Several explicit and recursive formulas for generalized Motzkin numbers

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\textbf{Abstract:} In the paper, the authors find two explicit formulas and recover a recursive formula for generalized Motzkin numbers. Consequently, the authors deduce two explicit formulas and a recursive formula for the Motzkin numbers, the Catalan numbers, and the restricted hexagonal numbers respectively.

\textbf{Keywords:} explicit formula; recursive formula; generalized Motzkin number; Motzkin number; restricted hexagonal number; Catalan number; generating function

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\section{1. Introduction}

The Motzkin numbers $M_n$ enumerate various combinatorial objects. In 1977, Donaghey and Shapiro [3] gave fourteen different manifestations of the Motzkin numbers $M_n$. In particular, the Motzkin numbers $M_n$ give the numbers of paths from $(0,0)$ to $(n,0)$ which never dip below the $x$-axis $y = 0$ and are made up only of the steps $(1,0)$, $(1,1)$, and $(1,-1)$.

The first seven Motzkin numbers $M_n$ for $0 \leq n \leq 6$ are $1, 1, 2, 4, 9, 21, 51$. All the Motzkin numbers $M_n$ can be generated by

\[ M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} = \frac{1}{1 - x + \sqrt{1 - 2x - 3x^2}} = \sum_{k=0}^{\infty} M_k x^k. \]

In 2007, Mansour \textit{et al} [12] introduced the $(u,l,d)$-Motzkin numbers $m^{(u,l,d)}_n$ and
obtained [12, Theorem 2.1] that

\[ M_{n,d,l}(x) = 1 - lx - \sqrt{(1 - lx)^2 - 4udx^2} \]

\[ = \sum_{n=0}^{\infty} m_n^{l,d,u} x^n, \tag{1.1} \]

and

\[ m_n^{l,d,u} = n \sum_{j=0}^{n/2} \frac{1}{j+1} \binom{n}{j} \binom{ud}{2j}^j. \tag{1.2} \]

From (1.1) and (1.2), it is easy to see that

\[ m_n^{l,d,u} = m_n^{u,l,d}. \]

In 2014, Sun [42] generalized the Motzkin numbers \( M_n \) to

\[ M_n(a, b) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k a^{n-2k} b^k \]

for \( a, b \in \mathbb{N} \) in terms of the Catalan numbers

\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]

and established the generating function

\[ M_{a,b}(x) = 1 - ax - \sqrt{(1 - ax)^2 - 4bx^2} \]

\[ = \frac{1}{1 - ax + \sqrt{(1 - ax)^2 - 4bx^2}} = \sum_{k=0}^{\infty} M_k(a, b)x^k, \tag{1.5} \]

where \( \lfloor \lambda \rfloor \) denotes the floor function defined by the largest integer less than or equal to \( \lambda \in \mathbb{R} \). Wang and Zhang pointed out in [43] that

\[ M_n(1, 1) = M_n, \quad M_n(2, 1) = C_{n+1}, \quad \text{and} \quad M_n(3, 1) = H_n, \tag{1.6} \]

where \( H_n \) denote the restricted hexagonal numbers described by Harary and Read [4].

For more information on many results, applications, and generalizations of the Motzkin numbers \( M_n \), please refer to the papers [3, 9, 10, 42, 43] and closely related references therein. For more information on many results, applications, and generalizations of the Catalan numbers \( C_n \), please refer to the monograph [5], the newly published papers [11, 17, 19, 26, 27, 31, 36–38, 40, 41], the survey articles [25, 29], and closely related references therein.

Comparing (1.1) with (1.5) reveals that \( M_k(a, b) \) and \( m_k^{l,d,u} \) are equivalent to each other and satisfy

\[ M_k(a, b) = m_k^{u,a,b} = m_k^{b,a,1} \quad \text{and} \quad m_k^{l,d,u} = M_k(l, u,d). \tag{1.7} \]

Therefore, it suffices to consider generalized Motzkin numbers \( M_k(a, b) \), rather than the \((u, l, d)\)-Motzkin numbers \( m_k^{u,l,d} \), in this paper.

By the second relation in (1.7), one can reformulated the formula (1.2) as

\[ M_n(a, b) = a^n \sum_{j=0}^{n/2} \frac{1}{j+1} \binom{n}{2j} \binom{n}{2j} b^j. \tag{1.8} \]
Substituting (1.4) into (1.3) recovers (1.8) once again.

In 2015, Wang and Zhang [43, Theorem 1] combinatorially obtained, among other things, the recursive formula

$$M_{n+2}(a, b) = aM_{n+1}(a, b) + b \sum_{\ell=0}^{n} M_{\ell}(a, b)M_{n-\ell}(a, b), \quad n \geq 0.$$  \hspace{1cm} (1.9)

It is not difficult to see that the function \((1 - ax)^2 - 4bx^2 = (a^2 - 4b)x^2 - 2ax + 1\) is nonnegative if and only if

1. either \(b = 0\) and \(x \in \mathbb{R}\),
2. or \(a^2 - 4b = 0, a \neq 0,\) and \(x \leq \frac{1}{2a}\),
3. or \(a^2 - 4b > 0, b < 0,\) and \(x \in \mathbb{R}\),
4. or \(a^2 - 4b > 0, b > 0,\) and \(x \geq \frac{1}{a-2\sqrt{b}}\) or \(x \leq \frac{1}{a+2\sqrt{b}}\).

Consequently, the generating function \(M_{a,b}(x)\) defined by (1.5) in [42] is defined for either \(b \leq 0\) or \(a \geq 2\sqrt{b} > 0\).

In this paper, we will find two explicit formulas, which are different from (1.8), and recover the recursive formula (1.9) for generalized Motzkin numbers \(M_n(a, b)\). Consequently, we will derive two explicit formula and a recursive formula for the Motzkin numbers \(M_n\), the Catalan numbers \(C_n\), and the restricted hexagonal numbers \(H_n\) respectively.

We can state our main results as the following three theorems.

**Theorem 1.** For \(n \geq 0\), we can compute generalized Motzkin numbers \(M_n(a, b)\) by

$$M_n(a, b) = \frac{1}{2b} \left( \frac{4b - a^2}{2a} \right)^{n+2} \sum_{\ell=0}^{n+2} \left( \frac{2a^2}{4b - a^2} \right)^{\ell} \left( \frac{2(\ell - 3)!!}{\ell!} \right) \left( \frac{\ell}{n - \ell + 2} \right).$$ \hspace{1cm} (1.10)

where \(\binom{n}{q} = 0\) for \(q > p \geq 0\) and the double factorial of negative odd integers \(-(2n + 1)\) is

$$[-(2n + 1)]!! = (-1)^n \frac{2^n n!}{(2n - 1)!!}, \quad n = 0, 1, \ldots .$$

Consequently, we can compute the Motzkin numbers \(M_n\) and the restricted hexagonal numbers \(H_n\) respectively by

$$M_n = \frac{9}{8} \left( \frac{3}{2} \right)^{n+2} \sum_{\ell=0}^{n+2} \left( \frac{2}{3} \right)^{\ell} \left( \frac{2(\ell - 3)!!}{\ell!} \right) \left( \frac{\ell}{n - \ell + 2} \right)$$ \hspace{1cm} (1.11)

and

$$H_n = (-1)^n \frac{25}{72} \left( \frac{5}{6} \right)^n \sum_{\ell=0}^{n+2} (-1)^{\ell} \left( \frac{18}{5} \right)^{\ell} \left( \frac{2(\ell - 3)!!}{\ell!} \right) \left( \frac{\ell}{n - \ell + 2} \right).$$ \hspace{1cm} (1.12)

**Theorem 2.** For \(n \geq 0\), we can compute generalized Motzkin numbers \(M_n(a, b)\) by

$$M_n(a, b) = \frac{- (a - 2\sqrt{b})^{n+2}}{2b} \sum_{\ell=0}^{n+2} \left( \frac{2(\ell - 3)!!}{(2\ell)!!} \right) \left( \frac{2(n - \ell + 2)!!}{[2(n - \ell + 2)]!!} \right) \left( \frac{a + 2\sqrt{b}}{a - 2\sqrt{b}} \right)^{\ell}.$$ \hspace{1cm} (1.13)
Consequently, we can compute the Motzkin numbers $M_n$ and the restricted hexagonal numbers $H_n$ respectively by

$$M_n = \frac{(-1)^{n+1}}{2} \sum_{\ell=0}^{n+2} (-1)^{\ell} \frac{(2\ell - 3)!! [2(n - \ell + 2) - 3]!!}{(2\ell)!! [2(n - \ell + 2)]!!}$$

and

$$H_n = -\frac{1}{2} \sum_{\ell=0}^{n+2} 5^\ell \frac{(2\ell - 3)!! [2(n - \ell + 2) - 3]!!}{(2\ell)!! [2(n - \ell + 2)]!!}.$$  

**Theorem 3.** For $n \geq 0$, generalized Motzkin numbers $M_n(a, b)$ satisfy

$$M_0(a, b) = 1, \quad M_1(a, b) = a, \quad (1.14)$$

and the recursive formula (1.9). Consequently, for $n \geq 0$, the Motzkin numbers $M_n$, the Catalan numbers $C_n$, and the restricted hexagonal numbers $H_n$ meet the recursive formulas

$$M_{n+2} = M_{n+1} + \sum_{\ell=0}^{n} M_\ell M_{n-\ell}, \quad (1.15)$$

$$C_{n+2} = 2C_{n+1} + \sum_{\ell=0}^{n} C_\ell C_{n-\ell}, \quad (1.16)$$

and

$$H_{n+2} = 3H_{n+1} + \sum_{\ell=0}^{n} H_\ell H_{n-\ell} \quad (1.17)$$

respectively.

**2. Lemmas**

In order to prove the explicit formula (1.10), we need the following lemmas.

**Lemma 1** ([1, p. 40, Exercise 5], [16, Section 2.2, p. 849], [22, p. 94], [34, Lemma 3], and [44, Lemma 2.1]). Let $u(x)$ and $v(x) \neq 0$ be two differentiable functions. Let $U_{(n+1)\times 1}(x)$ be an $(n + 1) \times 1$ matrix whose elements $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \leq k \leq n + 1$, let $V_{(n+1)\times n}(x)$ be an $(n + 1) \times n$ matrix whose elements

$$v_{i,j}(x) = \begin{cases} 
(i-1) y^{i-j}(x), & i - j \geq 0 \\
0, & i - j < 0 
\end{cases}$$

for $1 \leq i \leq n + 1$ and $1 \leq j \leq n$, and let $|W_{(n+1)\times(n+1)}(x)|$ denote the determinant of the $(n + 1) \times (n + 1)$ matrix

$$W_{(n+1)\times(n+1)}(x) = \begin{pmatrix} U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x) \end{pmatrix}.$$  

Then the $n$th derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$\frac{d^n}{dx^n} \left[ \frac{u(x)}{v(x)} \right] = (-1)^n \frac{|W_{(n+1)\times(n+1)}(x)|}{v^{n+1}(x)}.$$
Lemma 2 ([2, p. 134, Theorem A and p. 139, Theorem C]). The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind

\[ B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\ell_1 \leq \ell_2 \leq \cdots \leq \ell_n; \sum_{\ell_i=1}^{n} \ell_i = n} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i !} \prod_{i=1}^{n-k+1} \frac{(x_i)^{\ell_i}}{\ell_i !} \]

for \( n \geq k \geq 0 \) by

\[ \frac{d^n}{dt^n} [f \circ h(t)] = \sum_{k=0}^{n} f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \ldots, h^{(n-k+1)}(t)) \tag{2.1} \]

for \( n \geq 0 \).

Lemma 3 ([2, p. 135]). The Bell polynomials of the second kind \( B_{n,k} \) satisfy

\[ B_{n,k}(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \tag{2.2} \]

for \( n \geq k \geq 0 \).

Lemma 4. For \( n \geq k \geq 0 \), we have

\[ B_{n,k}(x_1, 1, 0, \ldots, 0) = \frac{(n-k)!}{2^{n-k}} \binom{n}{k} (n-k)^{2n-k}. \tag{2.3} \]

More generally, for \( n \geq k \geq 0 \) and \( \lambda, \alpha \in \mathbb{C} \), we have

\[ B_{n,k}(1, 1 - \lambda, (1 - \lambda)(1 - 2\lambda), \ldots, \prod_{\ell=0}^{n-k} (1 - \ell\lambda)) = (-1)^k \frac{k!}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) \tag{2.4} \]

or, equivalently,

\[ B_{n,k}(\langle \alpha \rangle_1, \langle \alpha \rangle_2, \ldots, \langle \alpha \rangle_{n-k+1}) = (-1)^k \frac{k!}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle \alpha \ell \rangle_n. \tag{2.5} \]

Proof. The formula (2.3) can be found in [24, Theorem 5.1], [35, p. 7, (19)], [39, Section 3], and [44, Lemma 2.5]. The explicit formula (2.4) was first established in [30, Remark 1] and then was applied in [18, Section 2], [20, First proof of Theorem 2], [21, Lemma 2.2], [24, Remark 6.1], [28, Lemma 4], and [32, Lemma 2.6]. The formula (2.5) and the equivalence were presented in [33, Theorems 2.1 and 4.1].

Remark 1. In recent years, there have been some literature such as [6–8, 13–15, 23, 24, 30, 35, 45–48] devoting to deep investigation and extensive applications of the Bell polynomials of the second kind \( B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \). Specially, in the papers [13, 14], the generalized Dyck paths (namely, various type of Motzkin paths) and the Bell polynomials were connected closely.
3. Proofs of Theorems 1 and 3

We are now in a position to prove our main results.

**Proof of Theorem 1.** By virtue of (2.1), (2.2), and (2.3), we obtain for \( k \geq 0 \) that

\[
\left[ \sqrt{(1 - ax)^2 - 4bx^2} \right]^{(k+2)} = \sum_{\ell=0}^{k+2} \binom{1}{\ell} \left[ (1 - ax)^2 - 4bx^2 \right]^{1/2-\ell} \\
\times B_{k+2,\ell}(-2[a + (4b - a^2)x], 2(a^2 - 4b), 0, \ldots, 0) \\
\rightarrow \sum_{\ell=0}^{k+2} \binom{1}{\ell} B_{k+2,\ell}(-2a, 2(a^2 - 4b), 0, \ldots, 0) \\
= \sum_{\ell=0}^{k+2} \binom{1}{\ell} [2(a^2 - 4b)]^{1/2} B_{k+2,\ell} \left( \frac{a}{4b - a^2}, 1, 0, \ldots, 0 \right) \\
= \sum_{\ell=0}^{k+2} \binom{1}{\ell} [2(a^2 - 4b)]^{(k-\ell+2)!/(\ell!(k-\ell+2))} \left( \frac{a}{4b - a^2} \right)^{2^{k-2}} \tag{3.1}
\]

as \( x \to 0 \), where

\[
\langle x \rangle_n = \begin{cases} 
  x(x-1) \cdots (x-n+1), & n \geq 1 \\
  1, & n = 0 
\end{cases}
\]

denotes the falling factorial of \( x \in \mathbb{R} \).

Letting \( u(x) = 1 - ax - \sqrt{(1 - ax)^2 - 4bx^2} \) and \( v(x) = x^2 \) in Lemma 1 gives

\[
\frac{d^n M_{a,b}(x)}{dx^n} = \frac{1}{2b} \frac{(-1)^n}{x^{2(n+1)}} \left[ \begin{array}{cccc}
  u(x) & \binom{0}{1} x^2 & 0 & \cdots & 0 & 0 \\
  u'(x) & 2\binom{0}{0} x & \binom{1}{1} x^2 & \cdots & 0 & 0 \\
  u''(x) & 2\binom{1}{1} x & \binom{2}{2} x & \cdots & 0 & 0 \\
  u^{(3)}(x) & 0 & 2\binom{2}{1} x & \cdots & 0 & 0 \\
  u^{(4)}(x) & 0 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  u^{(n-2)}(x) & 0 & 0 & \cdots & 2\binom{n-2}{n-3} x & (\binom{n-2}{n-2} x)^2 \\
  u^{(n-1)}(x) & 0 & 0 & \cdots & 2\binom{n-1}{n-3} x & (\binom{n-1}{n-2} x)^2 \\
  u^{(n)}(x) & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} x & (\binom{n}{n-1} x)^2 \\
\end{array} \right]
\]

\[
= \frac{1}{2b} \frac{(-1)^n}{x^{2(n+1)}} (-1)^n u^{(n)}(x)
\]

\[
\begin{array}{ccccccc}
\binom{0}{1} x^2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
2\binom{0}{0} x & \binom{1}{1} x^2 & 0 & \cdots & 0 & 0 & 0 \\
2\binom{1}{1} x & \binom{2}{2} x & \binom{2}{2} x & \cdots & 0 & 0 & 0 \\
0 & 2\binom{2}{1} x & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 2\binom{2}{2} x & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2\binom{n-2}{n-3} x & (\binom{n-2}{n-2} x)^2 & 0 \\
0 & 0 & 0 & \cdots & 2\binom{n-1}{n-3} x & (\binom{n-1}{n-2} x)^2 & 0 \\
0 & 0 & 0 & \cdots & 0 & 2\binom{n}{n-2} x & (\binom{n}{n-1} x)^2 \\
\end{array}
\]
\[ +2 \left( \frac{n}{n-1} \right)^x \]

\[ -2 \left( \frac{n}{n-2} \right) \left( \frac{n}{n-1} \right)^2 \]

\[ = \frac{1}{2b} \frac{u^{(n)}(x)}{x^2} - \frac{2n}{x} \frac{1}{2b} \frac{(-1)^{n-1}}{x^{2n}} \]

\[ -n(n-1) \frac{1}{x^2} \frac{1}{2b} \frac{(-1)^{n-2}}{x^{2n-1}} \]

\[ = 1 \frac{u^{(n)}(x)}{2b} \frac{1}{x^2} - \frac{2n}{x} \frac{d^{n-1} M_{a,b}(x)}{dx^{n-1}} - \frac{n(n-1)}{x^2} \frac{d^{n-2} M_{a,b}(x)}{dx^{n-2}} \]

\[ = \frac{1}{x^2} \left[ \frac{u^{(n)}(x)}{2b} - 2n \frac{d^{n-1} M_{a,b}(x)}{dx^{n-1}} - n(n-1) \frac{d^{n-2} M_{a,b}(x)}{dx^{n-2}} \right]. \]

Therefore, by L'Hôpital's rule, we have
\[
\lim_{x \to 0} \frac{d^n M_{a,b}(x)}{dx^n} = \lim_{x \to 0} \left( \frac{1}{x^2} \left[ u^{(n)}(x) - 2nx \frac{d^{n-1} M_{a,b}(x)}{dx^{n-1}} - n(n-1) \frac{d^{n-2} M_{a,b}(x)}{dx^{n-2}} \right] \right)
\]
\[
= \lim_{x \to 0} \left( \frac{1}{2x} \left[ u^{(n+1)}(x) - 2nx \frac{d^{n} M_{a,b}(x)}{dx^{n}} - n(n+1) \frac{d^{n-1} M_{a,b}(x)}{dx^{n-1}} \right] \right)
\]
\[
= \frac{1}{2} \lim_{x \to 0} \left( \frac{u^{(n+2)}(x)}{2b} - 2nx \frac{d^{n+1} M_{a,b}(x)}{dx^{n+1}} - n(n+3) \frac{d^{n} M_{a,b}(x)}{dx^{n}} \right)
\]
\[
= \frac{1}{2} \lim_{x \to 0} \left( \frac{u^{(n+2)}(x)}{2b} - n(n+3) \frac{d^{n} M_{a,b}(x)}{dx^{n}} \right)
\]
which is equivalent to
\[
\lim_{x \to 0} \frac{d^n M_{a,b}(x)}{dx^n} = \frac{1}{(n+1)(n+2)} \lim_{x \to 0} \frac{u^{(n+2)}(x)}{2b} = \frac{1}{2b(n+1)(n+2)} \lim_{x \to 0} u^{(n+2)}(x).
\]

Considering
\[
\lim_{x \to 0} \frac{d^n M_{a,b}(x)}{dx^n} = n! M_n(a, b),
\]

making use of (3.1), and simplifying lead to the explicit formula (1.10).

Letting \((a, b) = (1, 1)\) and \((a, b) = (3, 1)\) respectively in (1.10) and considering the three relations in (1.6) derive (1.11) and (1.12) immediately. The proof of Theorem 1 is complete. \(\square\)

**Proof of Theorem 2.** From (1.5), it is derived that
\[
\sqrt{(1-ax)^2 - 4bx^2} = 1 - ax - 2b \sum_{k=0}^{\infty} M_k(a, b)x^{k+2}.
\]

This implies that
\[
M_k(a, b) = \frac{1}{2b} \frac{1}{(k+2)!} \lim_{x \to 0} \sqrt{(1-ax)^2 - 4bx^2}^{(k+2)}, \quad k \geq 0. \tag{3.2}
\]

It is easy to see that
\begin{enumerate}
  \item when \(a^2 - 4b > 0\) and \(x \leq \min \left\{ \frac{1}{a+2\sqrt{b}}, \frac{1}{a-2\sqrt{b}} \right\} = \frac{1}{a+2\sqrt{b}}, \) we have
    \[
    \left[ \sqrt{(1-ax)^2 - 4bx^2} \right]^{(k+2)} = \left[ \sqrt{(a^2 - 4b) \left( x - \frac{1}{a+2\sqrt{b}} \right) \left( x - \frac{1}{a-2\sqrt{b}} \right) (x-\frac{1}{a+2\sqrt{b}})} \right]^{(k+2)}
    \]
    \[
    = \sqrt{a^2 - 4b} \left( \sqrt{a+2\sqrt{b}} - x \sqrt{a-2\sqrt{b}} \right)^{(k+2)}
    \]
    \[
    = \sqrt{a^2 - 4b} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \sqrt{a^2 + \ell \sqrt{b}} - x \right)^{\ell} \left( \sqrt{a^2 - \ell \sqrt{b}} - x \right)^{k+2-\ell}
    \]
    \[
    = (-1)^k \sqrt{a^2 - 4b} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{1}{2} \right)^{\ell} \left( \frac{1}{a+2\sqrt{b}} - x \right)^{1/2-\ell} \left( \frac{1}{2} \right)^{k+2-\ell} \left( \frac{1}{a-2\sqrt{b}} - x \right)^{k+2-\ell}
    \]
\end{enumerate}
\[
\rightarrow (-1)^k \sqrt{a^2 - 4b} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{1}{2} \right)^{1/2-\ell} \left( \frac{1}{2} \right)_{k-\ell+2} \left( a - 2\sqrt{b} \right)^{\ell-3/2} \\
= (k+2)! \left(a - 2\sqrt{b}\right)^{k+2} \sum_{\ell=0}^{k+2} \frac{(2\ell-3)!!}{(2\ell)!!} \frac{(2(k-\ell+2) - 3)!!}{(a + 2\sqrt{b})^{\ell}} \frac{1}{(a - 2\sqrt{b})}
\]

as \( x \to 0; \)

2. when \( a^2 - 4b < 0 \) and

\[
\frac{1}{a + 2\sqrt{b}} = \max\left\{ \frac{1}{a + 2\sqrt{b}}, \frac{1}{a - 2\sqrt{b}} \right\} > x > \min\left\{ \frac{1}{a + 2\sqrt{b}}, \frac{1}{a - 2\sqrt{b}} \right\} = \frac{1}{a - 2\sqrt{b}},
\]

we have

\[
\left[ \sqrt{(1-ax)^2 - 4bx^2} \right]^{(k+2)} = \left[ \sqrt{(4b - a^2) \left( \frac{1}{a + 2\sqrt{b}} - x \right) \left( \frac{1}{a - 2\sqrt{b}} \right)} \right]^{(k+2)}
\]

\[
= \sqrt{4b - a^2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{1}{2} \right)^{1/2-\ell} \left( \frac{1}{2} \right)_{k-\ell+2} \left( a - 2\sqrt{b} \right)^{\ell-3/2} \left( x - \frac{1}{a - 2\sqrt{b}} \right)
\]

\[
\rightarrow \sqrt{4b - a^2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{1}{2} \right)^{1/2-\ell} \left( \frac{1}{2} \right)_{k-\ell+2} \left( a - 2\sqrt{b} \right)^{\ell-3/2} \left( x - \frac{1}{a - 2\sqrt{b}} \right)
\]

\[
= (2 \sqrt{b} - a)^{k+1} \sqrt{4b - a^2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{2(k-\ell+2) - 3)!!}{2^{\ell}} \left( a + 2\sqrt{b} \right)^{(\ell-1)/2} \left( x - \frac{1}{a - 2\sqrt{b}} \right)
\]

\[
= (2 \sqrt{b} - a)^{k+2} \sum_{\ell=0}^{k+2} \binom{k+2}{\ell} \left( \frac{2(k-\ell+2) - 3)!!}{2^{\ell}} \left( a + 2\sqrt{b} \right)^{(\ell-1)/2} \left( x - \frac{1}{a - 2\sqrt{b}} \right)
\]

as \( x \to 0. \)

By virtue of (3.2), we obtain the formula (1.13) readily.

Letting \( (a, b) = (1, 1) \) and \( (a, b) = (3, 1) \) respectively in (1.13) and making use of the first and third relations in (1.6) lead to (1.11) and (1.12) immediately. The proof of Theorem 2 is complete.

**Proof of Theorem 3.** From (1.5), it is derived that

\[
\sqrt{(1-ax)^2 - 4bx^2} = 1 - ax - 2b \sum_{k=0}^{\infty} M_k(a, b)x^{k+2}.
\]
Squaring on both sides of the above equation gives

\[
(1 - ax)^2 - 4bx^2 = 1 - 2ax + (a^2 - 4b)x^2 = \left[1 - ax - 2b \sum_{k=0}^{\infty} M_k(a, b)x^{k+2}\right]^2
\]

\[
= 1 + a^2x^2 + 4b^2\left[\sum_{k=0}^{\infty} M_k(a, b)x^{k+2}\right]^2 - 2ax - 4b \sum_{k=0}^{\infty} M_k(a, b)x^{k+2} + 4ab \sum_{k=0}^{\infty} M_k(a, b)x^{k+2}
\]

\[
= 1 - 2ax + a^2x^2 + 4b^2x^4 \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} M_{\ell}(a, b)M_{k-\ell}x^k
\]

\[-4b \sum_{k=2}^{\infty} M_{k-2}(a, b)x^k + 4ab \sum_{k=3}^{\infty} M_{k-3}(a, b)x^k
\]

\[
= 1 - 2ax + a^2x^2 - 4b \sum_{k=2}^{\infty} M_{k-2}(a, b)x^k + 4ab \sum_{k=3}^{\infty} M_{k-3}(a, b)x^k
\]

\[
-4b \sum_{k=4}^{\infty} M_{k-2}(a, b)x^k + 4ab \sum_{k=4}^{\infty} M_{k-3}(a, b)x^k + 4b^2 \sum_{k=4}^{\infty} \sum_{\ell=0}^{k-4} M_{\ell}(a, b)M_{k-\ell-4}(a, b)x^k
\]

\[-4b \sum_{k=4}^{\infty} M_{k-2}(a, b) - aM_{k-3}(a, b) - b \sum_{\ell=0}^{k-4} M_{\ell}(a, b)M_{k-\ell-4}(a, b)x^k
\]

which means that

\[
a^2 - 4b = a^2 - 4bM_0(a, b), \quad 4b[aM_0(a, b) - M_1(a, b)] = 0,
\]

and

\[
M_{k-2}(a, b) - aM_{k-3}(a, b) - b \sum_{\ell=0}^{k-4} M_{\ell}(a, b)M_{k-\ell-4}(a, b) = 0, \quad k \geq 4.
\]

Consequently, the identities in (1.14) and the recursive formula (1.9) follow.

Taking \((a, b) = (1, 1), (a, b) = (2, 1), \) and \((a, b) = (3, 1)\) respectively in (1.9) and considering the three relations in (1.6) lead to (1.15), (1.16), and (1.17) immediately. The proof of Theorem 3 is complete.

\[\square\]

4. Two more remarks

Remark 2. From the proof of Theorem 1, we can conclude that

\[
x^n \frac{d^n M_{a,b}(x)}{dx^n} + 2nx \frac{d^{n-1}M_{a,b}(x)}{dx^{n-1}} + n(n-1) \frac{d^{n-2}M_{a,b}(x)}{dx^{n-2}} = \frac{u^{(n)}(x)}{2b}, \quad n \geq 2.
\]
This implies that the generating function \( M_{a,b}(x) \) expressed in (1.5) is an explicit solution of the linear ordinary differential equations

\[
x^2 f^{(n)}(x) + 2nf^{(n-1)}(x) + n(n-1)f^{(n-2)}(x) = F_{n,a,b}(x)
\]

for all \( n \geq 2 \), where, by (2.2) and (2.3) or (2.4),

\[
F_{n,a,b}(x) = \frac{n!(4b - a^2)^n}{2^{n+1}b} \sqrt{(1 - ax)^2 - 4bx^2} \sum_{\ell=1}^{n} \frac{2\ell(2\ell - 3)!!}{\ell!} \left( \frac{a + (4b - a^2)x}{(1 - ax)^2 - 4bx^2} \right)^{2\ell}.
\]

**Remark 3.** This paper is a continuation of the article [49] and a revised version of the preprint [28].

**Conflict of interest**

The authors declare that they have no conflict of interest in this paper.

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