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Research article

The existence and forms of solutions for some Fermat-type differential-difference equations

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Abstract: The main aim of this article is to investigate the existence and the forms of solutions for several complex differential-difference equations of Fermat-type. Our results about the existence and the forms of solutions for these Fermat type equations are great improvement of the previous theorems given by Liu, Yang, Cao, Zhang. Moreover, it is a very satisfactory fact that in some examples explicit solutions are given.

Keywords: Fermat type; entire function; existence; differential-difference equation
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1. Introduction and main results

The main purpose of this article is to deal with the existence of solutions for several complex differential-difference equations of Fermat type. The basic results and the standard symbols of Nevanlinna theory will be used in this paper (see [8, 28, 30]). A. Wiles and R. Taylor [23, 24] in 1995 pointed out: The Fermat equation $x^m + y^m = 1$ does not admit nontrivial solutions in rational numbers as $m \geq 3$, and this equation possesses nontrivial rational solutions as $m = 2$. About sixty years ago, Gross [4] investigated the existence of solutions for the Fermat-type functional equation $f^m + g^m = 1$, and obtained: For $m = 2$, the entire solutions are $f = \cos a(z), g = \sin a(z)$, where $a(z)$ is an entire function; for $m > 2$, there are no nonconstant entire solutions.

In the last twenty years, Nevanlinna theory (especially the difference analogues such as logarithmic derivative lemma, Tumura-Clunie theorem, etc.) has played an important role in studying the properties of solutions for complex difference equations, complex differential-difference equations, and there were a number of results about the existence and the form of solutions for some equations (including [1–3, 5–7, 9–13, 17–22, 25–27]). In 2009, Liu [14] proved that the Fermat type equation $f(z)^2 + [f(z + c) - f(z)]^2 = a^2$ has no nonconstant entire solutions of finite order, where a is a nonzero constant. In 2012, Liu et al. [15] pointed out that equation $f(z)^2 + f(z+c)^2 = 1$ has a transcendental entire solution of finite order. Furthermore, they also obtained that equation $f'(z)^2 + [f(z + c) - f(z)]^2 = 1$ admits the finite order transcendental entire solutions with the form $f(z) = 1/2 \sin(2z + Bi)$, where $c = (2k + 1)\pi$, and B is a constant (see [15]).

In 2018, Zhang [31] further discussed the existence of solutions for some Fermat type differential-difference equations, which forms are more general than those given by Liu [14], Liu etal. [15], and obtained

Theorem A (see [31, Theorem 1.3]). *Let f be a transcendental meromorphic function with finitely many poles and $\sigma(f) < \infty$. Then f can not be a solution of the difference equation*

$$f(z)^2 + [f(z + c) - f(z)]^2 = R,$$

where R is a nonzero rational function and c is a nonzero constant.

Theorem B (see [31, Theorem 1.4]). *Let f be a transcendental meromorphic function with finitely many poles and $\sigma(f) < \infty$. If f is a solution of the differential-difference equation*

$$f'(z)^2 + [f(z + c) - f(z)]^2 = R,$$

where R is a nonzero rational function and c is a nonzero constant, then R is a nonzero constant and f is of form

$$f(z) = c_1 e^{2iz} + c_2 e^{-2iz} + b, \quad c = k\pi + \pi/2,$$

where c_1, c_2 are two nonzero constants such that $16c_1c_2 = R$, b is a constant and k is an integer.

In this paper, we proceed to study the existence and the form of the solutions for some differential-difference equations with more general form than the previous form given by Liu, Liu et al. and Zhang [14, 15, 31]. Our results are listed as follows.

Theorem 1.1. *Let c be a nonzero constant, R be a nonzero rational function, and $\alpha, \beta \in \mathbb{C}$ satisfy $\alpha^2 - \beta^2 \neq 1$. Then the following difference equation of Fermat type*

$$f(z)^2 + [\alpha f(z + c) - \beta f(z)]^2 = R(z), \tag{1.1}$$

has no finite order transcendental meromorphic solutions with finitely many poles.

Theorem 1.2. *Let $c(\neq 0), \alpha(\neq 0), \beta \in \mathbb{C}$ and $P(z), Q(z)$ be nonzero polynomials satisfying one of two following cases*

(i) $\deg_z R \geq 1$ *or* $\deg_z Q \geq 1$*;*

(ii) P, Q are two constants and $P^2(\alpha^2 - \beta^2) \neq 1$. Then the following Fermat-type difference equation

$$f(z)^2 + P(z)^2[\alpha f(z + c) - \beta f(z)]^2 = Q(z), \tag{1.2}$$

has no transcendental entire solutions with finite order.

Theorem 1.3. *Let $\alpha(\neq 0), \beta \in \mathbb{C}$, and k is an integer. Let f be a transcendental meromorphic solution of difference-differential equation of Fermat type*

$$f'(z)^2 + [\alpha f(z+c) - \beta f(z)]^2 = R(z) \tag{1.3}$$

where $R(z)$ is a nonzero rational function and c is a nonzero constant. If f is of finite order and has finitely many poles, then $i\alpha c = \pm 1$ and $R(z)$ is a nonconstant polynomial with $\deg_z R \leq 2$ or $R(z)$ is a nonzero constant. Furthermore,

(i) *If $R(z)$ is a nonconstant polynomial and $\deg_z R \leq 2$, then f is of the form*

$$f(z) = \frac{s_1(z)e^{az+b} + s_2(z)e^{-(az+b)}}{2},$$

where $R(z) = -(as_1(z) + m_1)(as_2(z) + m_2)$, $a \neq 0, b \in \mathbb{C}$, and a, b, c, α, β satisfy $\alpha \neq \pm\beta$, $a = -i(\alpha + \beta)$, $c = \frac{(2k+1)\pi}{a} i$, $i\alpha c = -1$ or $a = i(\alpha - \beta)$, $c = \frac{2k\pi}{a} i$, $i\alpha c = 1$, where $s_j(z) = m_j z + n_j$, $m_j, n_j \in \mathbb{C}(j = 1, 2)$;

(ii) *If $R(z)$ is a nonzero constant, then f is of the form*

$$f(z) = \frac{n_1 e^{az+b} + n_2 e^{-(az+b)}}{2} + d,$$

$R = -a^2 n_1 n_2$, $a \neq 0, b \in \mathbb{C}$, and a, b, c, α, β satisfy the following cases:

(ii_1) *if $\alpha = \beta$, then $a = -2\alpha i, c = \frac{(2k+1)\pi}{a} i$ and $d \in \mathbb{C}$;*

(ii_2) *if $\alpha = -\beta$, then $a = 2\alpha i$, $c = \frac{2k\pi}{a} i$ and $d = 0$;*

(ii_3) *if $\alpha \neq \pm\beta$, then $d = 0$ and $a = -i(\alpha + \beta)$, $c = \frac{(2k+1)\pi}{a} i$ or $a = i(\alpha - \beta)$, $c = \frac{2k\pi}{a} i$.*

Next, we give some examples to explain the existence of solutions for Eq. (1.3) in the above cases.

- For Case (i), let $s_1(z) = 1$, $s_2 = z + 1$, $c = \pi i$, $a = 1$ and $b \in \mathbb{C}$. That is

$$f(z) = \frac{e^{z+b} + (z+1)e^{-(z+b)}}{2}.$$

Thus, $f(z)$ satisfies Eq. (1.3) with $c = \pi i$, $\alpha = \frac{1}{\pi}, \beta = i - \frac{1}{\pi}$ and $R(z) = -z$;
Let $s_1(z) = z + 1$, $s_2 = z - 1$, $a = 1$ and $b \in \mathbb{C}$. That is

$$f(z) = \frac{(z+1)e^{z+b} + (z-1)e^{-(z+b)}}{2}.$$

Thus, $f(z)$ satisfies Eq. (1.3) with $c = \pi i, \alpha = \frac{1}{\pi}$, $\beta = i - \frac{1}{\pi}$ and $R(z) = -z(z+2)$;

- For Case (ii_1), let $n_1(z) = 1$, $n_2 = 2$, $a = 1$ and $b, d \in \mathbb{C}$. That is

$$f(z) = \frac{e^{z+b} + 2e^{-(z+b)}}{2} + d.$$

Thus, $f(z)$ satisfies Eq. (1.3) with $c = \pi i$, $\alpha = \frac{i}{2}$, $\beta = \frac{i}{2}$ and $R(z) = -2$;
For Case (ii_2), let $n_1(z) = 1$, $n_2 = 1$, $a = 2$ and $b \in \mathbb{C}$. That is

$$f(z) = \frac{e^{2z+b} + e^{-(2z+b)}}{2}.$$

Thus, $f(z)$ satisfies Eq. (1.3) with $c = \pi i$, $\alpha = -i, \beta = i$ and $R(z) = -4$;

For Case (ii_3), let $n_1(z) = 2$, $n_2 = 1$, $a = 1$ and $b \in \mathbb{C}$. That is

$$f(z) = \frac{2e^{z+b} + e^{-(z+b)}}{2}.$$

Thus, $f(z)$ satisfies Eq. (1.3) with $c = 2\pi i$, $\alpha = 1, \beta = 1 + i$ and $R(z) = -2$.

Theorem 1.4. *Let $\alpha(\neq 0), \beta \in \mathbb{C}$, and k is an integer. Let f be a transcendental meromorphic solution of difference-differential equation of Fermat type*

$$f''(z)^2 + [\alpha f(z+c) - \beta f(z)]^2 = R(z) \tag{1.4}$$

where $R(z)$ is a nonzero rational function and c is a nonzero constant.

(i) If $\alpha = \pm\beta$, then Eq. (1.4) *has no finite order transcendental meromorphic solutions with finitely many poles;*

(ii) If $\alpha \neq \pm\beta$, and Eq. (1.4) *has a finite order transcendental meromorphic solution f with finitely many poles, then $R(z)$ must be a nonzero polynomial with* $\deg_z R \leq 1$. *Furthermore,*

(ii_1) *if $R(z)$ is a nonzero polynomial of degree one, then $f(z)$ is of the form*

$$f(z) = \frac{s_1(z)e^{az+b} + n_2 e^{-(az+b)}}{2},$$

where $a^4 = \alpha^2 - \beta^2$, $b \in \mathbb{C}$, $c = \frac{\log \frac{a^2+i\beta}{i\alpha} + 2k\pi i}{a}$, $e^{ac} = \frac{2a}{i\alpha c} \neq \pm 1$ and $R = a^3 n_2[as_1(z) + 2m_1], s_1(z) = m_1 z + n_1$, or $f(z)$ is of the form

$$f(z) = \frac{n_1 e^{az+b} + s_2(z)e^{-(az+b)}}{2},$$

where $a^4 = \alpha^2 - \beta^2$, $b \in \mathbb{C}$, $c = \frac{\log \frac{-a^2+i\beta}{i\alpha} + (2k+1)\pi i}{a}$, $e^{ac} = \frac{i\alpha c}{2a} \neq \pm 1$ and $R = a^3 n_1[as_2(z) - 2m_2]$, $s_2(z) = m_2 z + n_2$;

(ii_2) *if $R(z)$ is a nonzero constant, then $f(z)$ is of the form*

$$f(z) = \frac{c_1 e^{az+b} + c_2 e^{-(az+b)}}{2},$$

where $a, b, c, \alpha, \beta, c_1, c_2, R$ satisfy $a^4 = \alpha^2 - \beta^2$, $b \in \mathbb{C}$, $c = \frac{\log \frac{a^2+i\beta}{i\alpha} + 2k\pi i}{a}$ and $R = a^4 c_1 c_2$;

The following examples show that the existence of solutions for complex differential-difference equation of Theorem 1.4 (ii_1) and (ii_2).

Example 1.1. *Let $s_1(z) = z$, $n_2 = 1$, $a = 1$ and $b \in \mathbb{C}$. And let c_0 be a solution of equation $e^{2c}(1-c) = 1$, $\alpha = \frac{2}{ic_0 e^{c_0}}$, and $\beta = \frac{2-c_0}{ic_0}$. Then it follows that $i\alpha e^{c_0} - i\beta = 1$, $i\alpha e^{-c_0} - i\beta = 1$, $\alpha^2 - \beta^2 = 1$, and $i\alpha c_0 e^{c_0} = 2$. Thus, we can deduce that*

$$f(z) = \frac{ze^{z+b} + e^{-(z+b)}}{2}$$

satisfies the following equation

$$f''(z)^2 + [\alpha f(z+c) - \beta f(z)]^2 = z + 2.$$

Example 1.2. *Let $c_1 = 1$, $c_2 = 1$, $e^c = \sqrt{5} - 2$, $a = 1, b \in \mathbb{C}$, $\alpha = \frac{i}{2}$, and $\beta = \frac{\sqrt{5}}{2}i$. Thus*

$$f(z) = \frac{c_1 e^{z+b} + c_2 e^{-(z+b)}}{2}$$

satisfies the following equation

$$f''(z)^2 + \left[\frac{i}{2}f(z+c) - \frac{\sqrt{5}}{2}if(z)\right]^2 = 1.$$

2. Some Lemmas

To prove our theorems, we require the following lemmas.

Lemma 2.1. *([31, Lemma 2.1]). If f is a nonconstant rational function and it satisfies the following differential-difference equation*

$$f' = \eta\Delta_c f = \eta[f(z+c) - f(z)],$$

where η and c are two nonzero constants, then $\eta c = 1$ and f is a polynomial of degree one.

Lemma 2.2. *Let c, a, α be three nonzero constants satisfying $\eta c \neq n - 1$ and $n \geq 1$ be an integer. If f is a nonconstant rational solution of the following differential-difference equation*

$$i\alpha\Delta_c f(z) = f^{(n)}(z) + \binom{n}{1}\eta f^{(n-1)}(z) + \cdots + \binom{n}{j}\eta^j f^{(n-j)}(z) + \cdots + \binom{n}{n-1}\eta^{n-1} f'(z). \tag{2.1}$$

Then $i\alpha c = n\eta^{n-1}$ and f is a polynomial of degree one.

Proof. We firstly prove that $f(z)$ has no poles. On the contrary, suppose that z_0 is a pole of f. Since (2.1) can be rewritten (2.1) in the following form

$$\begin{aligned} &i\alpha f(z+c) \\ =&[f^{(n)}(z) + n\eta f^{(n-1)}(z) + \cdots + \binom{n}{j}\eta^j f^{(n-j)}(z) + \cdots + n\eta^{n-1} f'(z)] + i\alpha f(z). \end{aligned} \tag{2.2}$$

It is easy to see that $z_0 + c$ is also a pole of $f(z)$ by comparing the order of pole z_0 on both sides of Eq. (2.2). By the cyclic utilization of this operation, we can get that a sequence poles of $f(z)$ are $z_0 + 2c$, $z_0 + 3c, \ldots, z_0 + tc, \cdots$, this is impossible since $f(z)$ is a nonconstant rational function. Hence, $f(z)$ is a polynomial.

Let $f(z)$ be a polynomial of the form $f(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0$, where $k \geq 1$ and $a_k(\neq 0), a_{k-1}, \ldots, a_0$ are constants. Then

$$\begin{aligned} &f'(z) = ka_k z^{k-1} + (k-1)a_{k-1} z^{k-2} + \cdots, \\ &f''(z) = k(k-1)a_k z^{k-2} + (k-1)(k-2)a_{k-1} z^{k-3} + \cdots, \\ &\cdots, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\Delta_c f(z) =& a_k[(z+c)^k - z^k] + a_{k-1}[(z+c)^{k-1} - z^{k-1}] + \cdots + a_1 c\\ =& a_k(kcz^{k-1} + \binom{k}{2}c^2 z^{k-2} + \cdots) + a_{k-1}[(k-1)cz^{k-2} + \cdots]\\ &+ \cdots + a_1 c\\ =& a_k kcz^{k-1} + [a_k\binom{k}{2}c^2 + a_{k-1}(k-1)c]z^{k-2} + \cdots + a_1 c. \end{aligned} \tag{2.4}$$

Substituting (2.3), (2.4) into Eq. (2.2), we obtain

$$\begin{cases} i\alpha a_k kc = n\eta^{n-1} k a_k,\\ i\alpha[a_k\binom{k}{2}c^2 + a_{k-1}\binom{k-1}{1}c] = n\eta^{n-1}(k-1)a_{k-1} + \binom{n}{2}\eta^{n-2}k(k-1)a_k, \end{cases}$$

which means $\begin{cases} i\alpha c = n\eta^{n-1},\\ \frac{1}{2}a_k k(k-1)\eta^{n-2}(\eta c - n + 1) = 0. \end{cases}$ In view of $\eta c \neq n-1$, thus it follows that $i\alpha c = n\eta^{n-1}$ and $k = 1$.

Noting that $f(z)$ is a polynomial of degree one if $k = 1$, then the conclusions of this lemma are proved.

Therefore, this completes the proof of this lemma. □

Lemma 2.3. *Let α, a, c be three nonzero constants. If R_1, R_2 are two nonconstant rational functions satisfying the following differential-difference equations*

$$\begin{cases} i\alpha e^{ac}[R_1(z+c) - R_1(z)] = R_1''(z) + 2aR_1'(z),\\ i\alpha e^{-ac}[R_2(z+c) - R_2(z)] = -R_2''(z) + 2aR_2'(z), \end{cases} \tag{2.5}$$

Then $e^{ac} = \pm 1$ and R_1, R_2 are two polynomials of degree one.

Proof. Firstly, (2.5) can be written in the following form

$$\begin{cases} R_1(z+c) = \dfrac{1}{i\alpha e^{ac}}[R_1''(z) + 2aR_1'(z)] + R_1(z),\\ R_2(z+c) = \dfrac{1}{i\alpha e^{-ac}}[-R_2''(z) + 2aR_2'(z)] + R_2(z). \end{cases} \tag{2.6}$$

Similar to the argument in Lemma 2.2, we can prove that R_1, R_2 are two nonconstant polynomials. Let

$$R_1(z) = a_k z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0, R_2(z) = b_t z^t + b_{t-1}z^{t-1} + \cdots + b_1 z + b_0,$$

where $a_j, b_j \in \mathbb{C}$, $a_k \neq 0, b_t \neq 0$, $k \geq 1$ and $t \geq 1$. Substituting $R_1(z), R_2(z)$ into (2.5), and comparing the coefficients of $z^{k-1}, z^{k-2}, z^{t-1}$ and z^{t-2} both sides of such two equations, it yields

$$\begin{cases} i\alpha e^{ac} a_k kc = 2a a_k k,\\ i\alpha e^{-ac} b_t tc = 2a a_t t,\\ i\alpha e^{ac}[a_k\binom{k}{2}c^2 + a_{k-1}(k-1)c] = 2a(k-1)a_{k-1} + k(k-1)a_k,\\ i\alpha e^{-ac}[b_t\binom{t}{2}c^2 + b_{t-1}(t-1)c] = 2a(t-1)b_{t-1} - t(t-1)b_t, \end{cases}$$

which means

$$\begin{cases} e^{ac} = \pm 1, \\ k(k-1)(ac-1) = 0, \\ t(t-1)(ac+1) = 0. \end{cases} \tag{2.7}$$

Since $k \geq 1$ and $t \geq 1$, it follows $e^{ac} = \pm 1$ and $k = 1, s = 1$. Therefore, this completes the proof of Lemma 2.3 □

Lemma 2.4. *Let R be a nonconstant rational function and $p(z) = az+b$ $(a \neq 0)$. Denote $A_1 = R' + Rp'$, $A_n = A'_{n-1} + A_{n-1}p'$, $B_1 = R' - Rp'$, $B_n = B'_{n-1} + B_{n-1}(-p)'$. Then*

$$\lim_{|z|\to\infty} \frac{A'_n}{R} = 0, \quad \lim_{|z|\to\infty} \frac{A_n}{R} = a^n, \quad \lim_{|z|\to\infty} \frac{B'_n}{R} = 0, \quad \lim_{|z|\to\infty} \frac{B_n}{R} = (-a)^n.$$

Proof. We use the mathematical induction to prove it. When $k = 1$, since R is a nonconstant rational function and $p' = a$, then $\lim_{|z|\to\infty} \frac{A'_1}{R} = \lim_{|z|\to\infty} \frac{R''+R'a}{R} = 0$ and $\lim_{|z|\to\infty} \frac{A_1}{R} = \lim_{|z|\to\infty} \frac{R'+Rp'}{R} = a$.

Suppose that $\lim_{|z|\to\infty} \frac{A'_k}{R} = 0$, $\lim_{|z|\to\infty} \frac{A_k}{R} = a^k$. Thus, $\lim_{|z|\to\infty} \frac{A'_{k+1}}{R} = \lim_{|z|\to\infty} \frac{A'_k+A'_k a}{R} = 0$ and $\lim_{|z|\to\infty} \frac{A_{k+1}}{R} = \lim_{|z|\to\infty} \frac{A'_k+A_k p'}{R} = a^{k+1}$. Hence, we have $\lim_{|z|\to\infty} \frac{A'_n}{R} = 0$ and $\lim_{|z|\to\infty} \frac{A_n}{R} = a^n$.

Similar to the above argument, we can prove that $\lim_{|z|\to\infty} \frac{B'_n}{R} = 0$ and $\lim_{|z|\to\infty} \frac{B_n}{R} = (-a)^n$.

Therefore, this completes the proof of Lemma 2.4. □

Lemma 2.5. *([30, Theorem 1.51]). Suppose that $f_1, f_2, \ldots, f_n (n \geq 2)$ are meromorphic functions and $g_1, g_2, \ldots, g_n$ are entire functions satisfying the following conditions*

(i) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0$;

(ii) $g_j - g_k$ are not constants for $1 \leq j < k \leq n$;

(iii) For $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}$ $(r \to \infty, r \notin E)$, where E is a set of $r \in (0, \infty)$ with finite linear measure.

Then $f_j \equiv 0$ $(j = 1, 2, \ldots, n)$.

Lemma 2.6. *(see [30, Theorem 2.7]). Let f be a meromorphic function of finite order $\rho(f)$. Write*

$$f(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots, \quad (c_k \neq 0)$$

near $z = 0$ and let $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$ be the zeros and poles of f in $\mathbb{C}\backslash\{0\}$, respectively. Then

$$f(z) = z^k e^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

where $P_1(z)$ and $P_2(z)$ are the canonical products of f formed with the non-null zeros and poles of f, respectively, and $Q(z)$ is a polynomial of degree $\leq \rho(f)$.

3. The proof of Theorem 1.1

Proof. Suppose that Eq. (1.1) admits a finite order transcendental meromorphic solution $f(z)$ with finitely many poles. We can rewrite Eq. (1.1) as

$$[f(z) + i(\alpha f(z+c) - \beta f(z))][f(z) - i(\alpha f(z+c) - \beta f(z))] = R(z). \tag{3.1}$$

Since $f(z)$ has finitely many poles and R is a nonzero rational function, then $f(z)+i(\alpha f(z+c)-\beta f(z))$ and $f(z)-i(\alpha f(z+c)-\beta f(z))$ both have finitely many poles and zeros. Thus, in view of Lemma 2.6, (3.1) can be written as

$$\begin{cases} f(z)+i(\alpha f(z+c)-\beta f(z))=R_1e^{p(z)}, \\ f(z)-i(\alpha f(z+c)-\beta f(z))=R_2e^{-p(z)}, \end{cases} \tag{3.2}$$

where R_1, R_2 are two nonzero rational functions such that $R_1R_2 = R$ and $p(z)$ is a nonconstant polynomial. By solving the above equations system, we have

$$\begin{cases} f(z)=\dfrac{R_1e^{p(z)}+R_2e^{-p(z)}}{2}, \\ \alpha f(z+c)-\beta f(z)=\dfrac{R_1e^{p(z)}-R_2e^{-p(z)}}{2i}. \end{cases} \tag{3.3}$$

Substituting the first equation of system (3.3) into the second equation of system (3.3), we get

$$\begin{aligned} &e^{p(z)}\left[i\alpha R_1(z+c)e^{p(z+c)-p(z)}-i\beta R_1(z)-R_1(z)\right] \\ &\quad+e^{-p(z)}\left[i\alpha R_2(z+c)e^{-p(z+c)+p(z)}-i\beta R_2(z)+R_2(z)\right]=0. \end{aligned} \tag{3.4}$$

By Lemma 2.5, it yields from (3.4) that

$$\begin{cases} i\alpha R_1(z+c)e^{p(z+c)-p(z)}-i\beta R_1(z)-R_1(z)=0, \\ i\alpha R_2(z+c)e^{-p(z+c)+p(z)}-i\beta R_2(z)+R_2(z)=0. \end{cases} \tag{3.5}$$

Since R_1, R_2 are two nonzero rational functions and f is of finite order, we obtain that $p(z)$ is a polynomial of degree one. Otherwise, assume that $\deg_z p(z)\geq 2$. Thus, in view of (3.5), we have

$$e^{p(z+c)-p(z)}=\frac{i\beta R_1(z)+R_1(z)}{i\alpha R_1(z+c)},\quad e^{-p(z+c)+p(z)}=\frac{i\beta R_2(z)-R_2(z)}{i\alpha R_2(z+c)},$$

which imply $1=\rho\left(e^{p(z+c)-p(z)}\right)=\rho\left(\frac{i\beta R_1(z)+R_1(z)}{i\alpha R_1(z+c)}\right)=0$, a contradiction. Hence, $\deg_z p(z)=1$. Let $p(z)=az+b$, $a\neq 0, b\in\mathbb{C}$. Substituting this into (3.5), we can deduce

$$\begin{aligned} &\lim_{|z|\to\infty} i\left(\frac{R_1(z+c)}{R_1(z)}e^{p(z+c)-p(z)}-\beta\right)=i(\alpha e^{ac}-\beta)=1, \\ &\lim_{|z|\to\infty} i\left(\frac{R_2(z+c)}{R_2(z)}e^{p(z+c)-p(z)}-\beta\right)=i(\alpha e^{-ac}-\beta)=-1. \end{aligned}$$

Thus, it yields that $\alpha^2-\beta^2=1$, a contradiction.

Therefore, this completes the proof of Theorem 1.1. □

4. The proof of Theorem 1.2

Proof. Suppose that $f(z)$ is a transcendental entire solution with finite order of Eq. (1.2). Similar to the above argument as in the proof of Theorem 1.1, it follows that

$$\begin{cases} f(z)=\dfrac{Q_1e^{p(z)}+Q_2e^{-p(z)}}{2}, \\ \alpha f(z+c)-\beta f(z)=\dfrac{Q_1e^{p(z)}-Q_2e^{-p(z)}}{2iP(z)}, \end{cases} \tag{4.1}$$

where $p(z)$ is a nonconstant polynomial and $Q_1(z)Q_2(z) = Q(z)$, $Q_1(z)$, $Q_2(z)$ are nonzero polynomials. In view of (4.1), it yields that

$$\begin{aligned} &e^{p(z)}\left[i\alpha P(z)Q_1(z+c)e^{p(z+c)-p(z)} - i\beta P(z)Q_1(z) - Q_1(z)\right] \\ &\quad + e^{-p(z)}\left[i\alpha P(z)Q_2(z+c)e^{-p(z+c)+p(z)} - i\beta P(z)Q_2(z) + Q_2(z)\right] = 0. \end{aligned} \tag{4.2}$$

Since $p(z)$ is a nonconstant polynomial, then by Lemma 2.5, we conclude that

$$\begin{cases} i\alpha P(z)Q_1(z+c)e^{p(z+c)-p(z)} - i\beta P(z)Q_1(z) - Q_1(z) = 0, \\ i\alpha P(z)Q_2(z+c)e^{-p(z+c)+p(z)} - i\beta P(z)Q_2(z) + Q_2(z) = 0, \end{cases} \tag{4.3}$$

which implies that $p(z)$ is a polynomial of degree one. Set $p(z) = az + b$, $a \neq 0, b \in \mathbb{C}$. Thus, it follows from (4.3) that

$$\begin{cases} i\alpha P(z)Q_1(z+c)e^{ac} = (i\beta P(z) + 1)Q_1(z), \\ i\alpha P(z)Q_2(z+c)e^{-ac} = (i\beta P(z) - 1)Q_2(z), \end{cases}$$

which means that

$$\alpha^2 P(z)^2 Q_1(z+c)Q_2(z+c) = (\beta^2 P(z)^2 + 1)Q_1(z)Q_2(z).$$

Hence, it yields

$$P(z)^2[\alpha^2 Q(z+c) - \beta^2 Q(z)] = Q(z). \tag{4.4}$$

Set $\deg_z P = p$ and $\deg_z Q = q$, then $p \geq 0$, $q \geq 0$ and $p, q \in \mathbb{N}_+$.

Case 1. $p \geq 1$ and $\alpha = \pm\beta$. If $q \geq 1$. Since $\alpha \neq 0$, thus, by comparing the order both sides of Eq. (4.4), it follows $2p + q - 1 = q$, that is, $p = \frac{1}{2}$, a contradiction. If $q = 0$, that is, $Q(z)$ is a constant. Thus, we can conclude from (4.4) that $Q(z) \equiv 0$, a contradiction.

Case 2. $p \geq 1$ and $\alpha \neq \pm\beta$. If $q \geq 1$. Since $\alpha \neq 0$, thus, by comparing the order both sides of Eq. (4.4), it follows $2p + q = q$, that is, $p = 0$, a contradiction. If $q = 0$, that is, $Q(z)$ is a constant. Thus, we can conclude from (4.4) that $P(z)$ is a constant, a contradiction with $p \geq 1$.

Case 3. $p = 0$ and $\alpha = \pm\beta$. Thus, $P(z) = K(\neq 0)$. If $q \geq 1$. Since $\alpha \neq 0$, thus, by comparing the order both sides of Eq. (4.4), it follows $q - 1 = q$, a contradiction. If $q = 0$, it follows $Q(z) \equiv 0$, a contradiction.

Case 4. $p = 0$ and $\alpha \neq \pm\beta$. If $q \geq 1$, set $P(z) = K(\neq 0)$, $Q(z) = b_q z^q + b_{q-1}z^{q-1} + \cdots + b_0$, $b_q \neq 0, b_{q-1}, \ldots, b_0$ are constants. By comparing the coefficients of z^q, z^{q-1} both sides of (4.4), it yields that

$$K^2[\alpha^2 - \beta^2] = 1, \quad K^2[\alpha^2 qca_q + (\alpha^2 - \beta^2)a_{q-1}] = a_{q-1}, \tag{4.5}$$

which implies that $\alpha^2 qca_q = 0$, a contradiction. If $q = 0$, then $K^2(\alpha^2 - \beta^2) = 1$, a contradiction.

Therefore, this completes the proof of Theorem 1.2. □

5. The proof of Theorem 1.3

Proof. Suppose that Eq. (1.3) admits a finite order transcendental meromorphic solution $f(z)$ with finitely many poles. We can rewrite Eq. (1.3) in the following form

$$[f'(z) + i(\alpha f(z+c) - \beta f(z))][f'(z) - i(\alpha f(z+c) - \beta f(z))] = R(z). \tag{5.1}$$

Since $f(z)$ has finitely many poles and R is a nonzero rational function, then $f'(z) + i(\alpha f(z+c) - \beta f(z))$ and $f'(z) - i(\alpha f(z+c) - \beta f(z))$ both have finitely many poles and zeros. Thus, in view of Lemma 2.6, (5.1) can be written as

$$\begin{cases} f'(z) + i(\alpha f(z+c) - \beta f(z)) = R_1 e^{p(z)}, \\ f'(z) - i(\alpha f(z+c) - \beta f(z)) = R_2 e^{-p(z)}, \end{cases} \tag{5.2}$$

where R_1, R_2 are two nonzero rational functions such that $R_1 R_2 = R$, and $p(z)$ is a nonconstant polynomial. By solving the above equations system, we have

$$\begin{cases} f'(z) = \dfrac{R_1 e^{p(z)} + R_2 e^{-p(z)}}{2}, \\ \alpha f(z+c) - \beta f(z) = \dfrac{R_1 e^{p(z)} - R_2 e^{-p(z)}}{2i}. \end{cases} \tag{5.3}$$

In view of the second equation of (5.3), it follows that

$$\alpha f'(z+c) - \beta f'(z) = \frac{A_1 e^{p(z)} - B_1 e^{-p(z)}}{2i}, \tag{5.4}$$

where $A_1 = R_1' + R_1 p'$ and $B_1 = R_2' - R_2 p'$. Substituting the first equation of system (5.3) into (5.4), it yields that

$$\begin{aligned} &e^{p(z)}\left[i\alpha R_1(z+c)e^{p(z+c)-p(z)} - i\beta R_1(z) - A_1(z)\right] \\ &\quad + e^{-p(z)}\left[i\alpha R_2(z+c)e^{-p(z+c)+p(z)} - i\beta R_2(z) + B_1(z)\right] = 0. \end{aligned} \tag{5.5}$$

By Lemma 2.5, it yields from (5.5) that

$$\begin{cases} i\alpha R_1(z+c)e^{p(z+c)-p(z)} - i\beta R_1(z) - A_1(z) = 0, \\ i\alpha R_2(z+c)e^{-p(z+c)+p(z)} - i\beta R_2(z) + B_1(z) = 0. \end{cases} \tag{5.6}$$

Since R_1, R_2 are two nonzero rational functions and f is of finite order, similar to argument as in the proof of Theorem 1.1, it follows that $p(z)$ is a polynomial of degree one. Let $p(z) = az+b$, $a \neq 0, b \in \mathbb{C}$. Substituting $p(z), A_1, B_1$ into (5.6), and let $z \to \infty$, thus we can conclude from Lemma 2.4 that

$$\begin{aligned} &\lim_{|z|\to\infty} i\left(\frac{R_1(z+c)}{R_1(z)}e^{p(z+c)-p(z)} - \beta\right) = i(\alpha e^{ac} - \beta) = \frac{R_1'(z)}{R_1(z)} + a = a, \\ &\lim_{|z|\to\infty} i\left(\frac{R_2(z+c)}{R_2(z)}e^{p(z+c)-p(z)} - \beta\right) = i(\alpha e^{-ac} - \beta) = -\frac{R_2'(z)}{R_2(z)} + a = a, \end{aligned}$$

which means that

$$i(\alpha e^{ac} - \beta) = a, \quad i(\alpha e^{-ac} - \beta) = a. \tag{5.7}$$

Hence, it yields $e^{ac} = \pm 1$.

If $e^{ac} = 1$, then $a = i\alpha - i\beta$. Thus, we can rewrite (5.6) in the following form

$$\begin{cases} i\alpha[R_1(z+c) - R_1(z)] = R_1'(z), \\ i\alpha[R_2(z+c) - R_2(z)] = R_2'(z). \end{cases} \tag{5.8}$$

If R_1, R_2 are nonzero constants, then (5.8) holds and $R = R_1R_2$ is a constant.

If $R_j(j = 1, 2)$ is a nonzero rational function, then in view of Lemma 2.1, it follows that $i\alpha c = 1$ and $R_j(j = 1, 2)$ is a polynomial of $\deg_z R_j = 1$. In view of $R = R_1R_2$, thus R is a nonconstant polynomial with $\deg_z R \leq 2$.

If $e^{ac} = -1$, then $a = -i\alpha - i\beta$. Thus, we can rewrite (5.6) in the following form

$$\begin{cases} -i\alpha[R_1(z+c) - R_1(z)] = R_1'(z), \\ -i\alpha[R_2(z+c) - R_2(z)] = R_2'(z). \end{cases} \tag{5.9}$$

Like in the previous case, we can obtain that $i\alpha c = \pm 1$ and $R(z)$ is a nonconstant polynomial with $\deg_z R \leq 2$ or $R(z)$ is a nonzero constant.

Hence, we can conclude that R is a nonconstant polynomial with $\deg_z R \leq 2$ or R is a nonzero constant.

(i) Suppose that $R(z)$ is a nonconstant polynomial with $\deg_z R \leq 2$, then in view of the first equation of (5.3), it follows that $f(z)$ is of the form

$$f(z) = \frac{s_1(z)e^{az+b} + s_2(z)e^{-(az+b)}}{2} + \gamma, \tag{5.10}$$

where $s_j(z) = m_jz + n_j, m_j, n_j \in \mathbb{C}$ $(j = 1, 2)$ and $\gamma \in \mathbb{C}$.

Case 1. If $\deg_z R = 2$, then it follows that $m_j \neq 0(j = 1, 2)$. Substituting (5.10) into (5.3), it follows that $R(z) = -(as_1(z) + m_1)(as_2(z) + m_2)$, $i\alpha c = 1$ and $a = i(\alpha - \beta)$ or $i\alpha c = -1$ and $a = -i(\alpha + \beta)$.

If $i\alpha c = 1$ and $a = i(\alpha - \beta)$, then $e^{ac} = 1$, *i.e.*, $c = \frac{2k\pi i}{a}$. Obviously, $\alpha \neq \beta$ as $a \neq 0$. If $\alpha = -\beta$, then $a = 2i\alpha$. Thus, since $\alpha \neq 0$ and from (5.7), it follows $1 = e^{ac} = e^{2i\alpha c} = e^2$, a contradiction. Hence, $\alpha \neq \pm\beta$. Thus, substituting (5.10) into the second equation of (5.3), it follows $\gamma \equiv 0$.

If $i\alpha c = -1$ and $a = -i(\alpha + \beta)$, then $e^{ac} = -1$, *i.e.*, $c = \frac{(2k+1)\pi i}{a}$. Obviously, $\alpha \neq -\beta$ as $a \neq 0$. If $\alpha = \beta$, then $a = -2i\alpha$. Thus, since $\alpha \neq 0$ and from (5.7), it follows $-1 = e^{ac} = e^{-2i\alpha c} = e^2$, a contradiction. Hence, $\alpha \neq \pm\beta$.

Case 2. If $\deg_z R = 1$, then one of m_1, m_2 is zero, without loss of generality, assuming that $m_1 = 0$. Substituting (5.10) into (5.3), it follows that R_1 is a constant and R_2 is a polynomial of degree one, and $i\alpha c = 1$ and $a = i(\alpha - \beta)$ or $i\alpha c = -1$ and $a = -i(\alpha + \beta)$. Similar to the argument as in Case 1, it is easy to prove that $\alpha \neq \pm\beta$ and $\gamma \equiv 0$.

Therefore, $f(z)$ is of the form

$$f(z) = \frac{s_1(z)e^{az+b} + s_2(z)e^{-(az+b)}}{2},$$

where $R(z) = -(as_1(z) + m_1)(as_2(z) + m_2)$, $a \neq 0, b \in \mathbb{C}$, and a, b, c, α, β satisfy $\alpha \neq \pm\beta$, $a = -i(\alpha + \beta)$, $c = \frac{(2k+1)\pi}{a}i$, $i\alpha c = -1$ or $a = i(\alpha - \beta)$, $c = \frac{2k\pi}{a}i$, $i\alpha c = 1$.

(ii) If $R(z)$ is a nonzero constant, then in view of the first equation of (5.3), it follows that $f(z)$ is of the form

$$f(z) = \frac{n_1e^{az+b} + n_2e^{-(az+b)}}{2} + d, \tag{5.11}$$

where $n_1, n_2 \in \mathbb{C}$ and $\gamma \in \mathbb{C}$. Substituting (5.11) into the first equation of (5.3), it yields $R = -a^2n_1n_2$. For the sake of brevity and readability, we give the proof of Theorem 1.3 (ii_1)-(ii_3) in Appendix A.

Therefore, this completes the proof of Theorem 1.3. □

6. The proof of Theorem 1.4

Proof. Suppose that Eq. (1.4) admits a finite order transcendental meromorphic solution $f(z)$ with finitely many poles. We can rewrite Eq. (1.4) as

$$[f''(z) + i(\alpha f(z+c) - \beta f(z))][f''(z) - i(\alpha f(z+c) - \beta f(z))] = R(z). \tag{6.1}$$

Since $f(z)$ has finitely many poles and R is a nonzero rational function, then $f''(z)+i(\alpha f(z+c)-\beta f(z))$ and $f''(z) - i(\alpha f(z+c) - \beta f(z))$ both have finitely many poles and zeros. Thus, (6.1) can be written as

$$\begin{cases} f''(z) + i(\alpha f(z+c) - \beta f(z)) = R_1 e^{p(z)}, \\ f''(z) - i(\alpha f(z+c) - \beta f(z)) = R_2 e^{-p(z)}, \end{cases} \tag{6.2}$$

where R_1, R_2 are two nonzero rational functions such that $R_1R_2 = R$ and $p(z)$ is a nonconstant polynomial. By solving the above equations system, we have

$$\begin{cases} f''(z) = \dfrac{R_1 e^{p(z)} + R_2 e^{-p(z)}}{2}, \\ \alpha f(z+c) - \beta f(z) = \dfrac{R_1 e^{p(z)} - R_2 e^{-p(z)}}{2i}. \end{cases} \tag{6.3}$$

In view of the second equation of (6.3), it follows that

$$\alpha f''(z+c) - \beta f''(z) = \frac{A_2 e^{p(z)} - B_2 e^{-p(z)}}{2i}, \tag{6.4}$$

where $A_2 = A_1' + A_1 p'$ and $B_2 = B_1' - B_1 p'$. Substituting the first equation of system (6.3) into (6.4), it yields

$$\begin{aligned} &e^{p(z)} \left[i\alpha R_1(z+c) e^{p(z+c)-p(z)} - i\beta R_1(z) - A_2(z) \right] \\ &\quad + e^{-p(z)} \left[i\alpha R_2(z+c) e^{-p(z+c)+p(z)} - i\beta R_2(z) + B_2(z) \right] = 0. \end{aligned} \tag{6.5}$$

By Lemma 2.5, it yields from (6.5) that

$$\begin{cases} i\alpha R_1(z+c) e^{p(z+c)-p(z)} - i\beta R_1(z) - A_2(z) = 0, \\ i\alpha R_2(z+c) e^{-p(z+c)+p(z)} - i\beta R_2(z) + B_2(z) = 0. \end{cases} \tag{6.6}$$

Since R_1, R_2 are two nonzero rational functions and f is of finite order, we obtain that $p(z)$ is a polynomial of degree one. Let $p(z) = az + b$, $a \neq 0, b \in \mathbb{C}$. Substituting $p(z), A_2, B_2$ into (6.6), and let $z \to \infty$, thus we can conclude from Lemma 2.4 that

$$\begin{aligned} &\lim_{|z|\to\infty} i\left(\frac{R_1(z+c)}{R_1(z)} e^{p(z+c)-p(z)} - \beta \right) = i(\alpha e^{ac} - \beta) = \frac{A_1'(z)}{R_1(z)} + a^2 = a^2, \\ &\lim_{|z|\to\infty} i\left(\frac{R_2(z+c)}{R_2(z)} e^{p(z+c)-p(z)} - \beta \right) = i(\alpha e^{-ac} - \beta) = \frac{B_1'(z)}{R_2(z)} - a^2 = -a^2, \end{aligned}$$

which means that

$$i(\alpha e^{ac}-\beta)=a^2,\quad i(\alpha e^{-ac}-\beta)=-a^2. \tag{6.7}$$

Hence, it follows $a^4=\alpha^2-\beta^2$.

(i) If $\alpha=\pm\beta$, this is a contradiction with $a^4=\alpha^2-\beta^2$. Therefore, this proves the conclusion of Theorem 1.3 (i).

(ii) If $\alpha\neq\pm\beta$. Substituting $p(z)=az+b$ and (6.7) into (6.6), it yields

$$\begin{cases} i\alpha e^{ac}[R_1(z+c)-R_1(z)]=R_1''(z)+2aR_1'(z),\\ i\alpha e^{-ac}[R_2(z+c)-R_2(z)]=-R_2''(z)+2aR_2'(z). \end{cases} \tag{6.8}$$

Suppose that R_1,R_2 are nonconstant rational functions, in view of Lemma 2.3 and (6.8), we can conclude that $e^{ac}=\pm1$ and R_1,R_2 are nonconstant polynomials of degree one. Set $\deg_z R_1=k$ and $\deg_z R_2=s$.

If $k=1$ and $s=1$, in view of Lemma 2.3, we obtain (2.7). If $e^{ac}=1$, then from (6.7), it follows that $i\alpha-i\beta=a^2$ and $i\alpha-i\beta=-a^2$, a contradiction. If $e^{-ac}=1$, then from (6.7), it follows that $-i\alpha-i\beta=a^2$ and $-i\alpha-i\beta=-a^2$, a contradiction. Hence, there is at most a polynomial of degree one in R_1 and R_2. For the sake of brevity and readability, we give the proof of Theorem 1.4 (ii_1) and (ii_2) in Appendix B.

Therefore, this completes the proof of Theorem 1.4.

□

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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Appendix A

The proof of Theorem 1.3 $(ii_1)-(ii_3)$: (ii_1) If $\alpha=\beta$, in view of (5.7), it follows that $e^{ac}=\pm 1$. If $e^{ac}=1$, then $a=0$ as $i\alpha(e^{ac}-1)=a$, a contradiction. Thus, $e^{ac}=-1$. Hence, it follows that $c=\frac{(2k+1)\pi i}{2}$, $a=-2i\alpha$ and $d\in\mathbb{C}$.

(ii_2) If $\alpha=-\beta$, in view of (5.7), it follows that $e^{ac}=\pm 1$. If $e^{ac}=-1$, then $a=0$ as $i\alpha(e^{-ac}+1)=a$, a contradiction. Thus, $e^{ac}=1$. Hence, it follows that $c=\frac{2k\pi i}{2}$, $a=2i\alpha$ and $d\equiv 0$.

(ii_3) If $\alpha\neq\pm\beta$, substituting (5.11) into the second equation of (5.3), it yields $d\equiv 0$. in view of (5.7), it follows that $e^{ac}=\pm 1$. If $e^{ac}=-1$, it follows that $c=\frac{(2k+1)\pi i}{2}$ and $a=i(\alpha-\beta)$. If $e^{ac}=-1$, it follows that $c=\frac{2k\pi i}{2}$ and $a=-i(\alpha+\beta)$.

Appendix B

The proof of Theorem 1.4 (ii_1) **and** (ii_2): (ii_1) Suppose that $k=1$ and $s=0$. In view of (6.5), it follows that f is of the form

$$f(z)=\frac{s_1(z)e^{az+b}+s_2(z)e^{-(az+b)}}{2}+P(z), \tag{6.9}$$

where $a\neq 0, b\in\mathbb{C}$, $s_1(z)=m_1z+n_1$, $s_2(z)=n_2$, $m_1(\neq 0), n_1, n_2\in\mathbb{C}$ and $P(z)$ is a polynomial of degree one. Since $\alpha\neq\beta$, then it yields from the second equation of (6.5) that $P(z)\equiv 0$. And in view of the first equation in (6.8), it follows that $i\alpha e^{ac}c=2a$. Hence, $f(z)$ is of the form

$$f(z)=\frac{s_1(z)e^{az+b}+n_2e^{-(az+b)}}{2},$$

where $a^4=\alpha^2-\beta^2$, $b\in\mathbb{C}$, $c=\frac{\log\frac{a^2+i\beta}{i\alpha}+2k\pi i}{a}$, $e^{ac}=\frac{2a}{i\alpha c}\neq\pm 1$ and $R=a^3n_2[as_1(z)+2m_1]$.

Suppose that $k = 0$ and $s = 1$. Similar to the above argument as in (ii_1), we get that

$$f(z) = \frac{n_1 e^{az+b} + s_2(z)e^{-(az+b)}}{2},$$

where $a^4 = \alpha^2 - \beta^2$, $b \in \mathbb{C}$, $c = \frac{\log \frac{-a^2+i\beta}{i\alpha} + (2k+1)\pi i}{a}$, $e^{ac} = \frac{i\alpha c}{2a} \neq \pm 1$ and $R = a^3 n_1[as_2(z) - 2m_2]$.

(ii_2) Suppose that R_1, R_2 are two nonzero constants. In view of (6.5), it follows that f is of the form

$$f(z) = \frac{c_1 e^{az+b} + c_2 e^{-(az+b)}}{2} + P(z), \tag{6.10}$$

where $a \neq 0, b \in \mathbb{C}$, $c_1, c_2 \in \mathbb{C} - \{0\}$ and $P(z)$ is a polynomial of degree one. Since $\alpha \neq \beta$, then it yields from the second equation of (6.5) that $P(z) \equiv 0$. Hence, $f(z)$ is of the form

$$f(z) = \frac{c_1 e^{az+b} + c_2 e^{-(az+b)}}{2},$$

where $a^4 = \alpha^2 - \beta^2$, $b \in \mathbb{C}$, $c = \frac{\log \frac{a^2+i\beta}{i\alpha} + 2k\pi i}{a}$ and $R = a^4 c_1 c_2$.