&=\lim_{n\longrightarrow\infty}\Bigg[\sup_{u(t,x)\in X}\Big\{\Sigma_{k\geq n}|F_k\Big(t,s,(H_k)(u),(D_k)(u)\Big)-F_k\Big(t,s,0,0\Big)\\&+F_k\Big(t,s,0,0\Big)|^p\Big\}^{\frac{1}{p}}\Bigg]\leq 2^pe^{-\tau}\lim_{n\longrightarrow\infty}\Bigg[\sup_{u(t,x)\in X}\Big\{\Sigma_{k\geq n}\Big\{|(H_k)(u)|^p\\&+|(D_k)(u)|^p\Big\}\Big\}^{\frac{1}{p}}\Bigg]\leq 2^pe^{-\tau}\lim_{n\longrightarrow\infty}\Bigg[\sup_{u(t,x)\in X}\Big\{\Sigma_{k\geq n}\Big\{e^{-\tau}\|u(t,x)\|^p_{\ell_p}\varphi_n\\&+M^p|u_k(t,x)|^p\Big\}^{\frac{1}{p}}\Bigg]=2^pe^{-\tau}M\lim_{n\longrightarrow\infty}\Bigg[\sup_{u(t,x)\in X}\Big\{\Sigma_{k\geq n}|u_k(t,x)|^p\Big\}^{\frac{1}{p}}\Bigg].\end{aligned}\tag{3.8}$$

Hence,

$$\chi_{\ell_p}(G(X))(t,x)) \leq 2^p e^{-\tau} M \lim_{n\longrightarrow\infty}\left[\sup_{u(t,x)\in X}\left\{\Sigma_{k\geq n}|u_k(t,x)|^p\right\}^{\frac{1}{p}}\right]. \tag{3.9}$$

Consequently,

$$\begin{aligned}
&\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}_+}\chi_{\ell_p}(G(X))(t,x)) \\
&= \chi_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)}(GX) \\
&\leq \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}_+} 2^p e^{-\tau} M \lim_{n\longrightarrow\infty}\left[\sup_{u(t,x)\in X}\left\{\Sigma_{k\geq n}|u_k(t,x)|^p\right\}^{\frac{1}{p}}\right].
\end{aligned}$$

By passing to logarithms, we get

$$\ln\Big(\chi_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)})(GX)\Big) + \tau \leq \ln\Big(\chi_{BC(\mathbb{R}_+\times\mathbb{R}_+,\ell_p)}(X)\Big) \tag{3.10}$$

Now applying Theorem 2 with $f(t) = \ln(t)$ and $\theta(t) = \tau$, we obtain that G has a fixed point and the proof is completed. □

Example 1. Now, we investigate the following system of integral equations:

$$\begin{aligned}
u_n(t,x) \quad &= \tfrac{(e^{-\tau}-t-n)^{\frac{1}{p}}}{2}\sin\left(\frac{(e^{-t-\tau})^{\frac{1}{p}}\sin\Big(\|u(t,x)\|_{\ell_p}\Big)}{2}\right. \\
&\times \int_0^t\int_0^x \arctan\left(\tfrac{\frac{1}{2^n}\times e^{-3t+s}}{8+|x|+|y|+|u_n(t,x)|}\right)\tfrac{e^x}{1+y^2e^{2x}}\tfrac{e^t}{1+t^2}dyds \\
&\left.+\cos\left(\tfrac{1}{1+\|u(t,x)\|_{\ell_p}}\right)\int_0^\infty \tfrac{e^{-s}}{1+\frac{t}{8}}\sin\Big(|u_n(t,x)|\Big)ds\right);
\end{aligned} \tag{3.11}$$

Observe that Eq (3.11) is a special case of the infinite system (1.1) if we put

$$\begin{aligned}
F_n\Big(t,s,x,y\Big) \quad &= \frac{(e^{-\tau-t-n})^{\frac{1}{p}}}{2}\sin\Big(x+y\Big), \\
g_n(t,s,x,y,u(t,x)) \quad &= \arctan\left(\frac{\frac{1}{2^n}\times e^{-3t+s}}{8+|x|+|y|+|u_n(t,x)|}\right), \\
f_1(t,u(t,x)) \quad &= \frac{(e^{-t-\tau})^{\frac{1}{p}}\sin\Big(\|u(t,x)\|_{\ell_p}\Big)}{2}, \\
a_n(t,s) \quad &= \frac{1}{2^n}e^{-3t+s}, \\
g_1(t,s) \quad &= \frac{se^t}{1+t^2}, \\
g_2(x,y) \quad &= \arctan\Big(ye^x\Big), \\
V_n(t,s,u(t,x)) \quad &= \frac{e^{-s}}{1+\frac{t}{8}}\sin\Big(|u_n(t,x)|\Big), \\
k(t,s) &= \frac{e^{-s}}{1+\frac{t}{8}},
\end{aligned}$$

$$(Tu)(t,x) \qquad = \cos\Big(\frac{1}{1+\|u(t,x)\|_{l_p}}\Big).$$

Thus, it is easily seen that F_n and f_1 satisfy assumptions (H_1) and (H_2) with $N_1 = 0$ and $f_0 = 0$. Further, the operator T satisfies hypothesis (H_3). To justify assumption (H_5) , let $t, sx, y \in \mathbb{R}_+$ and $u, u \in \ell_p$. Then, we have

$$|g_n(t,s,x,y,u(t,x))| \qquad \leq \frac{1}{2^n} e^{-3t+s} = a_n(t,s).$$

Since $\frac{\partial g_1}{\partial s} = \frac{e^t}{1+t^2} > 0$, then $\bigvee_{q=0}^{s} g_1(t,q) = g_1(t,s) - g_1(t,0) = \frac{se^t}{1+t^2}$. Consequently, we have

$$\begin{aligned}\lim_{t\longrightarrow\infty}\int_0^t a_n(t,s)d_s\bigvee_{q=0}^{s} g_1(t,q) \qquad &= \lim_{t\longrightarrow\infty}\int_0^t \frac{1}{2^n}e^{-3t+s}(\frac{e^t}{1+t^2})ds\\ &= \lim_{t\longrightarrow\infty}\frac{1}{2^n}\frac{e^{-2t+s}}{1+t^2}|_0^t = 0\end{aligned}$$

Inconsequence,

$$\begin{aligned}&\lim_{t\longrightarrow\infty}\Sigma_{n\geq1}\int_0^t|g_n(t,s,x,y,u(t,x)) - g_n(t,s,x,y,v(t,x))|d_s\bigvee_{q=0}^{t} g_1(t,q) = 0,\\ &A = \sup\Big\{\Sigma_{i=1}^{\infty}\int_0^t a_n(t,s)d_s\bigvee_{p=0}^{s} g_1(t,s),\ \ t\in\mathbb{R}_+\Big\},\\ &\varphi_k = \sup\Big\{\Sigma_{n\geq k}\Big[\int_0^t\int_0^x g_n\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s)\Big];\\ &\quad t,s,x,y\in\mathbb{R}_+, u(t,x)\in\ell_p\Big\} \leq G(\frac{e^{-2t}}{1+t^2} - \frac{e^{-t}}{1+t^2})\Sigma_{n\geq k}\frac{1}{2^n}.\end{aligned}$$

So, $\varphi_k \longrightarrow 0$. On the other hand the function $V_n(t,s,u(t,x)) = \frac{e^{-s}}{1+\frac{t}{8}}\sin\Big(|u_n(t,x)|\Big)$ verifies assumption (H_6) with $k(t,s) = \frac{e^{-s}}{1+\frac{t}{8}}$ and $M = 1$. To show that the functions g_1 and g_2 satisfy assumption (H_4), let first note that the functions g_1 and g_2 are increasing on every interval of the form $[0,t]$ and g_2 is bounded on the triangle $\triangle_2$. Consequently, the function $y \longrightarrow g_2(x,y)$ has bounded variation on the interval $[0,x]$ and we have

$$\bigvee_{y=0}^{x} g_2(x,y) = g_2(x,y) - g_2(x,0) = g_2(x,y) \leq \frac{\pi}{4}.$$

So, $G \leq \frac{\pi}{4}$. We can take $G = \frac{\pi}{4}$. Consequently, all conditions of Theorem 3 are satisfied and Theorem 3 implies that the infinite system (3.11) has at least one solution which belongs to the space ℓ_p.

4. c_0-solvability of the infinite system (1.1)

Now the existence of solutions of the system (1.1) is studied in the space c_0. In this case, we need the following assumptions.

(D$_1$) $F_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist positive real numbers $\tau > 0$ such that

$$|F_n(t,s,x_1,y_1) - F_n(t,s,x_2,y_2)| \leq e^{-\tau}(|x_1 - x_2| + |y_1 - y_2|),$$

for all $t, s \in \mathbb{R}_+$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Moreover, assume that

$$\lim_{i \longrightarrow \infty} |F_i(t,s,0,0)| = 0, \quad M_1 = \sup\Big\{|F_i(t,s,0,0)|; t,s \in \mathbb{R}_+, i \geq 1\Big\}.$$

(D$_2$) $f_1 : \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is continuous with $f_0 = \sup_{t \in \mathbb{R}_+} |f(t,0)|$ and there exist positive real numbers $\tau > 0$ such that

$$\begin{aligned} |f_1(t,u(t,x)) - f_1(t,v(t,x))| &\leq e^{-\tau} \sup_{n \geq 1} \Big\{|u_i(t,x) - v_i(t,x)|; \;\; i \geq n\Big\}, \\ |f_1(t,u(t,x))| &\leq e^{-\tau} \sup_{n \geq 1} \Big\{|u_i(t,x)|; \;\; i \geq n\Big\} \end{aligned}$$

for all $t, x \in \mathbb{R}_+$ and $u(t,x) = \Big\{u_i(t,x)\Big\}, v(t,x) = \Big\{v_i(t,x)\Big\} \in c_0$

(D$_3$) $T : BC(\mathbb{R}_+ \times \mathbb{R}_+, c_0) \longrightarrow BC(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ is a continuous operator such that

$$\begin{aligned} &|(Tu)(t,x) - (Tv)(t,x)| \leq \sup_{n \geq 1} \Big\{|u_i(t,x) - v_i(t,x)|; \;\; i \geq n\Big\}, \\ &|(Tu)(t,x)| \leq 1. \end{aligned}$$

for all $u, v \in BC(\mathbb{R}_+ \times \mathbb{R}_+, c_0)$ and $t, x \in \mathbb{R}_+$.

(D$_4$) For any fixed $t > 0$ the function $s \longrightarrow g_i(t,s)$ has a bounded variation on the interval $[0,t]$ and the functions $t \longrightarrow \bigvee_{s=0}^{t} g_i(t,s)$ are bounded on $\mathbb{R}_+$. Moreover, for arbitrarily fixed $T > 0$ the function $w \longrightarrow \bigvee_{z=0}^{w} g_i(w,z)$ is continuous on the interval $[0,T]$ for $i = 1,2$.

(D$_5$) $g_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is continuous and there exist continuous functions $a_n : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned} &|g_n(t,s,x,y,u(t,x))| \leq a_n(t,s), \\ &\lim_{t \longrightarrow \infty} \int_0^t |g_n(t,s,x,y,u(t,x)) - g_n(t,s,x,y,v(t,x))| d_s \bigvee_{q=0}^{t} g_1(t,q) = 0, \end{aligned}$$

for all $t, s, x, y \in \mathbb{R}_+$ and $u, v \in \mathbb{R}^\infty$. Moreover, assume that

$$\lim_{n \longrightarrow \infty} \int_0^t a_n(t,s) d_s \bigvee_{p=0}^{s} g_1(t,p) = 0, \;\; A = \sup\Big\{ \int_0^t a_n(t,s) d_s \bigvee_{p=0}^{s} g_1(t,p); n \in \mathbb{N}\Big\},$$

$$G = \sup\Big\{ \bigvee_{y=0}^{x} g_2(x,y); \;\; x \in \mathbb{R}_+\Big\}, \;\; G_1 = \sup\Big\{ \bigvee_{z=0}^{w} g_1(w,z); \;\; w \in [0,T]\Big\}.$$

where $T > 0$ is arbitrarily fixed.

(D_6) $V_n : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is a continuous function and there exists continuous function $k : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the function $s \longrightarrow k(t,s)$ is integrable over $\mathbb{R}_+$ and the following conditions hold:

$$|V_n(t,s,u(t,x))| \le k(t,s) \sup_{n\ge 1}\Big\{|u_i(t,x)|;\ \ i \ge n\Big\},$$
$$|V_n(t,s,u(t,x)) - V_n(t,s,v(t,x)| \le \sup_{n\ge 1}\Big\{|u_i(t,x) - v_i(t,x); i \ge n\Big\}k(t,s).$$

for all $t, s, x \in \mathbb{R}_+$ and $u, v \in c_0$. Moreover, assume that

$$M = \sup_{t\in\mathbb{R}_+}\int_0^\infty k(t,s)ds < 1,\ \ e^{-2\tau}GA + f_0GAe^{-\tau} + Me^{-\tau} + Me^{-\tau} < 1.$$

Theorem 4. Under assumptions (D_1) − (D_6), the infinite system (1.1) has at least one solution $u(t) = \Big\{u_i(t,x)\Big\}_{i=1}^{\infty}$ belonging to the space c_0.

Proof. Define the operator G on the space $BC\Big(\mathbb{R}_+ \times \mathbb{R}_+, c_0\Big)$ as

$$\begin{aligned}&(Gu)(t,x)\\&= \Big\{F_n\Big(t,s,f_1(t,u(t,x))\int_0^t\int_0^x g_n\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s),\\&(Tu)(t,x)\int_0^\infty V_n(t,s,u(t,x))ds\Big)\Big\}\end{aligned}$$

where $t, x \in \mathbb{R}_+$. We show that

$$\overline{B_{r_0}} = \Big\{u \in BC\Big(\mathbb{R}_+ \times \mathbb{R}_+, c_0\Big);\ \ \|u\|_{BC\big(\mathbb{R}_+\times\mathbb{R}_+,c_0\big)} \le r_0\Big\}$$

is G-invariant where $i = 1, 2, ...$ and $t, x \in \mathbb{R}_+$. Assume that

$$\begin{aligned}(H_n)(u) &= \int_0^t\int_0^x g_n\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s),\\(D_n)(u) &= (Tu)(t,x)\int_0^\infty V_n(t,s,u(t,x))ds.\end{aligned}$$

For arbitrarily fixed $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have

$$\begin{aligned}&\|(Gu)(t,x)\|_{c_0}\\&= \sup_{n\ge1}|F_n\Big(t,s,(H_n)(u),(D_n)(u)\Big)|\\&\le \sup_{n\ge1}\Big[|F_n\Big(t,s,f_1(t,u(t,x))(H_n)(u),(D_n)(u)\Big) - F_n\Big(t,s,0,0\Big)| + |F_n(t,s,0,0)|\Big]\\&\le \sup_{n\ge1}\Big[e^{-\tau}|(f_1(t,u(t,x))H_n)(u)| + e^{-\tau}|(D_n)(u)|\Big] + \sup_{n\ge1}|F_n\Big(t,s,0,0\Big)|\end{aligned}$$

$$\leq \sup_{n\geq 1}\Big[e^{-\tau}\Big(|f_1(t,u(t,x))-f_1(t,0)|+|f_1(t,0)|\Big)(|H_n)(u)|$$
$$+e^{-\tau}\|u(t,x)|_{c_0}M\Big] \leq \sup_{n\geq 1}\Big[e^{-2\tau}\Big\{|u_i(t,x)|;\;\; i\geq n\Big\}GA$$
$$+f_0GAe^{-\tau}+e^{-\tau}\|u(t,x)\|_{c_0}M$$
$$\leq (e^{-2\tau}GA+e^{-\tau}f_0GA+Me^{-\tau})\|u(t,x)\|_{c_0}.$$

Consequently,

$$\|Gu\| \leq \|u(t,x)\|_{c_0} \tag{4.1}$$

By applying (4.1) , one can easily seen that G maps the ball $\overline{B_{r_0}}$ into itself. Next, the continuity property of the operator G will be proved on the ball $\overline{B_{r_0}}$. Let $u, v \in B_{r_0}$ and $\varepsilon > 0$ such that $\|u - v\|_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)} < \varepsilon$. Thus for all $t, x \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\|(Gu)(t,x)-(Gv)(t,x)\|_{c_0}\\
&=\sup\nolimits_{n\geq 1}|F_n\Big(t,s,f_1(t,u(t,x))H_n(u),(D_nu)\Big)\\
&-F_n\Big(t,s,f_1(t,v(t,x))H_n)(v),(D_nv)\Big)|\\
&\leq \sup\nolimits_{n\geq 1}\Big\{e^{-\tau}|f_1(t,u(t,x))H_n)(u)-f_1(t,v(t,x))H_n)(v)|\\
&+e^{-\tau}|(D_n)(u)-(D_n)(v)|\Big\}.
\end{aligned} \tag{4.2}$$

Besides, we have

$$\begin{aligned}
&|f_1(t,u(t,x))H_n)(u)-f_1(t,v(t,x))H_n)(v)|\\
&\leq 2^pGAe^{-\tau}\sup\nolimits_{n\geq 1}\Big\{|u_i(t,x)-v_i(t,x)|;\;\; i\geq n\Big\}\\
&+2^pe^{-\tau}G\sup\nolimits_{n\geq 1}\Big\{|v_i(t,x)|;\;\; i\geq n\Big\}\\
&\times\textstyle\int_0^t|g_n(t,s,x,y,u(t,x))-g_n(t,s,x,y,v(t,x))d_s(\bigvee_{q=0}^t g_1(t,q).
\end{aligned} \tag{4.3}$$

By assumption (D_5), there exists $T > 0$ such that for $t > T$, we have

$$\int_0^t |g_n(t,s,x,y,u(t,x))-g_n(t,s,x,y,v(t,x))|d_s(\bigvee_{q=0}^{t} g_1(t,q)<\varepsilon.$$

Further, the assumptions (D_3) and (D_6) give us the following eastimates

$$\begin{aligned}
&|(Tu)(t,x)\textstyle\int_0^\infty V_n(t,s,u(t,x))ds-(Tv)(t,x)\int_0^\infty V_n(t,s,v(t,x))ds|\\
&\leq M\|u(t,x)-v(t,x)\|_{c_0}\|u(t,x)\|_{c_0}\\
&+|(Tv)(t,x)|\textstyle\int_0^\infty|V_n(t,s,u(t,x))-V_n(t,s,v(t,x))|ds\\
&\leq M\|u(t,x)-v(t,x)\|_{c_0}\|u(t,x)\|_{c_0}\\
&+M\|u(t,x)-v(t,x)\|_{c_0}.
\end{aligned} \tag{4.4}$$

Applying (4.2), (4.3) and (4.4),we have

$$\begin{aligned}
&\|(Gu)(t,x)-(Gv)(t,x)\|_{c_0}\\
&\leq 2^p e^{-2\tau} GA \sup_{n\geq 1}\Big\{|u_i(t,x)-v_i(t,x)|;\ \ i\geq n\Big\}\\
&+2^p e^{-2\tau} G \sup_{n\geq 1}\Big\{|v_i(t,x)|;\ \ i\geq n\Big\}\varepsilon + M\|u-v\|_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)}\\
&+M(\|u(t,x)\|_{c_0})\|u-v\|_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)}\\
&\leq 2^p e^{-2\tau} GA\varepsilon + 2^p e^{-\tau} G\|v(t,x)\|_{c_0}\varepsilon\\
&+e^{-\tau}M\varepsilon + Me^{-\tau}\|u(t,x)\|_{c_0})\varepsilon.
\end{aligned} \tag{4.5}$$

For $t\in[0,T]$, we have

$$\begin{aligned}
&\|(Gu)(t,x)-(Gv)(t,x)\|_{c_0}\\
&\leq 2^p e^{-2\tau} GA \sup_{n\geq 1}\Big\{|u_i(t,x)-v_i(t,x)|;\ \ i\geq n\Big\}\\
&+2^p e^{-\tau} G \sup_{n\geq 1}\Big\{|u_i(t,x);\ \ i\geq n\Big\}G_1\omega(g_n,\varepsilon)\\
&+M\|u-v\|_{c_0} + M\|u\|_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)}\|u-v\|_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)}\\
&\leq e^{-\tau}GA\varepsilon + e^{-\tau}GG_1\|v(t,x)\|_{c_0}\omega(g_n,\varepsilon) + M\varepsilon\\
&+M\|u\|_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)}\varepsilon,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
\omega(g_n,\varepsilon) \qquad &= \sup\Big\{|g_n(t,s,x,y,u(t,x)) - g_n(t,s,x,y,v(t,x))|;\\
&(t,s)\in\Delta_1,(x,y)\in\Delta_2,u,v\in\mathbb{R}^\infty;\|u-v\|_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)}<\varepsilon\Big\}.
\end{aligned}$$

Moreover, in light of the continuity of V on $\triangle_1\times\triangle_2\times\mathbb{R}^\infty$, we have $\omega(g_n,\varepsilon)\longrightarrow 0$. Now, combining (4.5) and (4.6) implies that G is continuous on the Ball $\overline{B_{r_0}}$. In what follows let X be a nonempty subset of the ball $\overline{B_{r_0}}$, In view of the formula (2.3) and our assumptions, we have

$$\begin{aligned}
&\chi_{c_0}(GX)(t,x)\\
&=\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\max_{i\geq n}|F_i\Big(t,s,(H_i)(u),(D_i)(u)|\Big)\Big\}\\
&\leq \lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\max_{i\geq n}|F_i\Big(t,s,(H_i)(u),(D_i)(u)|\Big)-F_i\Big(t,s,0,0\Big)|\\
&+|F_i\Big(t,s,0,0\Big)|\Big\}\leq \lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\max_{i\geq n}\Big(e^{-\tau}|(H_i)(u)|+e^{-\tau}|(D_i)(u)|\Big)\Big)\Big\}\\
&\leq \lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\max_{i\geq n}\Big(e^{-\tau}|f_1(t,u(t,x))-f_1(t,0)|(H_i)(u)|
\end{aligned}$$

$$+e^{-\tau}|f_1(t,0)||(H_i)(u)| + e^{-\tau}|(D_i)(u)|\Big)\Big)\Big)\Big\}$$

$$\leq \lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\max_{i\geq n}\Big(e^{-2\tau}\sup_{n\geq 1}\Big\{|u_i(t,x)|;\ \ i\geq n\Big\}GA$$

$$+f_0GA + e^{-\tau}\sup_{n\geq 1}\Big\{|u_i(t,x)|;\ \ i\geq n\Big\}M\Big)\Big\}.$$

Consequently,

$$\chi_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)}(GX) \leq Me^{-\tau}\sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}_+}\lim_{n\longrightarrow\infty}\Big\{\sup_{u\in X}\Big(\max_{i\geq n}|u_i(t,x)|\Big)\Big\}.$$

As, $M<1$, by passing to logarithms, we have

$$\tau + \ln(\chi_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)}(GX)\Big) \leq \ln\Big(\chi_{BC\left(\mathbb{R}_+\times\mathbb{R}_+,c_0\right)}(X))\Big).$$

Thus all conditions of Theorem 2 hold true with $f(t)=\ln(t)$ and $\theta(t)=\tau$ and by Theorem 2 there exists $\Big\{u_i(t,x)\Big\}_{i=1}^{\infty}\in c_0$ such that

$$\begin{aligned} u_n(t,x) \ &= F_n\Big(t,s,f_1(t,u(t,x))\int_0^t\int_0^x g_n\Big(t,s,x,y,u(t,x)\Big)d_yg_2(x,y)d_sg_1(t,s),\\ &(Tu)(t,x)\int_0^\infty V_n(t,s,u(t,x))ds\Big). \end{aligned} \tag{4.7}$$

Hence, the proof is completed. □

Example 2.

$$Now we investigate \quad \begin{aligned} &u_n(t,x)\\ &= e^{-t-s-\tau-n}\sqrt[3]{\sqrt[5]{\arctan\Big(e^{-\tau}\Sigma_{k\geq n}\frac{|u_k(t,x)|}{1+k^2}\Big)(H_n)(u)} + \sqrt[7]{(D_n)(u)}} \end{aligned} \tag{4.8}$$

on the space c_0. Taking

$$\begin{aligned} (D_n)(u) &= e^{-100}\Sigma_{k\geq n}\frac{\sin\Big(|u_k(t,x)|\Big)}{(1+k^2)}\int_0^\infty e^{-t-s-n}\Sigma_{k\geq n}\frac{|u_k(t,x)|}{10^n(1+k^2)}ds,\\ (H_n)(u) &= \int_0^t\int_0^x \arctan\Big(\frac{e^{s+t}2^{-n}}{8+|u(t,x)|}\Big)\frac{e^{-2t}}{1+t^2}\times\frac{e^x}{1+y^2e^{2x}}dyds,\\ F_n(t,s,x,y) &= e^{-\tau-t-s-n}\sqrt[3]{\sqrt[5]{x}+\sqrt[7]{y}},\\ f_1(t,u(t,x)) &= \arctan\Big(e^{-\tau}\Sigma_{k\geq n}\frac{|u_k(t,x)|}{1+k^2}\Big),\\ g_n(t,s,x,y,u(t,x)) &= \arctan\Big(\frac{e^{s+t}2^{-n}}{8+|u(t,x)|}\Big), \end{aligned}$$

$$
\begin{aligned}
g_1(t,s) &= \frac{se^{-2t}}{1+t^2},\\
g_2(x,y) &= \arctan\Big(ye^x\Big),\\
V_n(t,s,u(t,x)) &= e^{-t-s-n}\Sigma_{k\ge n}\frac{|u_k(t,x)|}{10^n(1+k^2)},\\
k(t,s) &= e^{-t-s},\\
(Tu)(t,x) &= e^{-100}\Sigma_{k\ge n}\frac{\sin\Big(|u_k(t,x)|\Big)}{(1+k^2)}\ n\in\mathbb{N},
\end{aligned}
$$

in the system (1.1) , the system of integral Eq (4.8) is obtained. Note that the functions F_n and f_1 satisfy conditions (D_1) and (D_2). Indeed, we have

$$
\begin{aligned}
|F_n(t,x_1,y_1)-F_n(t,x_2,y_2)| \quad &= e^{-\tau-n-t}\Bigg[|\sqrt[3]{\sqrt[5]{x_1}+\sqrt[7]{y_1}}-\sqrt[3]{\sqrt[5]{x_1}+\sqrt[7]{y_1}|}\Bigg]\\
&\le e^{-\tau}\Bigg[\sqrt[3]{|\ \sqrt[5]{x_1}+\sqrt[7]{y_1}-\sqrt[5]{x_2}-\sqrt[7]{y_2}\ |}\ \Bigg]\\
&\le e^{-\tau}\Bigg[\sqrt[3]{\sqrt[5]{|\ x_1-x_2\ |}+\sqrt[7]{|\ y_1-y_2|}}\Bigg]\\
&\le e^{-\tau}\Bigg[|x_1-x_2|+|y_1-y_2|\Bigg],\\
M_1 \quad &= 0,\ \lim_{n\longrightarrow\infty}F_n\Big(t,s,0,0\Big)=0,\\
|f_1\Big(t,u(t,x)\Big)| \quad &\le \sup_{n\ge1}\Big\{|u_i(t,x)|;\ \ i\ge n\Big\},\\
|f_1\Big(t,u(t,x)\Big)-f_1\Big(t,v(t,x)\Big)| \quad &\le \sup_{n\ge1}\Big\{|u_i(t,x)|-|v_i(t,x);\ \ i\ge n\Big\}
\end{aligned}
$$

Also, it can easily be seen that the operator T satisfies assumption (D_3) and

$$
\begin{aligned}
|(Tu)(t,x)| \quad &\le e^{-100}\frac{\pi^2}{6}\sup_{n\ge1}\Big\{|u_i(t,x)|;\ \ i\ge n\Big\},\\
|(Tu)(t,x)-(Tv)(t,x)| \quad &\le e^{-\tau}\frac{\pi^2}{6}\sup_{n\ge1}\Big\{|u_i(t,x)-v_i(t,x)|;\ \ i\ge n\Big\}.
\end{aligned}
$$

Moreover, since $\frac{\partial g_1}{\partial s}=\frac{e^{-2t}}{1+t^2}>0$, so g_1 is increasing and we have

$$
\bigvee_{q=0}^{s} g_1(t,q)=g_1(t,s)-g_1(t,0)=g_1(t,s)=\frac{se^{-2t}}{1+t^2}>0
$$

Consequently,

$$
|g_n(t,s,x,y,u(t,x))|\le e^{s+t}2^{-n},
$$

$$\lim_{t\longrightarrow\infty}\int_0^t |g_n(t,s,x,y,u(t,x)) - g_n(t,s,x,y,v(t,x))| d_s \bigvee_{q=0}^{t} g_1(t,q)$$
$$\leq\ 2\lim_{t\longrightarrow\infty}\int_0^t e^{t+s}\frac{e^{-2t}}{1+t^2}ds = 0$$

Again, we have

$$\bigvee_{q=0}^{y} g_2(x,y) = g_2(x,y) - g_2(x,0) = g_2(x,y) \leq \frac{\pi}{4},$$
$$\lim_{n\longrightarrow\infty}\int_0^t a_n(t,s)d_s \bigvee_{q=0}^{s} g_1(t,q) = \lim_{n\longrightarrow\infty} 2^{-n}(\frac{1}{1+t^2} - \frac{e^{-t}}{1+t^2}) = 0.$$

So, $G = \frac{\pi}{4}$ and $A < \infty$. On the other hand the function $V_n(t,s,u(t,x)) = e^{-t-s-n}\Sigma_{k\geq n}\frac{|u_k(t,x)|}{10^n(1+k^2)}$ verifies assumption (D_6) with $k(t,s) = e^{-t-s}$ and $M = 1$. By applying the continuity of the function $h \longrightarrow \bigvee_{z=0}^{w} g_i(h,z)$ on the interval $[0,T]$ we can take $G_1 = \sup\left\{ \bigvee_{z=0}^{w} g_1(w,z) :\ \ w\in[0,T]\right\}$ where $T > 0$ is arbitrarily fixed. Thus all conditions of Theorem 4 are satisfied and by applying Theorem 4, infinite system (4) has at least one solution in the space c_0

5. Conclusions

We studied the existence of solutions for an infinite system of integral equations of Volterra-Stieltjes type of the following form in the Banach sequence spaces ℓ_p and c_0 via the techniques of measures of noncompactness and Darbo's fixed point theorem.

$$\begin{aligned} u_n(t,x) &= F_n\Big(t,s,f_1(t,u(t,x)) \textstyle\int_0^t\int_0^x g_n\Big(t,s,x,y,u(t,x)\Big)d_y g_2(x,y)d_s g_1(t,s), \\ &\quad (Tu)(t,x)\textstyle\int_0^\infty V_n(t,s,u(t,x))ds\Big); \\ u(t,x) &= \Big\{u_i(t,x)\Big\}_{i=1}^{\infty}, \quad u_i(t,x)\in BC(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}), \end{aligned}$$

where $BC(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R})$ is the space of all real functions $u(t,x) = u : \mathbb{R}_+\times\mathbb{R}_+ \longrightarrow \mathbb{R}$, which are defined, continuous and bounded on the set $\mathbb{R}_+\times\mathbb{R}_+$ with a supremum norm $\|u\| = \sup\Big\{|u(t,x)| : (t,x) \in \mathbb{R}_+\times\mathbb{R}_+\Big\}$. Some examples in the Banach sequence spaces ℓ_p and c_0 are also given to ascertain the usefulness of our main result.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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