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Research article

Hermite-Hadamard type inequalities in the setting of k-fractional calculus theory with applications

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Abstract: The main objective of this paper is to derive some new k-fractional refinements of Hermite-Hadamard like inequalities. We also discuss some new special cases of the main results. In the last section, we discuss applications, which shows the significance of the obtained results.

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1. Introduction and preliminaries

A function $\Lambda : I \subset \mathbb{R} \to \mathbb{R}$ is said to be a convex, if

$$\Lambda((1-\varrho)u + \varrho v) \leq (1-\varrho)\Lambda(u) + \varrho\Lambda(v), \quad \forall u, v \in I, \varrho \in [0, 1].$$

Recently convexity has received special attention by many researchers and consequently it has been extended and generalized in different directions, for instance, see [3] in which authors introduced and discussed several new types of convexities. Varosanec [15] introduced h-convex functions. It has been noticed that for different suitable choices of function $h(.)$ it contains the classes of Breckner type of s-convexity [2], Godunova-Levin type of s-convexity [5], P-functions [7], Q-functions [8] and tgs-type of convex functions [14] etc.

Definition 1.1. Let $h : (0, 1) \to \mathbb{R}$ be a real function. We say that $\Lambda : I \to \mathbb{R}$ is an h-convex function, if

$$\Lambda(\varrho u + (1-\varrho)v) \leq h(\varrho)\Lambda(u) + h(1-\varrho)\Lambda(v), \quad \forall u, v \in I, \varrho \in (0, 1).$$

Theory of convexity played significant role in the development of theory of inequalities. Many know results in theory of inequalities can be obtained using the convexity property of functions. A simple yet a powerful result in this regard is Hermite-Hadamard's (HH) integral inequality, which provides us a necessary and sufficient condition for a function to be convex. It reads as:

Theorem 1.1. *Let* $\Lambda : I = [e, f] \subset \mathbb{R} \to \mathbb{R}$ *be a convex, if* $\Lambda \in L[e, f]$*, then*

$$\Lambda\left(\frac{e+f}{2}\right) \leq \frac{1}{f-e}\int_{e}^{f}\Lambda(u)\mathrm{d}u \leq \frac{\Lambda(e)+\Lambda(f)}{2}.$$

For more information, see [6].

Recently Sarikaya et al. [13] utilized the concepts of Riemann-Liouville (RL) fractional integrals which are defined as:

Definition 1.2 ([9])**.** Let $\Lambda \in L_1[e, f]$. Then the RL integrals $\mathrm{J}^{\alpha}_{e^+}\Lambda$ and $\mathrm{J}^{\alpha}_{f^-}\Lambda$ of order $\alpha > 0$ with $e \geq 0$ are defined by

$$\mathrm{J}^{\alpha}_{e^+}\Lambda(u) = \frac{1}{\Gamma(\alpha)}\int_{e}^{u}(u-\varrho)^{\alpha-1}\Lambda(\varrho)\mathrm{d}\varrho, \quad u > e, \tag{1.1}$$

and

$$\mathrm{J}^{\alpha}_{f^-}\Lambda(u) = \frac{1}{\Gamma(\alpha)}\int_{u}^{f}(\varrho-u)^{\alpha-1}\Lambda(\varrho)\mathrm{d}\varrho, \quad u < f, \tag{1.2}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\mathrm{d}u.$$

and obtained a new version of HH inequality in the setting of fractional calculus.

Theorem 1.2 ([13])**.** *Let* $\Lambda : [e, f] \to \mathbb{R}$ *be a positive function with* $0 \leq e < f$ *and* $\Lambda \in L[e, f]$*. If* Λ *is convex function on* $[e, f]$*, then*

$$\Lambda\left(\frac{e+f}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(f-e)^{\alpha}}[\mathrm{J}^{\alpha}_{e^+}\Lambda(f) + \mathrm{J}^{\alpha}_{f^-}\Lambda(e)] \leq \frac{\Lambda(e)+\Lambda(f)}{2}.$$

Note that if $\alpha = 1$, then the above version of HH inequality reduces to the classical HH inequality as described in Theorem 1.1. We now recall some preliminary concepts of k-fractional calculus. Generalized k-gamma and k-beta functions were introduced and studied by Diaz et al. [4] respectively as:

$$\Gamma_k(u) = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{u}{k}-1}}{(u)_{n,k}}, \quad k > 0, u \in \mathbb{C}\setminus k\mathbb{Z}^-. \tag{1.3}$$

Γ_k is one parameter deformation of the classical gamma function as $\Gamma_k \to \Gamma$ when $k \to 1$. Γ_k is based on the repeated appearance of the expression of:

$$\phi(\phi+k)(\phi+2k)(\phi+3k)\ldots(\phi+(n-1)k).$$

This above statement is a function of the variable ϕ and is denoted by $(\phi)_{n,k}$. It is known as Pochhammer k-symbol, which reduces to classical Pochhammer symbol $(\phi)_n$ by taking $k = 1$. The integral form of Γ_k is given by $\Gamma_k(u) = \int\limits_0^\infty \varrho^{u-1} e^{-\frac{\varrho^k}{k}} \mathrm{d}\varrho, \quad \Re(u) > 0$ and

$$B_k(u, v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}, \quad \Re(u) > 0, \Re(v) > 0. \tag{1.4}$$

Integral form of k-Beta function is given by:

$$B_k(u, v) = \frac{1}{k} \int\limits_0^1 \varrho^{\frac{u}{k}-1}(1-\varrho)^{\frac{v}{k}-1} \mathrm{d}\varrho.$$

Sarikaya et al. [12] discussed the concept of k-RL fractional integrals and discussed some of its interesting aspects and applications.
Let Λ be piecewise continuous on $I^* = (0, \infty)$ and integrable on any finite subinterval of $I = [0, \infty]$. Then for $\varrho > 0$, we consider k-RL fractional integral of Λ of order α

$${}_kJ_e^\alpha \Lambda(u) = \frac{1}{k\Gamma_k(\alpha)} \int\limits_e^u (u-\varrho)^{\frac{\alpha}{k}-1} \Lambda(\varrho) \mathrm{d}\varrho, \quad u > e, k > 0.$$

Note that when $k \to 1$ k-RL fractional integrals become classical RL fractional integral. For details on fractional inequalities and k-fractional inequalities, see [1, 12, 16].

The motivation of this paper is to derive some new parametric estimations of Hermite-Hadamard like inequalities via h-convex functions involving the concepts of k-fractional calculus. We also discuss in detail the special cases which can be deduced from the main results of our paper. In the last section, we discuss some applications. This shows the significance of our results. It is expected that the results of this paper may inspire interested readers.

2. Main results

We now derive a new k-fractional integral identity which will play significant role in obtaining main results of the article.

Lemma 2.1. *Let $\Lambda : I = [e, f] \subset \mathbb{R} \to \mathbb{R}$ be k-fractional differentiable function on I°. If $\Lambda' \in L[e, f]$, then*

$$|\Xi(e, f; \alpha, k)| = \frac{f-e}{4}\left[\int_0^1 \varrho^{\frac{\alpha}{k}} \Lambda'\left(\varrho\frac{e+f}{2} + (1-\varrho)e\right) \mathrm{d}\varrho - \int_0^1 \varrho^{\frac{\alpha}{k}} \Lambda'\left(\varrho\frac{e+f}{2} + (1-\varrho)f\right) \mathrm{d}\varrho\right],$$

where

$$\begin{aligned}&\Xi(e, f; \alpha, k)\\ &= \Lambda\left(\frac{e+f}{2}\right) + \frac{\Gamma_k(\alpha+1)}{f-e}\left[\left(\frac{2}{e-f}\right)^{\frac{\alpha}{k}-1} {}_k\mathrm{J}^\alpha_{\left(\frac{e+f}{2}\right)^+}\Lambda(e) - \left(\frac{2}{f-e}\right)^{\frac{\alpha}{k}-1} {}_k\mathrm{J}^\alpha_{\left(\frac{e+f}{2}\right)^-}\Lambda(f)\right].\end{aligned}$$

Proof. Take

$$\frac{f-e}{4}\left[\int_0^1 \varrho^{\frac{\alpha}{k}}\Lambda'\left(\varrho\frac{e+f}{2}+(1-\varrho)e\right)\mathrm{d}\varrho-\int_0^1 \varrho^{\frac{\alpha}{k}}\Lambda'\left(\varrho\frac{e+f}{2}+(1-\varrho)f\right)\mathrm{d}\varrho\right]=I_1-I_2. \tag{2.1}$$

Integration by parts and using k-fractional integral, we have

$$\begin{aligned} I_1 &= \int_0^1 \varrho^{\frac{\alpha}{k}}\Lambda'\left(\varrho\frac{e+f}{2}+(1-\varrho)e\right)\mathrm{d}\varrho \\ &= \frac{2}{f-e}\Lambda\left(\frac{e+f}{2}\right)+\frac{\alpha}{k}\left(\frac{2}{e-f}\right)^{\frac{\alpha}{k}+1}\int_{\frac{e+f}{2}}^{e}(e-u)^{\frac{\alpha}{k}-1}\Lambda(u)\mathrm{d}u \\ &= \frac{2}{f-e}\Lambda\left(\frac{e+f}{2}\right)+\Gamma_k(\alpha+1)\left(\frac{2}{f-e}\right)^{\frac{\alpha}{k}+1}\ {}_k\mathrm{J}^{\alpha}_{\left(\frac{e+f}{2}\right)^+}\Lambda(e). \end{aligned} \tag{2.2}$$

Also

$$\begin{aligned} I_2 &= \int_0^1 \varrho^{\frac{\alpha}{k}}\Lambda'\left(\varrho\frac{e+f}{2}+(1-\varrho)f\right)\mathrm{d}\varrho \\ &= -\frac{2}{f-e}\Lambda\left(\frac{e+f}{2}\right)+\Gamma_k(\alpha+1)\left(\frac{2}{f-e}\right)^{\frac{\alpha}{k}+1}\ {}_k\mathrm{J}^{\alpha}_{\left(\frac{e+f}{2}\right)^-}\Lambda(f). \end{aligned} \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) completes the proof. □

Note that for $k=1$, we have Lemma 2.1 [10].

Now utilizing Lemma 2.1, we derive our main results.

Theorem 2.2. *Let $\Lambda : I=[e,f]\subset\mathbb{R}\to\mathbb{R}$ be k-fractional differentiable function on I° and $\Lambda'\in L[e,f]$. If $|\Lambda'|$ is h-convex function, then*

$$|\Xi(e,f;\alpha,k)|\leq\frac{f-e}{4}\left\{2\left|\Lambda'\left(\frac{e+f}{2}\right)\right|\int_0^1\varrho^{\frac{\alpha}{k}}h(\varrho)\mathrm{d}\varrho+\{|\Lambda'(e)|+|\Lambda'(f)|\}\int_0^1\varrho^{\frac{\alpha}{k}}h(1-\varrho)\mathrm{d}\varrho\right\}.$$

Proof. Using Lemma 2.1, the hypothesis of the theorem and the property of modulus, we have

$$\begin{aligned} &|\Xi(e,f;\alpha)| \\ &\leq\frac{f-e}{4}\left\{\int_0^1\varrho^{\frac{\alpha}{k}}\left|\Lambda'\left(\varrho\frac{e+f}{2}+(1-\varrho)e\right)\right|\mathrm{d}\varrho+\int_0^1\varrho^{\frac{\alpha}{k}}\left|\Lambda'\left(\varrho\frac{e+f}{2}+(1-\varrho)f\right)\right|\mathrm{d}\varrho\right\} \\ &\leq\frac{f-e}{4}\left\{\int_0^1\varrho^{\frac{\alpha}{k}}\left[h(\varrho)\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+h(1-\varrho)|\Lambda'(e)|\right]\mathrm{d}\varrho\right. \\ &\quad\left.+\int_0^1\varrho^{\frac{\alpha}{k}}\left[h(\varrho)\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+h(1-\varrho)|\Lambda'(f)|\right]\mathrm{d}\varrho\right\} \\ &=\frac{f-e}{4}\left\{2\left|\Lambda'\left(\frac{e+f}{2}\right)\right|\int_0^1\varrho^{\frac{\alpha}{k}}h(\varrho)\mathrm{d}\varrho+\{|\Lambda'(e)|+|\Lambda'(f)|\}\int_0^1\varrho^{\frac{\alpha}{k}}h(1-\varrho)\mathrm{d}\varrho\right\}. \end{aligned}$$

This completes the proof. □

We now discuss some special cases of Theorem 2.2.
(i) If $h(\varrho) = \varrho$, then we have result for convex functions.

Corollary 2.3. *Under the conditions of Theorem 2.2, if $|\Lambda'|$ is convex function, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{f-e}{4}\left\{\Big(\frac{2k}{\alpha+2k}\Big)\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + \left(\frac{k^2}{(\alpha+k)(\alpha+2k)}\right)\{|\Lambda'(e)| + |\Lambda'(f)|\}\right\}.$$

(ii) If $h(\varrho) = \varrho^s$, then we have result for s-convex functions of Breckner type.

Corollary 2.4. *Under the conditions of Theorem 2.2, if $|\Lambda'|$ is s-convex function of Breckner type, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{f-e}{4}\left\{\Big(\frac{2k}{\alpha+ks+k}\Big)\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + kB_k(\alpha+k, ks+k)\{|\Lambda'(e)| + |\Lambda'(f)|\}\right\}.$$

(iii) If $h(\varrho) = \varrho^{-s}$, then we have result for s-convex functions of Godunova-Levin type.

Corollary 2.5. *Under the conditions of Theorem 2.2, if $|\Lambda'|$ is s-convex function of Godunova-Levin type, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{f-e}{4}\left\{\Big(\frac{2k}{\alpha-ks+k}\Big)\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + kB_k(\alpha+k, k-ks)\{|\Lambda'(e)| + |\Lambda'(f)|\}\right\}.$$

(iv) If $h(\varrho) = 1$, then we have result for P-convex functions.

Corollary 2.6. *Under the conditions of Theorem 2.2, if $|\Lambda'|$ is P-convex function, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{k(f-e)}{4(\alpha+k)}\left\{|\Lambda'(e)| + 2\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + |\Lambda'(f)|\right\}.$$

(v) If $h(\varrho) = \varrho(1-\varrho)$, then we have result for tgs-convex functions.

Corollary 2.7. *Under the conditions of Theorem 2.2, if $|\Lambda'|$ is tgs-convex function, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{k^2(f-e)}{4(\alpha+2k)(\alpha+3k)}\left\{|\Lambda'(e)| + 2\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + |\Lambda'(f)|\right\}.$$

Theorem 2.8. *Let $\Lambda : I = [e, f] \subset \mathbb{R} \to \mathbb{R}$ be k-fractional differentiable function on I° and $\Lambda' \in L[e, f]$. If $|\Lambda'|^q$ is h-convex function and if $q > 1, q \geq r \geq 0$, then*

$$\begin{aligned}|\Xi(e, f; \alpha, k)| \leq{}& \frac{f-e}{4}\left(\frac{k(q-1)}{\alpha(q-r)+k(q-1)}\right)^{1-\frac{1}{q}} \\ &\times\left\{2\left(\int_0^1 \varrho^{\frac{\alpha r}{k}} h(\varrho)\mathrm{d}\varrho\right)^{\frac{1}{q}}\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + \left(\int_0^1 \varrho^{\frac{\alpha r}{k}} h(1-\varrho)\mathrm{d}\varrho\right)^{\frac{1}{q}}[|\Lambda'(e)| + |\Lambda'(f)|]\right\}.\end{aligned}$$

Proof. Using Lemma 2.1 and the hypothesis of the theorem, we have

$$|\Xi(e, f; \alpha, k)|$$

$$\leq \frac{f-e}{4}\left\{\left(\int_0^1 \varrho^{\frac{\alpha(q-r)}{k(q-1)}}\mathrm{d}\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho^{\frac{\alpha r}{k}}\left|\Lambda'\left(\varrho\frac{e+f}{2}+(1-\varrho)e\right)\right|^q \mathrm{d}\varrho\right)^{\frac{1}{q}}+\left(\int_0^1 \varrho^{\frac{\alpha(q-r)}{k(q-1)}}\mathrm{d}\varrho\right)^{1-\frac{1}{q}}\right.$$
$$\left.\left(\int_0^1 \varrho^{\frac{\alpha r}{k}}\left|\Lambda'\left(\varrho\frac{e+f}{2}+(1-\varrho)f\right)\right|^q \mathrm{d}\varrho\right)^{\frac{1}{q}}\right\}$$
$$\leq \frac{f-e}{4}\left\{\left(\frac{k(q-1)}{\alpha(q-r)+k(q-1)}\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho^{\frac{\alpha r}{k}}\left[h(\varrho)\left|\Lambda'\left(\frac{e+f}{2}\right)\right|^q+h(1-\varrho)|\Lambda'(e)|^q\right]\mathrm{d}\varrho\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{k(q-1)}{\alpha(q-r)+k(q-1)}\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho^{\frac{\alpha r}{k}}\left[h(\varrho)\left|\Lambda'\left(\frac{e+f}{2}\right)\right|^q+h(1-\varrho)|\Lambda'(f)|^q\right]\mathrm{d}\varrho\right)^{\frac{1}{q}}\right\}$$
$$\leq \frac{f-e}{4}\left(\frac{k(q-1)}{\alpha(q-r)+k(q-1)}\right)^{1-\frac{1}{q}}$$
$$\times\left\{2\left(\int_0^1 \varrho^{\frac{\alpha r}{k}}h(\varrho)\mathrm{d}\varrho\right)^{\frac{1}{q}}\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+\left(\int_0^1 \varrho^{\frac{\alpha r}{k}}h(1-\varrho)\mathrm{d}\varrho\right)^{\frac{1}{q}}\left[|\Lambda'(e)|+|\Lambda'(f)|\right]\right\},$$

here we have used the fact that $\sum_{i=1}^{n}(e_i+f_i)^w \leq \sum_{i=1}^{n} e_i^w + \sum_{i=1}^{n} f_i^w$ where $0<w<1$, $e_1, e_2, \ldots, e_n \geq 0$ and $f_1, f_2, \ldots, f_n \geq 0$. This completes the proof. □

We now discuss some special cases of Theorem 2.8.
(i) If $h(\varrho)=\varrho$, then we have result for convex functions.

Corollary 2.9. *Under the assumptions of Theorem 2.8 if $|\Lambda'|^q$ is convex function and if $q>1, q\geq r\geq 0$, then*

$$|\Xi(e,f;\alpha,k)| \leq \frac{f-e}{4}\left(\frac{k(q-1)}{\alpha(q-r)+k(q-1)}\right)^{1-\frac{1}{q}}$$
$$\times\left\{2\left(\frac{k}{\alpha r+2k}\right)^{\frac{1}{q}}\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+\left(\frac{k^2}{(\alpha r+k)(\alpha r+2k)}\right)^{\frac{1}{q}}\left[|\Lambda'(e)|+|\Lambda'(f)|\right]\right\}.$$

(ii) If $h(\varrho)=\varrho^s$, then we have result for s-convex functions of Breckner type.

Corollary 2.10. *Under the assumptions of Theorem 2.8 if $|\Lambda'|^q$ is s-convex function of Breckner type and if $q>1, q\geq r\geq 0$, then*

$$|\Xi(e,f;\alpha,k)| \leq \frac{f-e}{4}\left(\frac{k(q-1)}{\alpha(q-r)+k(q-1)}\right)^{1-\frac{1}{q}}$$
$$\times\left\{2\left(\frac{k}{\alpha r+ks+k}\right)^{\frac{1}{q}}\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+(kB_k(k+\alpha r,k+ks))^{\frac{1}{q}}\left[|\Lambda'(e)|+|\Lambda'(f)|\right]\right\}.$$

(iii) If $h(\varrho)=\varrho^{-s}$, then we have result for s-convex functions of Godunova-Levin type.

Corollary 2.11. *Under the assumptions of Theorem 2.8 if $|\Lambda'|^q$ is s-convex function of Godunova-Levin type and if $q > 1, q \geq r \geq 0$, then*

$$|\Xi(e,f;\alpha,k)| \leq \frac{f-e}{4}\left(\frac{k(q-1)}{\alpha(q-r)+k(q-1)}\right)^{1-\frac{1}{q}} \times\left\{2\left(\frac{k}{\alpha r-ks+k}\right)^{\frac{1}{q}}\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + (kB_k(k+\alpha r, k-ks))^{\frac{1}{q}}\left[|\Lambda'(e)|+|\Lambda'(f)|\right]\right\}.$$

(iv) If $h(\varrho) = 1$, then we have result for P-convex functions.

Corollary 2.12. *Under the assumptions of Theorem 2.8 if $|\Lambda'|^q$ is P-convex function and if $q > 1, q \geq r \geq 0$, then*

$$|\Xi(e,f;\alpha,k)| \leq \frac{k(f-e)}{4(\alpha r+k)^{\frac{1}{q}}}\left(\frac{k(q-1)}{\alpha(q-r)+k(q-1)}\right)^{1-\frac{1}{q}}\left\{|\Lambda'(e)| + 2\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + |\Lambda'(f)|\right\}.$$

(v) If $h(\varrho) = \varrho(1-\varrho)$, then we have result for tgs-convex functions.

Corollary 2.13. *Under the assumptions of Theorem 2.8 if $|\Lambda'|^q$ is tgs-convex function and if $q > 1, q \geq r \geq 0$, then*

$$|\Xi(e,f;\alpha,k)| \leq \frac{k^2(f-e)}{4[(\alpha r+2k)(\alpha r+3k)]^{\frac{1}{q}}}\left(\frac{q-1}{\alpha(q-r)+q-1}\right)^{1-\frac{1}{q}}\left\{|\Lambda'(e)| + 2\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + |\Lambda'(f)|\right\}.$$

Theorem 2.14. *Let $\Lambda : I = [e, f] \subset \mathbb{R} \to \mathbb{R}$ be k-fractional differentiable function on I° and $\Lambda' \in L[e, f]$. If $|\Lambda'|^q$ is h-convex function and if $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$|\Xi(e,f;\alpha,k)| \leq \frac{f-e}{4}\left(\frac{k(q-1)}{\alpha q+k(q-1)}\right)^{\frac{1}{p}} \times\left\{2\left(\int_0^1 h(\varrho)\mathrm{d}\varrho\right)^{\frac{1}{q}}\left|\Lambda'\left(\frac{e+f}{2}\right)\right| + \left(\int_0^1 h(1-\varrho)\mathrm{d}\varrho\right)^{\frac{1}{q}}\left[|\Lambda'(e)|+|\Lambda'(f)|\right]\right\}.$$

Proof. Using Lemma 2.1 and the hypothesis of the theorem, we have

$$\begin{aligned}
&|\Xi(e,f;\alpha,k)| \\
&\leq \frac{f-e}{4}\left\{\left(\frac{k(q-1)}{\alpha q+k(q-1)}\right)^{\frac{1}{p}}\left(\int_0^1\left[h(\varrho)\left|\Lambda'\left(\frac{e+f}{2}\right)\right|^q + h(1-\varrho)|\Lambda'(e)|^q\right]\mathrm{d}\varrho\right)^{\frac{1}{q}}\right. \\
&\quad\left. + \left(\frac{k(q-1)}{\alpha q+k(q-1)}\right)^{\frac{1}{p}}\left(\int_0^1\left[h(\varrho)\left|\Lambda'\left(\frac{e+f}{2}\right)\right|^q + h(1-\varrho)|\Lambda'(f)|^q\right]\mathrm{d}\varrho\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

$$= \frac{f-e}{4}\left(\frac{k(q-1)}{\alpha q+k(q-1)}\right)^{\frac{1}{p}}\left\{2\left(\int_0^1 h(\varrho)\mathrm{d}\varrho\right)^{\frac{1}{q}}\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+\left(\int_0^1 h(1-\varrho)\mathrm{d}\varrho\right)^{\frac{1}{q}}[|\Lambda'(e)|+|\Lambda'(f)|]\right\},$$

here we have used the fact that $\sum_{i=1}^{n}(e_i+f_i)^w \leq \sum_{i=1}^{n} e_i^w + \sum_{i=1}^{n} f_i^w$ where $0 < w < 1$, $e_1, e_2, \ldots, e_n \geq 0$ and $f_1, f_2, \ldots, f_n \geq 0$. This completes the proof. □

We now discuss some special cases of Theorem 2.14.

(i) If $h(\varrho) = \varrho$, then we have result for convex functions.

Corollary 2.15. *Under the assumptions of Theorem 2.14 if $|\Lambda'|^q$ is convex function and if $q > 1$, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{f-e}{2^{2+\frac{1}{q}}}\left(\frac{k(q-1)}{\alpha q+k(q-1)}\right)^{1-\frac{1}{q}}\left\{|\Lambda'(e)|+2\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+|\Lambda'(f)|\right\}.$$

(ii) If $h(\varrho) = \varrho^s$, then we have result for s-convex functions of Breckner type.

Corollary 2.16. *Under the assumptions of Theorem 2.14 if $|\Lambda'|^q$ is s-convex function of Breckner type and if $q > 1$, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{f-e}{4(s+1)^{\frac{1}{q}}}\left(\frac{k(q-1)}{\alpha q+k(q-1)}\right)^{1-\frac{1}{q}}\left\{|\Lambda'(e)|+2\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+|\Lambda'(f)|\right\}.$$

(iii) If $h(\varrho) = \varrho^{-s}$, then we have result for s-convex functions of Godunova-Levin type.

Corollary 2.17. *Under the assumptions of Theorem 2.14 if $|\Lambda'|^q$ is s-convex function of Godunova-Levin type and if $q > 1$, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{f-e}{4(1-s)^{\frac{1}{q}}}\left(\frac{k(q-1)}{\alpha q+k(q-1)}\right)^{1-\frac{1}{q}}\left\{|\Lambda'(e)|+2\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+|\Lambda'(f)|\right\}.$$

(iv) If $h(\varrho) = 1$, then we have result for P-convex functions.

Corollary 2.18. *Under the assumptions of Theorem 2.14 if $|\Lambda'|^q$ is P-convex function and if $q > 1$, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{f-e}{4}\left(\frac{k(q-1)}{\alpha q+k(q-1)}\right)^{1-\frac{1}{q}}\left\{|\Lambda'(e)|+2\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+|\Lambda'(f)|\right\}.$$

(v) If $h(\varrho) = \varrho(1-\varrho)$, then we have result for tgs-convex functions.

Corollary 2.19. *Under the assumptions of Theorem 2.8 if $|\Lambda'|^q$ is tgs-convex function and if $q > 1$, then*

$$|\Xi(e, f; \alpha, k)| \leq \frac{f-e}{2^{2+\frac{1}{q}}3^{\frac{1}{q}}}\left(\frac{k(q-1)}{\alpha(q-r)+k(q-1)}\right)^{1-\frac{1}{q}}\left\{|\Lambda'(e)|+2\left|\Lambda'\left(\frac{e+f}{2}\right)\right|+|\Lambda'(f)|\right\}.$$

3. Applications

In this section, we discuss some applications for our results to means of special numbers. Let us recall some previously known concepts.

Definition 3.1 ([11]). Recall the following definitions:

(1) For arbitrary $e > 0$, $f > 0$ and $e \neq f$

$$\mathcal{L}(e, f) = \frac{f - e}{\log f - \log e},$$

is the logarithmic mean.

(2) For arbitrary $e, f \in \mathbb{R}$ and $e \neq f$

$$\mathcal{A}(e, f) = \frac{e + f}{2},$$

is the arithmetic mean.

(3) For arbitrary $e, f \in \mathbb{R}$ and $e \neq f$

$$\mathcal{L}_p(e, f) = \left[\frac{f^{p+1} - e^{p+1}}{(f - e)(p + 1)}\right]^{\frac{1}{p}},$$

is the generalized log-mean, $p \neq -1, 0$.

We now present our applications.

Proposition 3.1. *Let $e, f \in \mathbb{R}$, $e < f$ and $n \in \mathbb{R}$, $n \geq 2$, then*

$$\left|\mathcal{A}^n(e, f) - \mathcal{L}_n^n(e, f)\right| \leq \frac{|n|(f - e)}{12}\left\{2|\mathcal{A}^{n-1}(e, f)| + \mathcal{A}(|e|^{n-1}, |f|^{n-1})\right\}.$$

Proof. The proof is immediate from Corollary 2.3 applied for $\alpha = 1 = k$ and $\Lambda(u) = u^n$, $u \in \mathbb{R}$. □

Proposition 3.2. *Let $e, f \in \mathbb{R}$, $e < f$ and $0 < s < 1$, then*

$$\left|\mathcal{A}^s(e, f) - \mathcal{L}_s^s(e, f)\right| \leq \frac{|s|(f - e)}{2}\left\{\left(\frac{1}{s + 2}\right)|\mathcal{A}^{s-1}(e, f)| + \left(\frac{1}{(s + 1)(s + 2)}\right)\mathcal{A}(|e|^{s-1}, |f|^{s-1})\right\}.$$

Proof. The proof is immediate from Corollary 2.4 applied for $\alpha = 1 = k$ and $\Lambda : [0, 1] \to [0, 1]$, $\Lambda(u) = u^s$. □

Proposition 3.3. *Let $e, f \in \mathbb{R}$, $e < f$ and $n \in \mathbb{R}$, $n \geq 2$, then*

$$\begin{aligned}&\left|\mathcal{A}^n(e, f) - \mathcal{L}_n^n(e, f)\right| \\ &\leq \frac{|n|(f - e)}{2}\left(\frac{q - 1}{2q - r - 1}\right)^{1-\frac{1}{q}}\left\{\left(\frac{1}{r + 2}\right)^{\frac{1}{q}}|\mathcal{A}^{n-1}(e, f)| + \left(\frac{1}{(r + 1)(r + 2)}\right)^{\frac{1}{q}}\mathcal{A}(|e|^{n-1}, |f|^{n-1})\right\}.\end{aligned}$$

Proof. The proof is immediate from Corollary 2.9 applied for $\alpha = 1 = k$ and $\Lambda(u) = u^n$, $u \in \mathbb{R}$. □

Proposition 3.4. *Let* $e, f \in \mathbb{R}$, $e < f$ *and* $0 < s < 1$, *then*

$$\left|\mathcal{A}^s(e,f) - \mathcal{L}_s^s(e,f)\right| \leq \frac{|s|(f-e)}{4}\left(\frac{q-1}{2q-r-1}\right)^{1-\frac{1}{q}} \times \left\{2\left(\frac{1}{r-s+1}\right)^{\frac{1}{q}} |\mathcal{A}^{s-1}(e,f)| + B^{\frac{1}{q}}(r+1, s+1)\mathcal{A}(|e|^{s-1}, |f|^{s-1})\right\}.$$

Proof. The proof is immediate from Corollary 2.10 applied for $\alpha = 1 = k$ and $\Lambda : [0,1] \to [0,1]$, $\Lambda(u) = u^s$. □

Proposition 3.5. *Let* $e, f \in \mathbb{R}$, $e < f$ *and* $n \in \mathbb{R}$, $n \geq 2$, *then*

$$\left|\mathcal{A}^n(e,f) - \mathcal{L}_n^n(e,f)\right| \leq \frac{|n|(f-e)}{2^{1+\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left\{\mathcal{A}^{n-1}(e,f)| + \mathcal{A}(|e|^{n-1}, |f|^{n-1})\right\}.$$

Proof. The proof is immediate from Corollary 2.15 applied for $\alpha = 1 = k$ and $\Lambda(u) = u^n$, $u \in \mathbb{R}$. □

Proposition 3.6. *Let* $e, f \in \mathbb{R}$, $e < f$ *and* $0 < s < 1$, *then*

$$\left|\mathcal{A}^s(e,f) - \mathcal{L}_s^s(e,f)\right| \leq \frac{|s|(f-e)}{2(s+1)^{\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left\{\mathcal{A}^{s-1}(e,f)| + \mathcal{A}(|e|^{s-1}, |f|^{s-1})\right\}.$$

Proof. The proof is immediate from Corollary 2.16 applied for $\alpha = 1 = k$ and $\Lambda : [0,1] \to [0,1]$, $\Lambda(u) = u^s$. □

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Conflict of interest

Authors declare that they have no conflict of interest.

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