Research article

On analytic multivalent functions associated with lemniscate of Bernoulli

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Abstract: In this paper, we establish some sufficient conditions for analytic functions associated with lemniscate of Bernoulli. In particular, we determine conditions on $\alpha$ such that

$$1 + \alpha \frac{z^{2+p(j-1)}g^{\prime}(z)}{pg^j(z)}, \text{ for each } j = 0, 1, 2, 3,$$

are subordinated by Janowski function, then $\frac{g(z)}{z^p} < \sqrt{1 + z}$, $(z \in \mathbb{D})$. By choosing particular values of functions $g$, we obtain some sufficient conditions for multivalent starlike functions associated with lemniscate of Bernoulli.

Keywords: multivalent functions; subordination; lemniscate of Bernoulli; Janowski functions

Mathematics Subject Classification: 30C45, 30C50

1. Introduction and definitions

To understand in a clear way the notions used in our main results, we need to add here some basic literature of Geometric function theory. For this we start first with the notation $\mathcal{A}$ which denotes the class of holomorphic or analytic functions in the region $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and if a function $g \in \mathcal{A}$, then the relations $g(0) = g^{\prime}(0) - 1 = 0$ must hold. Also, all univalent functions will be in a subfamily $\mathcal{S}$ of $\mathcal{A}$. Next we consider to define the idea of subordinations between analytic functions $g_1$ and $g_2$, indicated by $g_1(z) < g_2(z)$, as; the functions $g_1, g_2 \in \mathcal{A}$ are connected by the relation of subordination, if there exists an analytic function $w$ with the restrictions $w(0) = 0$ and $|w(z)| < 1$ such that $g_1(z) = g_2(w(z))$. Moreover, if the function $g_2 \in \mathcal{S}$ in $\mathbb{D}$, then we obtain:

$$g_1(z) < g_2(z) \iff [g_1(0) = g_2(0) \text{ & } g_1(\mathbb{D}) \subset g_1(\mathbb{D})].$$
In 1992, Ma and Minda [16] considered a holomorphic function \( \varphi \) normalized by the conditions \( \varphi(0) = 1 \) and \( \varphi'(0) > 0 \) with \( \text{Re} \varphi > 0 \) in \( \mathbb{D} \). The function \( \varphi \) maps the disc \( \mathbb{D} \) onto region which is star-shaped about 1 and symmetric along the real axis. In particular, the function \( \varphi(z) = (1 + Az)/(1 + Bz) \), \((-1 \leq B < A \leq 1)\) maps \( \mathbb{D} \) onto the disc on the right-half plane with centre on the real axis and diameter end points \( \frac{1-A}{1-B} \) and \( \frac{1+A}{1-B} \). This interesting familiar function is named as Janowski function [10].

Recently, Ali et al. [1] have obtained sufficient conditions on \( \alpha \) such that
\[
1 + zg'(z)/g^n(z) < \sqrt{1 + z} \quad \Rightarrow \quad g(z) < \sqrt{1 + z}, \quad \text{for } n = 0, 1, 2.
\]

Similar implications have been studied by various authors, for example see the works of Halim and Omar [6], Haq et al [7], Kumar et al [13, 15], Paprocki and Sokól [18], Raza et al [20] and Sharma et al [22].

In 1994, Hayman [8] studied multivalent \((p\text{-valent})\) functions which is a generalization of univalent functions and is defined as: an analytic function \( g \) in an arbitrary domain \( D \subset \mathbb{C} \) is said to be \( p\text{-valent} \), if for every complex number \( \omega \), the equation \( g(z) = \omega \) has maximum \( p \) roots in \( D \) and for a complex number \( \omega_0 \) the equation \( g(z) = \omega_0 \) has exactly \( p \) roots in \( D \). Let \( \mathcal{A}_p \) \((p \in \mathbb{N} = \{1, 2, \ldots\})\) denote the class of functions, say \( g \in \mathcal{A}_p \), that are multivalent holomorphic in the unit disc \( \mathbb{D} \) and which have the following series expansion:
\[
g(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (z \in \mathbb{D}). \tag{1.1}
\]

Using the idea of multivalent functions, we now introduce the class \( \mathcal{SL}_p^* \) of multivalent starlike functions associated with lemniscate of Bernoulli and as given below:
\[
\mathcal{SL}_p^* = \left\{ g(z) \in \mathcal{A}_p : \frac{zg'(z)}{pg(z)} < \sqrt{1 + z}, \quad (z \in \mathbb{D}) \right\}.
\]

In this article, we determine conditions on \( \alpha \) such that for each
\[
1 + \alpha^{2+\ell(p-1)}zg'(z)/pg(z), \quad \text{for } j = 0, 1, 2, 3,
\]
are subordinated to Janowski functions implies \( \frac{\xi(z)}{\varphi} < \sqrt{1 + z}, \quad (z \in \mathbb{D}) \). These results are then utilized to show that \( g \) are in the class \( \mathcal{SL}_p^* \).

1.1. Lemma

Let \( w \) be analytic non-constant function in \( \mathbb{D} \) with \( w(0) = 0 \). If
\[
|w(z_0)| = \max \{|w(z)|, \quad |z| \leq |z_0|\}, \quad z \in \mathbb{D},
\]
then there exists a real number \( m \) \((m \geq 1)\) such that
\[
z_0 w'(z_0) = mw(z_0).
\]

This Lemma is known as Jack’s Lemma and it has been proved in [9].
2. Main results

2.1. Theorem

Let \( g \in \mathcal{A}_p \) and satisfying
\[
1 + \frac{\alpha z^{1-p} g'(z)}{p} < \frac{1 + Az}{1 + Bz},
\]
with the restriction on \( \alpha \) is
\[
|\alpha| \geq \frac{2^{\frac{1}{2}} p (A - B)}{1 - |B| - 4 p (1 + |B|)}.
\] (2.1)

Then
\[
\frac{g(z)}{z^p} < \sqrt{1 + z}.
\]

Proof

Let us define a function
\[
p(z) = 1 + \frac{\alpha z^{1-p} g'(z)}{p},
\]
where the function \( p \) is analytic in \( \mathbb{D} \) with \( p(0) = 1 \). Also consider
\[
\frac{g(z)}{z^p} = \sqrt{1 + w(z)}.
\] (2.3)

Now to prove our result we will only require to prove that \( |w(z)| < 1 \). Logarithmically differentiating (2.3) and then using (2.2), we get
\[
p(z) = 1 + \frac{\alpha z w'(z)}{2 p \sqrt{1 + w(z)}} + \alpha \sqrt{1 + w(z)},
\]
and so
\[
\left| \frac{p(z) - 1}{A - B p(z)} \right| = \left| \frac{\frac{\alpha z w'(z)}{2 p \sqrt{1 + w(z)}} + \alpha \sqrt{1 + w(z)}}{A - B \left( 1 + \frac{\alpha z w'(z)}{2 p \sqrt{1 + w(z)}} + \alpha \sqrt{1 + w(z)} \right)} \right| = \left| \frac{\alpha z w'(z) + 2 p \alpha (1 + w(z))}{2 p (A - B) \sqrt{1 + w(z)} - B (\alpha z w'(z) + 2 p \alpha (1 + w(z)))} \right|.
\]

Now, we suppose that a point \( z_0 \in \mathbb{D} \) occurs such that
\[
\max_{|z| = 1} |w(z)| = |w(z_0)| = 1.
\]

Also by Lemma 1.1, a number \( m \geq 1 \) exists with \( z_0 w'(z_0) = mw(z_0) \). In addition, we also suppose that \( w(z_0) = e^{i\theta} \) for \( \theta \in [-\pi, \pi] \). Then we have
\[
\left| \frac{p(z_0) - 1}{A - B p(z_0)} \right| = \left| \frac{amw(z_0) - 2 p \alpha (1 + w(z_0))}{2 p (A - B) \sqrt{1 + w(z_0)} - B (amw(z_0) + 2 p \alpha (1 + w(z_0)))} \right|,
\]
\[
\frac{|\alpha|m - 2p|\alpha|(|1 + e^{i\theta}|)}{2p(A - B) \sqrt{|1 + e^{i\theta}| + |B|(|\alpha|m + 2p\alpha|1 + e^{i\theta}|)}} \geq \frac{|\alpha|m - 2p|\alpha|}{2^{\frac{3}{2}}p(A - B) + |B||\alpha|(m + 4p)}.
\]

Now if
\[
\phi(m) = \frac{|\alpha|(m - 4p)}{2^{\frac{3}{2}}p(A - B) + |B||\alpha|(m + 4p)},
\]

then
\[
\phi'(m) = \frac{2^{\frac{3}{2}}p(A - B)|\alpha| + 8|\alpha|^2p|B|}{(2^{\frac{3}{2}}p(A - B) + |B||\alpha|(m + 4p))^2} > 0,
\]

which illustrates that the function \(\phi(m)\) is increasing and hence \(\phi(m) \geq \phi(1)\) for \(m \geq 1\), so
\[
\frac{|p(z_0) - 1|}{A - Bp(z_0)} \geq \frac{|\alpha|(1 - 4p)}{2^{\frac{3}{2}}p(A - B) + |B||\alpha|(1 + 4p)}.
\]

Now, by using (2.1), we have
\[
\frac{|p(z_0) - 1|}{A - Bp(z_0)} \geq 1
\]

which contradicts the fact that \(p(z) < \frac{1 + Az}{1 + Bz}\). Thus \(|w(z)| < 1\) and so we get the desired result.

Taking \(g(z) = \frac{z^{n+1}}{p f(z)}\) in the last result, we obtain the following Corollary:

2.2. Corollary

Let \(f \in \mathcal{A}_p\) and satisfying
\[
1 + \frac{\alpha z f'(z)}{p^2 f(z)} \left( p + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < \frac{1 + Az}{1 + Bz},
\]

with the condition on \(\alpha\) is
\[
|\alpha| \geq \frac{2^{\frac{3}{2}}p(A - B)}{1 - |B| - 4p(1 + |B|)}.
\]

Then \(f \in SL_p^*\).

2.3. Theorem

If \(g \in \mathcal{A}_p\) such that
\[
1 + \frac{\alpha z g'(z)}{p g(z)} < \frac{1 + Az}{1 + Bz},
\]

with
\[
|\alpha| \geq \frac{8p(A - B)}{1 - |B| - 4p(1 + |B|)},
\]

then
\[
\frac{g(z)}{z^p} < \sqrt{1 + z}.
\]
Proof

Let us choose a function $p$ by

$$p(z) = 1 + \alpha \frac{z g'(z)}{p g(z)},$$

in such a way that $p$ is analytic in $\mathbb{D}$ with $p(0) = 1$. Also consider

$$g(z) = \frac{e^{\sqrt{1 + w(z)}}}{z^p},$$

$$w(z) = \sqrt{1 + w(z)}.$$

Using some simple calculations, we obtain

$$p(z) = 1 + \frac{\alpha z w'(z)}{2p(1 + w(z))} + \alpha,$$

and so

$$\left| \frac{p(z) - 1}{A - B p(z)} \right| = \left| \alpha \frac{z w'(z)}{2p(1 + w(z))} \right| + \alpha \left| \frac{1 + w(z)}{2p(1 + w(z))} \right|.

Let a point $z_0 \in \mathbb{D}$ exists in such a way

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, by virtue of Lemma 1.1, a number $m \geq 1$ occurs such that $z_0 w'(z_0) = mw(z_0)$. In addition, we set $w(z_0) = e^{i\theta}$, so we have

$$\left| \frac{p(z_0) - 1}{A - B p(z_0)} \right| = \left| \alpha m w(z_0) + 2 \alpha p (1 + w(z_0)) \right|,

\left| \frac{A m - 2p |a| |1 + e^{i\theta}|}{|A m - 2p |a| |1 + e^{i\theta}| + |B| |a| m + 2p |B| |a| |1 + e^{i\theta}|} \right|

\geq \frac{2 (A - B) |1 + e^{i\theta}| + |B| |a| m + 2p |B| |a| |1 + e^{i\theta}|}{|a| (m - 4p)}

\geq \frac{2 (A - B) + |B| |a| |m|}{|a| (m - 4p)}.

Now let

$$\phi(m) = \frac{|a| (m - 4p)}{4p (2 (A - B) + |B| |a| |m|)},$$

it implies

$$\phi'(m) = \frac{|a| 8p ((A - B) + |a| |B|)}{(4p (2 (A - B) + |B| |a|) + |B| |a| |m|)} > 0,$$

which illustrates that the function $\phi(m)$ is increasing and so $\phi(m) \geq \phi(1)$ for $m \geq 1$, hence

$$\left| \frac{p(z_0) - 1}{A - B p(z_0)} \right| \geq \frac{|a| (1 - 4p)}{4p (2 (A - B) + |B| |a|) + |B| |a| |m|}.$$

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Now, by using (2.6), we have
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq 1,
\]
which contradicts (2.5). Thus \(|w(z)| < 1\) and so the desired proof is completed.
Putting \(g(z) = \frac{z^{n+1}f''(z)}{pf(z)}\) in last Theorem, we get the following Corollary:

\section*{2.4. Corollary}

If \(f \in \mathcal{A}_p\) and satisfying
\[
1 + \alpha \left( p + 1 + \frac{zf''(z)}{f(z)} - \frac{zf'(z)}{f(z)} \right) < \frac{1 + A z}{1 + B z},
\]
with
\[
|\alpha| \geq \frac{8 p (A - B)}{1 - |B| - 4 p (1 + |B|)},
\]
then \(f \in S L^*_p\).

\section*{2.5. Theorem}

If \(g \in \mathcal{A}_p\) and satisfy the subordination relation
\[
1 + \frac{\alpha z^{1-p}g'(z)}{p(g(z))^2} < \frac{1 + A z}{1 + B z},
\]
with the condition on \(\alpha\)
\[
|\alpha| \geq \frac{2^5 p (A - B)}{1 - |B| - 4 p (1 + |B|)}
\]
is true, then
\[
\frac{g(z)}{z^p} < \sqrt{1 + z}.
\]

\textbf{Proof}

Let us define a function
\[
p(z) = 1 + \alpha \frac{z^{1-p}g'(z)}{p(g(z))^2}.
\]
Then \(p\) is analytic in \(D\) with \(p(0) = 1\). Also let us consider
\[
\frac{g(z)}{z^p} = \sqrt{1 + w(z)}.
\]

Using some simplification, we obtain
\[
p(z) = 1 + \frac{\alpha zw'(z)}{2p(1 + w(z))^2} + \frac{\alpha}{\sqrt{1 + w(z)}},
\]

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and so
\[
\left| \frac{p(z) - 1}{A - Bp(z)} \right| = \left| \frac{\frac{\alpha zw'(z)}{2p(1+w(z))^2} + \frac{\alpha}{\sqrt{1+w(z)}}}{A - B \left( 1 + \frac{\alpha zw'(z)}{2p(1+w(z))^2} + \frac{\alpha}{\sqrt{1+w(z)}} \right)} \right|.
\]

Let us choose a point \(z_0 \in \mathbb{D}\) such a way that
\[
\max_{|z|=1} |w(z)| = |w(z_0)| = 1.
\]

Then, by the consequences of Lemma 1.1, a number \(m \geq 1\) occurs such that \(z_0 w'(z_0) = mw(z_0)\) and also put \(w(z_0) = e^{i\theta}\), for \(\theta \in [-\pi, \pi]\), we have
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| = \left| \frac{\alpha mw(z_0) + 2p\alpha (1 + w(z_0))}{2p (A - B) \left( 1 + w(z_0) \right)^{\frac{3}{2}} - Bmw(z_0) - 2p\alpha B (1 + w(z_0))} \right|,
\]
\[
\geq \frac{\alpha m |w(z_0)| + 2p\alpha |1 + w(z_0)|}{2p (A - B) \left( 1 + |w(z_0)| \right)^{\frac{3}{2}} - Bm|w(z_0)| - 2p\alpha B (1 + |w(z_0)|)},
\]
\[
= \frac{2^{\frac{3}{2}} p (A - B) + |B| |m + 4p| |B|}{|m - 4p|} \geq \frac{2^{\frac{3}{2}} p (A - B) + |B| |m + 4p| |B|}{|m - 4p|} = \phi(m) \quad \text{(say)}.
\]

Then
\[
\phi'(m) = \frac{2^{\frac{3}{2}} p (A - B) + 8 |\alpha|^2 |B| p}{\left( 2^{\frac{3}{2}} p (A - B) + |B| |m + 4p| |B| \right)^{\frac{3}{2}}} > 0,
\]

which demonstrates that the function \(\phi(m)\) is increasing and thus \(\phi(m) \geq \phi(1)\) for \(m \geq 1\), hence
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq \frac{|\alpha| (1 - 4p)}{2^{\frac{3}{2}} p (A - B) + |B| |m + 4p| |B|}.
\]

Now, using (2.8), we have
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| \geq 1,
\]

which contradicts (2.7). Thus \(|w(z)| < 1\) and so we get the required proof.

If we set \(g(z) = \frac{z^{p+1}f'(z)}{pf(z)}\) in last theorem, we easily have the following Corollary:

2.6. **Corollary**

Assume that
\[
|\alpha| \geq \frac{2^{\frac{3}{2}} p (A - B)}{1 - |B| - 4p (1 + |B|)}.
\]
and if \( f \in A_p \) satisfy
\[
1 + \frac{\alpha f(z)}{z^{p+1} f'(z)} \left( p + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < \frac{1 + Az}{1 + Bz},
\]
then \( f \in SL_p^* \).

2.7. Theorem

If \( g \in A_p \) satisfy the subordination
\[
1 + \frac{z^{1-2p} g'(z)}{p (g(z))^3} < \frac{1 + Az}{1 + Bz},
\]
with restriction on \( \alpha \) is
\[
|\alpha| \geq \frac{8p (A - B)}{1 - |B| - 4p (1 + |B|)},
\]
then
\[
g(z)z^p < \sqrt{1 + z}.
\]

Proof. Let us define a function
\[
p(z) = 1 + \frac{\alpha z^{1-2p} g'(z)}{p (g(z))^3},
\]
where \( p \) is analytic in \( \mathbb{D} \) with \( p(0) - 1 = 0 \). Also let
\[
g(z)z^p = \sqrt{1 + w(z)}.
\]

Using some simple calculations, we obtain
\[
p(z) = 1 + \frac{\alpha zw'(z)}{2p (1 + w(z))^2} + \frac{\alpha}{1 + w(z)},
\]
and so
\[
\left| \frac{p(z) - 1}{A - Bp(z)} \right| = \left| \frac{\alpha zw'(z)}{2p (1 + w(z))^2} + \frac{\alpha}{1 + w(z)} \right|.
\]

Let us pick a point \( z_0 \in \mathbb{D} \) in such a way that
\[
\max_{|z| = 1} |w(z)| = |w(z_0)| = 1.
\]

Then, by using Lemma 1.1, a number \( m \geq 1 \) exists such that \( z_0w'(z_0) = mw(z_0) \) and put \( w(z_0) = e^{i\theta} \), for \( \theta \in [-\pi, \pi] \), we have
\[
\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| = \left| \frac{\alpha mw(z_0) + 2pa (1 + w(z_0))}{2p (A - B) (1 + w(z_0))^2 - Bamw(z_0) - 2paB (1 + w(z_0))} \right|.
\]
Now let
\[ \phi(m) = \frac{|\alpha|(m - 4p)}{8p(A - B) + |B||\alpha|m + 4p|\alpha||B|^2}, \]
then
\[ \phi'(m) = \frac{8p|\alpha|(A - B) + 8|\alpha|^2|B|p}{(8p(A - B) + |B||\alpha|m + 4p|\alpha||B|^2)} > 0 \]
which shows that \( \phi(m) \) is an increasing function and hence it will have its minimum value at \( m = 1 \), so
\[ \frac{p(z_0) - 1}{A - Bp(z_0)} \geq \frac{|\alpha|(1 - 4p)}{8p(A - B) + |B||\alpha| + 4p|\alpha||B|^2}. \]

Using (2.9), we easily obtain
\[ \frac{p(z_0) - 1}{A - Bp(z_0)} \geq 1, \]
which is a contradiction to the fact that \( p(z) < \frac{1 + Az}{1 + Bz} \), and so \(|w(z)| < 1\). Hence we get the desired result.

If we put \( g(z) = \frac{zp^{f(z)}}{p[f(z)]} \) in last Theorem, we achieve the following result:

2.8. Corollary

If \( f \in \mathcal{A}_p \) and satisfy the condition
\[ |\alpha| \geq \frac{8p(A - B)}{1 - |B| - 4p(1 + |B|)}, \]
and
\[ 1 + \alpha \frac{p(f(z))^2}{z^{3p+2}(f'(z))^2} \left( p + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \leq \frac{1 + Az}{1 + Bz}, \]
then \( f \in \mathcal{S}L_p^* \).

Conflict of interest

All authors declare no conflict of interest in this paper.
References


